THE THOM CLASS OF MATHAI AND QUILLEN AND PROBABILITY THEORY

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As we will explain in this talk, the construction by Mathai and Quillen of explicit differential form representatives of the Thom class and Euler class of a vector bundle [5] gives a framework for understanding in unified way a number of ideas in stochastic differential geometry. We will also show briefly how Witten's topological quantum field theories fit into this formalism. For the moment, the picture that he envisages is inaccessible to rigourous methods; but even in the more humdrum world of Brownian motion, the point of view presented here illuminates quite a few of the other talks that were given at this conference.

For a more thorough presentation of the mathematical portion of this talk, we refer the reader to Chapters 1 and 7 of the book [2].

1. The Thom class

Recall that the de Rham complex of differential forms on a manifold M is consists of the infinite dimensional vector spaces $\mathcal{A}^i(M)$ of i-forms,with the exterior differential

$$d : \mathcal{A}^i(M) \to \mathcal{A}^{i+1}(M).$$

Let $E \xrightarrow{\pi} M$ be an orientable vector bundle with fibre $\mathbb{R}^m$ over a manifold M. A **Thom class** for the bundle E is a differential form $\mu(E) \in \mathcal{A}^m(E)$ on E such that

(1) $\mu(E)$ is closed;
(2) $\pi_*\mu(E)$, the integral of $\mu(E)$ over the fibres of π, equals 1.

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To construct a Thom class, we suppose that we are given three pieces of data:

(1) an orientation of E, that is, a nowhere-vanishing section of the highest exterior power $\Lambda^m E$ of E;
(2) a metric on the bundle E, that is, a bundle map
$$E \otimes E \longrightarrow M \times \mathbb{R}$$
which induces positive definite inner products $(\cdot,\cdot)_x$ on each fibre E_x of E;
(3) a connection D compatible with the metric $(\cdot,\cdot)$, that is, a map
$$D : \mathcal{A}^i(M,E) \longrightarrow \mathcal{A}^{i+1}(M,E)$$
such that
$$D(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^j \alpha \wedge (D\beta)$$
for all $\alpha \in \mathcal{A}^i(M)$ and $\beta \in \mathcal{A}^j(M,E)$ and such that
$$d(s_1, s_2) = (Ds_1, s_2) + (s_1, Ds_2)$$
for all s_1 and $s_2 \in \Gamma(M,E)$.

Since E has a metric, we can reformulate the orientation of the bundle E as an isometry
$$\mathsf{B}\{\cdot\} : \Lambda^m E \longrightarrow M \times \mathbb{R}$$
of the highest exterior power of E with the trivial line bundle over M. The map $\mathsf{B}\{\cdot\}$ is the **Berezin integral**.

Let π^*E be the pull-back of the bundle E over M to a bundle over E, whose fibre at $e \in E$ is $E_{\pi(e)}$; this bundle has a metric $\pi^*(\cdot,\cdot)$ with compatible connection π^*D; we will usually write these as $(\cdot,\cdot)$ and D. Let $\Lambda^*\pi^*E \longrightarrow E$ be the bundle of exterior algebras of π^*E, and let $\mathcal{A}^*(E, \Lambda^*\pi^*E)$ be the algebra of differential forms on E with values in $\Lambda^*\pi^*E$; it is a bigraded algebra, with graded subspaces
$$\mathcal{A}^{i,j} = \mathcal{A}^i(E, \Lambda^j \pi^* E).$$
The Berezin integral $\mathsf{B}\{\cdot\} : \Lambda^m E \longrightarrow M \times \mathbb{R}$ extends to a linear form
$$\mathsf{B}\{\cdot\} : \mathcal{A}^i(E, \Lambda^j \pi^* E) \longrightarrow \mathcal{A}^i(E)$$
which vanishes unless $j = m$.

Let x be the tautological section of π^*E, that is, the section which maps a point $e \in E$ to the corresponding point $e \in \pi^*E_e = E_{\pi(e)}$. We may think of x as an element of $\mathcal{A}^{0,1}$, and its covariant derivative Dx is an element of $\mathcal{A}^{1,1}$. Define an operator $\iota(x) : \mathcal{A}^{i,j} \longrightarrow \mathcal{A}^{i,j-1}$, characterized by the following properties:

(1) if $w \in \mathcal{A}^{0,1} = \Gamma(E, \pi^*E)$, then $\iota(x)w = (x, w)$ (for example, $\iota(x)x = |x|^2$);
(2) $\iota(x)$ is a derivation, that is,
$$\iota(x)(\alpha \wedge \beta) = (\iota(x)\alpha) \wedge \beta + (-1)^{i+j} \alpha \wedge (\iota(x)\beta)$$
for $\alpha \in \mathcal{A}^{i,j}$ and $\beta \in \mathcal{A}^{k,l}$.

Let D_t be the operator $D + t\iota(x)$, where $t > 0$.

Lemma 1.1. *If $\alpha \in \mathcal{A}^*(E, \Lambda^*\pi^*E)$, then $d\mathsf{B}\{\alpha\} = \mathsf{B}\{D_t\alpha\}$.*

Proof. This follows from the fact that D is compatible with the metric, so that $d\mathsf{B}\{\alpha\} = \mathsf{B}\{D\alpha\}$, combined with the obvious fact that $\iota(x)\alpha$ has no component in $\mathcal{A}^*(E, \Lambda^m\pi^*E)$. $\square$

The curvature $\Omega = D^2$ of the connection D on E is an element of $\mathcal{A}^2(M, \mathfrak{so}(E))$, where $\mathfrak{so}(E)$ is the bundle of skew-symmetric endomophisms of E. We may identify the bundle $\mathfrak{so}(E)$ with Λ^2E; if $A \in \mathfrak{so}(E_y)$ and e_i is an orthonormal basis of E_y, we map A to

$$\sum_{1\le i<j\le m} (e_i, Ae_j)\, e_i \wedge e_j \in \Lambda^2 E_y,$$

so that $\iota(x)A = -Ax$. The pull-back $\pi^*\Omega$ of Ω from M to E is then an element of $\mathcal{A}^{2,2}$, which we will usually denote by Ω. Consider the following differential form (here, t is a positive real number):

$$\omega_t = \tfrac{1}{2}t^2|x|^2 + tDx + \Omega.$$

Lemma 1.2. *The following formula holds: $D_t\omega_t = 0$.*

Proof. Note that $D\Omega = 0$, by Bianchi's identity, and that $D|x|^2 = -2\iota(x)Dx$. From this, we see that

$$D\omega_t = \frac{t^2}{2}D|x|^2 + tD^2x = -\iota(x)\big(t^2Dx + t\Omega\big) \in \mathcal{A}^{1,0} \oplus \mathcal{A}^{2,1}.$$

The lemma follows, since it is clear that $\iota(x)|x|^2 = 0$. $\square$

Suppose f is a polynomial in one variable. It is easy to see that $f(\omega_t)$ is the element of $\mathcal{A}^{*,*}$ given by the Taylor expansion

$$f(\omega_t) = \sum_{k=0}^{m} \frac{f^{(k)}(t^2|x|^2/2)}{k!}(tDx + \Omega)^k.$$

(The expansion terminates at $k = m$ because $(tDx+\Omega)^k = 0$ for $k > m$.) We adopt this formula as the definition of $f(\omega_t)$ for any smooth function on $\mathbb{R}$.

Proposition 1.3.

(1) *The differential form $\mathsf{B}\{f(\omega_t)\}$ is a closed m-form on E.*

(2) *If f decays at infinity, the integral over the fibres $\pi_*f(\omega_t)$ is the constant*

$$\pi_*\mathsf{B}\{f(\omega_t)\} = (-1)^{m(m+1)/2}\int_{\mathbb{R}^m} f^{(m)}(|x|^2/2)\, dx.$$

(3) *(transgression formula)*

$$\frac{d\mathsf{B}\{f(\omega_t)\}}{dt} = d\mathsf{B}\{xf'(\omega_t)\}$$

Proof. To see that $\mathsf{B}\{f(\omega_t)\} \in \mathcal{A}^m(E)$, we simply note that

$$f(\omega_t) \in \sum_{k=0}^{m} \mathcal{A}^{k,k},$$

and hence that only the component in $\mathcal{A}^{m,m}$ contributes to the Berezin integral.

The proof that $\mathsf{B}\{f(\omega_t)\}$ is closed is an easy consequence of Leibniz's rule and Lemmas 1.1 and 1.2:

$$d\mathsf{B}\{f(\omega_t)\} = \mathsf{B}\{f'(\omega_t)\, D_t\omega_t\} = 0.$$

The formula for $\pi_*\mathsf{B}\{f(\omega_t)\}$ may be checked in the special case in which M is a point, and hence we may take E to equal $\mathbb{R}^m$. If x_i is the standard orthonormal basis of $\mathbb{R}^m$, we see that

$$Dx = \sum_{i=1}^{m} dx_i \otimes x_i,$$

and hence that

$$\begin{aligned}(Dx)^m &= m!(dx_1 \otimes x_1) \dots (dx_m \otimes x_m) \\ &= (-1)^{m(m+1)/2} m!(x_1 \wedge \dots \wedge x_m) \otimes (dx_1 \dots dx_m).\end{aligned}$$

We see that

$$\begin{aligned}\mathsf{B}\{f(\omega_t)\} &= t^m \mathsf{B}\left\{f^{(m)}(t^2|x|^2/2)(dx_1 \otimes x_1) \dots (dx_m \otimes x_m)\right\} \\ &= (-1)^{m(m+1)/2} t^m\, \mathsf{B}\left\{f^{(m)}(t^2|x|^2/2) x_1 \dots x_m\right\} dx_1 \dots dx_m \\ &= (-1)^{m(m+1)/2} t^m\, f^{(m)}(t^2|x|^2/2)\, dx_1 \dots dx_m,\end{aligned}$$

since all of the other terms in the Taylor expansion have vanishing Berezin integral. From this, (2) follows easily.

The proof of the transgression formula is similar to the proof that $d\mathsf{B}\{f(\omega_t)\} = 0$. On the one hand,

$$\frac{d\mathsf{B}\{f(\omega_t)\}}{dt} = \mathsf{B}\left\{(t|x|^2 + Dx) f(\omega_t)\right\},$$

while on the other,

$$d\mathsf{B}\{x f'(\omega_t)\} = \mathsf{B}\{(D_t x) f(\omega_t)\} = \mathsf{B}\left\{(Dx + t|x|^2) f(\omega_t)\right\}. \quad \square$$

Being probabilists, we are especially fond of the Gaussian function, so we choose as our Thom class the following differential form:

$$\mu_t(E) = (-1)^{m(m-1)/2} (2\pi)^{-m/2} \mathsf{B}\left\{e^{-\omega_t}\right\} \in \mathcal{A}^m(E).$$

Let s be a section of the bundle E. We can form the pull-back $s^*\mu_t(E) \in \mathcal{A}^m(M)$; it is called the **Euler class** of the bundle E. The following formula for the differential form $s^*\mu_t(E)$ is obvious from the definition of $\mu_t(E)$:

$$s^*\mu_t(E) = (-1)^{m(m-1)/2}(2\pi)^{-m/2}\mathsf{B}\left\{e^{-t^2|s|^2/2-tDs-\Omega}\right\},$$

where $t^2|s|^2/2 + tDs + \Omega/2$ is thought of as a section of $\mathcal{A}^*(M,\Lambda^*E)$, and $\mathsf{B}\{\cdot\}$ is the Berezin integral from $\mathcal{A}^*(M,\Lambda^*E)$ to $\mathcal{A}^*(M)$.

Proposition. *The cohomology class of the differential form $s^*\mu_t(E) \in \mathcal{A}^m(M)$ is independent of the section s.*

Proof. Let $s_t = s + \tau$ be an affine one-parameter family of sections of E. We see that

$$\begin{aligned}
(-1)^{m(m-1)/2}(2\pi)^{m/2}\frac{ds_\tau^*\mu_t(E)}{d\tau} &= \frac{d}{d\tau}\mathsf{B}\left\{e^{-t^2|s_\tau|^2/2-tDs_\tau-\Omega}\right\} \\
&= -s_\tau^*\mathsf{B}\left\{(t^2(x,u)+tDu)e^{-t^2|x|^2/2-tDx-\Omega}\right\} \\
&= -s_\tau^*\mathsf{B}\left\{(tD_tu)e^{-t^2|x|^2/2-tDx-\Omega}\right\} \\
&= -ts_\tau^*d\mathsf{B}\left\{ue^{-t^2|x|^2/2-tDx-\Omega}\right\} \\
&= -td\mathsf{B}\left\{ue^{-t^2|s_\tau|^2/2-tDs_\tau-\Omega}\right\} \qquad \square
\end{aligned}$$

2. The Gauss-Bonnet theorem

In this section, we explain the geometric significance of the Euler class: its Poincaré dual represents the homology class of the zero-set of a non-degenerate section of E. We will assume for simplicity that M is compact, since otherwise we would have to formulate general conditions on the section s to increase at infinity, and these are better done on a case-by-case basis.

Suppose that the section $s \in \Gamma(M,E)$ has the special property that its zero-set M_0 is a submanifold of M, and furthermore that $\nabla s \in \Gamma(M,\mathrm{Hom}(TM,E))$ is surjective along M_0; we call such a section **non-degenerate**. In particular, it follows that $\dim(M_0) = \dim(M) - m$. Let δ_{M_0} be the current on M whose value on a differential form $\alpha \in \mathcal{A}^*(M)$ is

$$\langle \delta_{M_0}, \alpha\rangle = \int_{M_0} \alpha\big|_{M_0}.$$

Thus, δ_{M_0} vanishes on α unless $\alpha \in \mathcal{A}^{\dim(M)-m}(M)$. The following theorem explains the significance of the Euler class.

Theorem 2.1. *The differential forms $s^*\mu_t(E) \in \mathcal{A}^m(M)$ converge, as $t \to \infty$, to a current of the form*
$$\lim_{t\to\infty} s^*\mu_t(E) = \varepsilon\delta_{M_0},$$
where $\varepsilon : M_0 \longrightarrow \{\pm 1\}$ is a continuous function which measures whether the map ∇s preserves or reverses orientation.

Proof. By (1.3), we see that
$$s^*\mu_t(E) = (-1)^{m(m-1)/2}(2\pi)^{-m/2}\mathsf{B}\left\{e^{-t^2|s|^2/2-tDs-\Omega}\right\}.$$
To take the limit as $t \to \infty$ in this formula, we observe that there is clearly no contribution from the region in which $|s| >$, for some small $c > 0$. Thus, the limiting current is supported on the submanifold M_0. We may assume that M is an open subset of $\mathbb{R}^n$, parametrized by coordinates
$$(x_1, \dots, x_m, y_1, \dots, y_{n-m}),$$
that $M_0 = U \cap \mathbb{R}^m$, where
$$\mathbb{R}^m = \{(x, y) \in \mathbb{R}^n \mid x = 0\}$$
and that E is the trivial bundle with fibre $\mathbb{R}^m$. In a neighbourhood of M_0, we see that the endomorphism $\nabla s : M \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is surjective, and hence, possibly replacing M by a smaller neighbourhood of M_0 in $\mathbb{R}^n$, we may choose as a frame of the bundle E the sections
$$e_i = D_{\partial/\partial x_i} s, \quad 1 \le i \le m.$$
For the moment, assume that the resulting frame of E is oriented.

Let $A_{ij} = (e_i, e_j)$ be the $m \times m$-matrix of inner products in this frame. By changing the coordinate system on M to
$$\tilde{x}_i = \left(A(0, y)^{-1/2}\right)_{ij} x_j,$$
we may assume that $(e_i, e_j) = \delta_{ij} + O(|x|)$.

We see that the section s may be written
$$s(x, y) = \sum_{i,j=1}^m x_i f_{ij}(x, y) e_j(x, y),$$

where $f_{ij} \in C^\infty(M)$ are functions satisfying $f_{ij}(0,y) = \delta_{ij}$. It follows that

$$Ds = \sum_{j=1}^m \Big(\sum_{i=1}^m g_{ij} dx_i \otimes e_j + \sum_{i=1}^{n-m} h_{ij} dy_i \otimes e_j \Big),$$

where $g_{ij}(0,y) = \delta_{ij}$ and $h_{ij}(0,y) = 0$. Finally, let $\Omega_0 \in \mathcal{A}^2(M_0, \Lambda^2 E)$ be the restriction of the curvature $\Omega \in \mathcal{A}^2(M, \Lambda^2 E)$ to $M_0 \subset M$.

If $\varphi \in C_0^\infty(M)$ is a function on M with compact support, we have

$$\int_M \varphi\, \mu_t\, dx\, dy = (-1)^{m(m-1)/2} (2\pi)^{-m/2} \int_M \varphi\, \mathsf{B} \left\{ e^{-t^2|s|^2/2 - tDs - \Omega} \right\} dx\, dy.$$

Pull this differential form back by the map $\rho_t(x,y) = (t^{-1}x, y)$. Since the integral of a differential form is invariant under pull-back, we see that

$$\int_M \varphi\, \mu_t\, dx = (-1)^{m(m-1)/2} (2\pi)^{-m/2} \int_{\mathbb{R}^m \times M_0} \varphi(t^{-1}x, y)\, \mathsf{B} \left\{ e^{-t^2|s(t^{-1}x,y)|^2/2 - t\rho_t^* Ds - \rho_t^* \Omega} \right\} dx\, dy.$$

Under these deformations, the exponent becomes

$$\frac{1}{2} \sum_{i=1}^m x_i^2 + \sum_{i=1}^m dx_i \otimes e_i + \Omega_0(0,y) + O(t).$$

By the dominated convergence theorem, the integral over $\mathbb{R}^m \times M_0$ converges to

$$(2\pi)^{-m/2} \int_{\mathbb{R}^m \times M_0} \varphi(0,y)\, e^{-|x|^2/2} \mathsf{B} \left\{ \exp\big(- \textstyle\sum_{i=1}^m dx_i \otimes e_i(0,y) - \Omega_0(0,y) \big) \right\} dx\, dy.$$

It is easily seen that Ω_0 cannot contribute to this integral, since any term involving Ω_0 will not have enough powers of dx_i to give a non-zero answer. This shows that this limit equals $\int_{M_0} \varphi(0,y)\, dy$ as desired.

This handles the case in which the basis e_i is an oriented basis of E. If it is not an oriented basis, there is an additional sign $\varepsilon = -1$, but otherwise the above calculation is unchanged. □

Theorem 2.1 implies as a special case the Gauss-Bonnet-Chern theorem. Let M be a compact, oriented, even-dimensional Riemannian manifold, and let $E = T^*M$. We choose a Morse function f on M, and consider the section $s = df$. By

Proposition 1.4, the differential forms $s^*\mu_t(E)$ are cohomologous. Setting $t = 0$, we obtain the differential form

$$s^*\mu_0(T^*M) = (-2\pi)^{-m/2}\mathsf{B}\left\{e^{-\Omega}\right\} = (2\pi)^{-m/2}\,\mathrm{Pf}(\Omega),$$

where $\mathrm{Pf}(\Omega) = \mathsf{B}\left\{e^{\Omega}\right\}$ is the Pfaffian of Ω. On the other hand, as $t \to \infty$, the differential forms $s^*\mu_t(T^*M)$ converge in the distributional sense to the differential form

$$s^*\mu_\infty(T^*M) = \sum_{df(x)=0} \operatorname{sgn}\det(\mathrm{Hess}_x(f))\,\delta_x.$$

Here, $\mathrm{Hess}_x(f) = \nabla df(x)$ is the Hessian of the function f at $x \in M$, which may be considered to be an element of $\mathrm{End}(T^*_xM)$.

In this way, we see that

$$\int_M s_0^*\mu(T^*M) = (2\pi)^{-m/2}\int_M \mathrm{Pf}(\Omega)$$

and

$$\lim_{t\to\infty}\int_M s^*\mu_t(T^*M) = \sum_{df(x)=0} \operatorname{sgn}\det(\mathrm{Hess}_x(f))$$

are equal. By Morse Theory, this second sum is just the Euler number of the manifold, and we obtain the Gauss-Bonnet-Chern theorem.

3. An infinite-dimensional example

In this section, we will see that the Euler class constructed in Section 1 can be used to understand the functional integral of the Laplace-Beltrami operator on a compact manifold X. This is a simple example of a topological quantum field theory: the Hilbert space is the space of harmonic forms on X, which is a topological invariant of the manifold.

If X is a compact orientable Riemannian manifold, let LX be the loop space of X; this is the Hilbert manifold of all maps $\gamma : S^1 \longrightarrow X$ of finite energy,

$$E(\gamma) = \int_{S^1} |\dot\gamma(t)|^2\,dt < \infty.$$

The tangent space $T_\gamma LX$ of the loop space at a loop γ is the space of all finite energy tangent vector fields along γ.

If $\mathrm{SO}(X)$ is the orthonormal frame bundle of X, then the loop space $L\,\mathrm{SO}(X)$ is a principal bundle over LX with structure group $L\,\mathrm{SO}(n)$. The tangent bundle of

LX is the bundle associated to $L\,\mathrm{SO}(X)$ with fibre $L^2_1(S^1, \mathbb{R}^n)$; however, the action of $L\,\mathrm{SO}(n)$ is not unitary, so this representation does not give the Hilbert bundle structure of $T(LX)$.

The principal bundle $L\,\mathrm{SO}(X)$ has a connection which is derived from the Levi-Civita connection on $\mathrm{SO}(X)$ as follows: if $\theta \in \mathcal{A}^1(\mathrm{SO}(X); \mathfrak{so}(n))$ is the connection one-form on $\mathrm{SO}(X)$, then the connection form on $L\,\mathrm{SO}(X)$ is just the element of $\mathcal{A}^1(L\,\mathrm{SO}(X); L\mathfrak{so}(n))$ given by integrating θ around the circle. It is not very difficult to calculate the curvature of this connection; it is just the element of $\mathcal{A}^2(L\,\mathrm{SO}(X); \mathfrak{so}(n))$ given by integrating the curvature form $R \in \mathcal{A}^2(\mathrm{SO}(X); \mathfrak{so}(n))$ around the circle:

$$(\Omega(X,Y)Z, W) = \int_{S^1} \big(R_{\gamma(t)}(X_t, Y_t)Z_t, W_t\big)\, dt.$$

Over the manifold LX, we will also consider the Hilbert bundle E whose fibre at $\gamma \in LX$ is the space of square-integrable vector fields along γ; this is the bundle associated to $L\,\mathrm{SO}(X)$ with fibre $L^2(S^1, \mathbb{R}^n)$. On this Hilbert space, the structure group $L\,\mathrm{SO}(n)$ acts in a unitary fashion, so that E is a Hilbert bundle with compatible covariant derivative, derived from the connection on $L\,\mathrm{SO}(X)$.

There is a smooth section of the bundle E, given by taking a finite energy loop γ to the L^2-vector field $\dot\gamma \in E_\gamma$, the tangent vector field to the loop. It is clear that the zero-set of this section is the space of constant loops, which we may think of as a manifold $X \subset LX$. Another class of sections of E are determined by smooth functions $f \in C^\infty(M)$: we map $\gamma \in LX$ to the tangent vector field

$$\big(t \mapsto \operatorname{grad} f(\gamma(t))\big) \in E_\gamma.$$

Call this section $\operatorname{grad} f$.

Lemma 3.1. *If f is a Morse function, the section $s_f = \dot\gamma + \operatorname{grad} f$ has as its zero set the finite set of constant loops taking values at a critical point of f.*

Proof. Let us calculate the L^2-norm of $\dot\gamma + \operatorname{grad} f$:

$$\begin{aligned}\int_{S^1} |\dot\gamma(t) + \operatorname{grad}_{\gamma(t)} f|^2\, dt &= \int_{S^1} \Big(|\dot\gamma(t)|^2 + 2(\dot\gamma(t), \operatorname{grad}_{\gamma(t)} f) + |\operatorname{grad}_{\gamma(t)} f|^2\Big)\, dt \\ &= E(\gamma) + \int_{S^1} V(\gamma(t))\, dt,\end{aligned}$$

where $V = |\operatorname{grad} f|^2 \in C^\infty(X)$. Here we have used the fact that the cross-term vanishes by integration by parts:

$$\int_{S^1} (\dot\gamma(t), \operatorname{grad}_{\gamma(t)} f)\, dt = \int_{S^1} \frac{d}{dt} f(\gamma(t))\, dt = 0. \quad \square$$

This calculation is familiar from the study of the Nicolai map (see [3] for references, and also [4]). Indeed, we see that the Nicolai map is just a section of a vector bundle over the space of fields, and that the supersymmetric quantum field theories with Nicolai maps are just those whose finite temperature (periodic time) functional integral is an Euler class: these are the **topological quantum field theories**.

Let us now try to make some sense of the Euler class of E obtained by pulling back the Thom form by the section s_f. According to the formulas of Section 1, this is the differential form on E given by the formula

$$(-1)^{m(m-1)/2}(2\pi)^{-m/2}\exp\left(-\tfrac{1}{2}\textstyle\int_{S^1}V(\gamma(t))\right)e^{-E(\gamma)/2}\mathsf{B}\left\{e^{-Ds_f-\Omega}\right\}.$$

The difficulty with this expression is that since the bundle E is infinite dimensional, $\mathsf{B}\{\cdot\}$ makes no sense. The constant $(-1)^{m(m-1)/2}(2\pi)^{-m/2}$ is also meaningless, of course; we are not even sure why the rank of the bundle E is even. Of course, the solution of these problems is tied together.

Let us look at the case in which $X=\mathbb{R}^n$, which was discussed in [3]. Here, $\Omega=0$ while Ds_f may be thought of as the infinite-dimensional generalization of

$$\sum_{ij}A_{ij}dx_i\otimes e_j,$$

where A_{ij} is the operator

$$a\in C^\infty(S^1,\mathbb{R}^n)\mapsto\frac{da(t)}{dt}+\mathrm{Hess}_{\gamma(t)}(f)a(t).$$

If we set $f=0$, we may interpret the Euler class as being the Brownian bridge measure on $L\mathbb{R}^n$, thought of as a volume form. For this measure to be defined, we must of course replace the manifold $L\mathbb{R}^n$ by the space of all continuous loops in $\mathbb{R}^n$. However, we will be somewhat lax in keeping track of this.

The discussion of [3] justifies the following result (which is more a definition than a theorem, since it relates two objects, one of which is a ill-defined). Let $d_f=e^{-f}\cdot d\cdot e^f$, and let Δ_f be Witten's twisted Laplace-Beltrami operator

$$\Delta_f=(d_f+d_f^*)^2=\Delta+|\operatorname{grad}f|^2+\mathrm{Hess}(f),$$

acting on $\mathcal{A}^*(\mathbb{R}^n)$, where $\mathrm{Hess}(f)$ is the derivation of the algebra of differential forms which vanishes on functions and equals $\mathrm{Hess}(f)\alpha$ for a one-form α (see [6]).

If $\alpha=f_0df_1\dots df_k$ is a differential form on $\mathbb{R}^n$, let $c(\alpha)$ be the operator of Clifford multiplication on differential forms given by the formula

$$c(f_0df_1\dots df_k)=\frac{1}{k!}\sum_{\sigma\in S_k}(-1)^{\varepsilon(\sigma)}f_0\,(\varepsilon(df_{\sigma_1})-\varepsilon(df_{\sigma_1})^*)\dots(\varepsilon(df_{\sigma_k})-\varepsilon(df_{\sigma_k})^*).$$

Finally, let Str be the supertrace on the Hilbert space of L^2-differential forms on $\mathbb{R}^n$, that is,

$$\mathrm{Str}_{\mathcal{A}(\mathbb{R}^n)}(A) = \mathrm{Tr}_{\mathcal{A}^{2*}(\mathbb{R}^n)}(A) - \mathrm{Tr}_{\mathcal{A}^{2*+1}(\mathbb{R}^n)}(A).$$

A cylinder form is a differential form on $L\mathbb{R}^n$ obtained by taking differential forms $\{\alpha_i \in \mathcal{A}^*(\mathbb{R}^n) \mid 1 \le i \le n\}$, and times $0 \le t_1 < \cdots < t_n < 1$, and forming

$$\gamma_{t_1}^* \alpha_1 \dots \gamma_{t_n}^* \alpha_n \in \mathcal{A}(L\mathbb{R}^n).$$

"Theorem". *The integral of a cylinder form against the Euler form $s_f^*\mu(E)$ is given by the formula*

$$\begin{aligned}\int_{L\mathbb{R}^m} &\gamma_{t_1}^* \alpha_1 \dots \gamma_{t_n}^* \alpha_n \wedge s_f^*\mu(E) \\ &= \mathrm{Str}_{\mathcal{A}(\mathbb{R}^m)}\big(e^{-t_1\Delta_f} c(\alpha_1) e^{-(t_1-t_2)\Delta_f} \dots e^{-(t_{n-1}-t_n)\Delta_f} c(\alpha_n) e^{-(1-t_n)\Delta_f}\big),\end{aligned}$$

where $e^{-t\Delta}$ is the heat kernel of the operator Δ_f.

Emboldened by this success, it is clear that there is only one plausible possibility for the Euler class on LX for arbitrary X, given by exactly the same formula as above except that we replace $\mathbb{R}^n$ throughout by the Riemannian manifold X. We now see that the curvature Ω in the exponential used to define the Thom class corresponds in a precise way to the curvature in the Feynman-Kac formula for the Laplace-Beltrami operator, which may be seen by writing out the Weitzenböck formula for Δ_f:

$$\Delta_f = \nabla^*\nabla + |\operatorname{grad} f|^2 + \sum_{ijkl} R_{ijkl}\, \varepsilon^i \iota^j \varepsilon^k \iota^l + \sum_{ij} \partial_i \partial_j f\, \varepsilon^i \iota^j.$$

(This formula is with respect to an orthonormal frame of the cotangent bundle.) In this way, we can understand better some of the calculations made by Alvarez-Gaumé in his heuristic proof of the Gauss-Bonnet-Chern theorem for X [1]: he was repeating on the loop-space LX the calculations of the last section.

4. Another example: Donaldson polynomials

We will briefly indicate one more example of the formalism of Euler classes, introduced by Witten to give an analytic approach to Donaldson polynomials [7]. Let us recall the setting of the Yang-Mills equation in four dimensions. We will ignore questions of Sobolev spaces and regularity: this section is physics!

Let X be a compact oriented four dimensional Riemannian manifold, and let $P \to X$ be a principal G-bundle over X, where G is a compact Lie group, with Lie algebra $\mathfrak{g}$. Let $\mathcal{A}$ be the space of all connections on the bundle P: if we choose a background connection ∇, this space may be realized as the affine space

$$\nabla + \mathcal{A}^1(X, P \times_{\mathrm{ad}} \mathfrak{g}).$$

Let $\mathcal{G}_*$ be the restricted gauge group of the bundle P: this is the space of sections

$$\mathcal{G}_* = \{g \in \Gamma(X, P \times_{\mathrm{Ad}} G) \mid g(*) = 1\},$$

where $*$ is a basepoint in X. It acts freely on the space $\mathcal{A}$ by the action

$$g \cdot A = g^{-1}Ag + g^{-1}dg.$$

We will think of $\mathcal{A}$ as a principal bundle, with structure group $\mathcal{G}_*$ and base $\mathcal{B}$:

$$\begin{tikzcd} \mathcal{G}_* \arrow[r] & \mathcal{A} \arrow[d] \\ & \mathcal{B} \end{tikzcd}$$

Incidentally, this shows that $\mathcal{B}$ is a classifying space for the group $\mathcal{G}_*$.

The second exterior power $\Lambda^2 T^*X$ of the cotangent bundle of X splits into two pieces

$$\Lambda^2 T^*X \cong \Lambda^+ T^*X \oplus \Lambda^- T^*X,$$

the self-dual and anti-self-dual bundles respectively: these are defined by the equation

$$\star\alpha = \pm\alpha,$$

where $\star$ is the Hodge dual operator. Let $\mathcal{H} = \Omega^-(X, P \times_{\mathrm{ad}} \mathfrak{g})$ be the space of anti-self-dual two-forms with values in the adjoint bundle. Since the gauge group acts on the bundle $P \times_{\mathrm{ad}} \mathfrak{g}$, we may form an associated bundle

$$E = \mathcal{A} \times_{\mathcal{G}_*} \mathcal{H} \to \mathcal{B}.$$

This bundle has a natural metric, since $\mathcal{G}_*$ preserves the L^2-metric on $\mathcal{H}$, and a natural connection compatible with this metric, coming from the natural invariant Riemannian structure on $\mathcal{A}$. The curvature of this connection is not pretty: it may be given in terms of a Green's kernel associated to the covariant derivative $\nabla + A$ acting on the space $\Omega^1(X, P \times_{\mathrm{ad}} \mathfrak{g})$.

To construct an Euler class, we need a section of the bundle E. This is easily obtained. The curvature F of the connection $\nabla + A$ is an element of $\Omega^2(X, P \times_{\mathrm{ad}} \mathfrak{g})$, so its anti-self-dual component F^- lies in $\Omega^-(X, P \times_{\mathrm{ad}} \mathfrak{g})$. Since the curvature transforms correctly under the gauge group, we see that F^- defines a section of the bundle E, whose zero-set is the space of self-dual connections. In [7], Witten presents evidence that the Donaldson polynomials, which are invariants of the bundle $P \longrightarrow X$, may be obtained by applying a formal version of Theorem 2.1 to this situation. It would be interesting to understand the extent to which his arguments are more than formal.

For the sake of accuracy, we should mention one further point: the above structures are all acted on by a finite-dimensional group, namely the quotient group $\mathcal{G}/\mathcal{G}_* \cong G$, where $\mathcal{G}$ is the gauge group with no restriction on its value at the base-point $*$. The spaces $\mathcal{A}$, $\mathcal{H}$, $\mathcal{B}$ and E all carry compatible actions of this group, and the section F^- of E is equivariant; the quotient of its zero-set by the action of G is the moduli space $\mathcal{M}$. However, the action of G on $\mathcal{B}$ is not in general free. It turns out that to formulate correctly the Donaldson invariants, one must consider a generalization of the formalism of Section 1, in which we replace the algebra of differential forms on E by the algebra of equivariant differential forms: this is the space of maps from the Lie algebra $\mathfrak{g}$ of G to $\Omega(E)$ invariant under the action of G. For more details, we refer the reader to [2].

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