JOURNAL OF FUNCTIONAL ANALYSIS 74, 121–138 (1987)

The Degree of the Nicolai Map in Supersymmetric Quantum Mechanics

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Communicated by the Editors

Received January 1986

0. INTRODUCTION

In an earlier paper, we developed a degree theory for Wiener maps [0]. In this paper, we will give an application of this theory which arises in the study of path integrals for supersymmetric Hamiltonians in quantum mechanics. Before defining the map which we will be looking at, we must specify the Wiener space on which it acts. This is the space of Hölder loops $C^\alpha(S, \mathbf{R}^M)$, $\alpha < \frac{1}{2}$, with the Ornstein-Uhlenbeck measure $d\lambda$—the Gaussian measure corresponding to the Hilbert subspace $L^{2,1}(S, \mathbf{R}^M)$, which is the Hilbert space of functions on the circle for which the inner product

$$\|f\|_{2,1}^2 = \int_S |f'|^2 + |f|^2. \tag{0.1}$$

is finite. The measure $d\lambda$ is related to the Brownian measure $d\mu$ on $C^\alpha(S, \mathbf{R}^M)$ by

$$\frac{d\lambda}{d\mu} = Z^{-1} \exp\left(-\int |f|^2/2 \right), \tag{0.2}$$

where $Z = \prod_{n \in Z} (n^2+1)^{-1}$ is the normalizing constant which makes $d\lambda$ into a probability measure.

We will need another Wiener space: the Banach space $C^{\alpha-1}(S, \mathbf{R}^M)$, for $\alpha < \frac{1}{2}$, carries a Gaussian measure $d\kappa$, known as the white noise measure, corresponding to the Hilbert subspace $L^2(S, \mathbf{R}^M)$. A nice way to construct $d\kappa$ is to use the invertible operator $\Lambda = d/dt + 1$: $C^\alpha \to C^{\alpha-1}$ to identify the

* Alfred P. Sloan Foundation Doctoral Dissertation Fellow. Supported in part by the National Science Foundation under Grant PHY-82-03669.

0022-1236/87 $3.00

two Banach spaces. Since Λ carries $L^{2,1}$ isometrically into L^2, that is, $\|f\|_{2,1} = \|\Lambda f\|_2$, it follows that we can simply define $d\kappa$ as the push-forward of $d\lambda$ by Λ.

Let V be a C^∞ function on $\mathbf{R}^M$, satisfying

(a) for each $m > 0$,

$$|\nabla^m V| < e^{O(|x|)} \tag{0.3}$$

and

(b) for any $c > 0$, and $|x|$ large enough,

$$|\nabla^2 V(x)| < c\, |\nabla V(x)|.$$

Thinking of ∇V as a smooth map from $\mathbf{R}^M$ to itself, then the map that we will discuss in this article is the nonlinear map from $C^\alpha(S, \mathbf{R}^M)$ to $C^{\alpha-1}(S, \mathbf{R}^M)$ given by

$$A(f)(t) = \frac{df(t)}{dt} + \nabla V(f(t)). \tag{0.4}$$

The map A is not quite a Wiener map as it stands, since it does not go from a Wiener space to itself. However, we can generalize the notion of a Wiener map as follows. Let B_i, $i = 1, 2$, be two Wiener spaces, with Gaussian measures $d\mu_i$ and associated Hilbert spaces H_i. Then a Wiener map from B_1 to B_2 is defined to be a map of the form $\Lambda + F$, where Λ is an invertible linear map from B_1 to B_2 (possibly only defined and invertible on a domain whose complement has zero capacity) which is an orthogonal isomorphism between H_1 and H_2, and $F \in W^\infty(B_1, H_2)$. Of course, we could equally well consider the Wiener map $1 + \Lambda^{+1} F$ from B_1 to itself, and this is what we shall do in practice.

With these definitions out of the way, we can state our first results, which are proved in Section 1.

THEOREM A. *The map A is a Wiener map from C^α to $C^{\alpha-1}$, $0 < \alpha < \frac{1}{2}$. Thus, it satisfies Sard's theorem: the set of critical values of A in $C^{\alpha-1}$ has κ-measure zero.*

Following the method of [0], we define the pullback of the white noise measure $d\kappa$ by A, using the inverse function theorem on regular open sets in C^α to define the measure $d(A^*\kappa)$ there as the pushforward $\pm(A^{-1})_* \, d\mu$, the sign depending on whether A preserves or reverses orientation. We can now state the main result of this paper.

THEOREM B. *There is an integer* $\deg(A)$ *such that for all* $\phi \in L^{\infty}(C^{\alpha-1})$,

$$\int_{C^{\alpha}} A^{*}(\phi\, d\kappa) = \deg(A) \int_{C^{\alpha-1}} \phi\, d\kappa.$$

In effect, we are considering $\phi\, d\kappa$ to be a volume form on $C^{\alpha-1}$ which we pull back to calculate the degree. By taking ϕ to equal various characteristic functions, we can translate this into a more geometric result.

COROLLARY. *For* κ*-a.e.* $g \in C^{\alpha-1}(S, \mathbf{R}^M)$, *the set* $A^{-1}(g)$ *is finite,* g *is a regular value, and*

$$\sum_{A(f)=g} \operatorname{sgn}(\nabla_f A) = \deg(A).$$

This corollary shows that A has a quite satisfactory degree theory, even though it does not satisfy the assumptions of Leray-Schauder theory—namely, it is not proper. This degree is a measure theoretic, or probabilistic, notion, quite distinct from the Leray-Schauder degree.

Cecotti and Girardello [2] gave the following heuristic calculation of $\deg(A)$, assuming that V is a Morse function, that is, has only a finite number of non-degenerate critical points, and $|\nabla V|: \mathbf{R}^M \to \mathbf{R}^M$ is proper. If f is a solution of $A(f)=0$, then we have

$$\int_S |A(f)|^2 = \int_S |f'|^2 + 2\,\frac{dV(f)}{dt} + |\nabla V(f)|^2 = 0. \tag{0.5}$$

Since the middle term vanishes, we see that f is a constant equal to a zero of ∇V, that is, a critical point of V. It is easy to show that $\operatorname{sgn}(\nabla_f A)$ equals $\operatorname{sgn}(\nabla_f^2 V)$, so that

$$\sum_{A(f)=0} \operatorname{sgn}(\nabla f A) = \sum_{\nabla_x V=0} \operatorname{sgn}(\nabla_x^2 V) = \deg(\nabla V: \mathbf{R}^M \to \mathbf{R}^M). \tag{0.6}$$

As it stands, this argument is not a calculation of $\deg(A)$, because $\{0\}$ is a set of zero measure in $C^{\alpha-1}$. On the other hand, it is not hard to turn this heuristic into a proof. Let $d\kappa_\varepsilon$, $\varepsilon>0$, be the measure on $C^{\alpha-1}(S, \mathbf{R}^M)$ obtained by scaling $d\kappa$,

$$\int_{C^{\alpha-1}} \phi(g)\, d\kappa_\varepsilon(g) = \int_{C^{\alpha-1}} \phi(\varepsilon g)\, d\kappa(g). \tag{0.7}$$

Instead of using the Gaussian measure $d\kappa$ in the definition of $\deg(A)$, we could as well have used $d\kappa_\varepsilon$. It is easy to see that the resulting degree is a continuous function of ε, and being an integer, is independent of ε. Thus, assuming that the family of measures $A^* \, d\kappa_\varepsilon$ has a weak limit as $\varepsilon \to 0$, we can write

$$\deg(A) = \int_{C^{\alpha-1}} w\text{-}\lim_{\varepsilon \to 0} A^* \, d\kappa_\varepsilon. \tag{0.8}$$

Now, as $\varepsilon \to 0$, the family $d\kappa_\varepsilon$ approaches a delta measure at 0, and once we can show that the measure w-lim $A^* \, d\kappa_\varepsilon$ is concentrated at $A^{-1}(0)$, it is easy to calculate this limit.

THEOREM C. *If V is a Morse function on $\mathbf{R}^M$, then*

$$w\text{-}\lim_{\varepsilon \to 0} A^* \, d\kappa_\varepsilon = \sum_{\nabla_x V = 0} \operatorname{sgn}(\nabla_x^2 V) \cdot \delta_x,$$

where δ_x is the measure of mass 1 concentrated at the constant function x in $C^\alpha(S, \mathbf{R}^M)$.

COROLLARY. *If $|\nabla V|$ is proper, then*

$$\deg(d/dt + \nabla V) = \deg(\nabla V \colon \mathbf{R}^M \to \mathbf{R}^M).$$

Proof. Any such function V may be perturbed slightly so that it becomes a Morse function without changing $\deg(A)$. Q.E.D.

This calculation of $\deg(A)$ has many similarities to the calculation of the index of the Dirac operator by path integral methods (Atiyah [1] and Getzler [4]). In fact $\deg(A)$ is actually the index of an elliptic operator on $\mathbf{R}^M$, which is the Hamiltonian for a supersymmetric quantum theory. It was discovered by Nicolai [7] that for any supersymmetric quantum field theory, the path integral of the theory is formally the pullback by a map A of a Gaussian measure. Following this, Parisi and Sourlas [8] and Cecotti and Girardello [2] showed that the map A given above in formula (0.4) is actually the Nicolai map for a certain supersymmetric quantum mechanical theory that is well known on account of its having been used by Witten to prove the Morse inequalities [9]. The Hamiltonian of this theory is a self-adjoint operator on the space of L^2 differential forms on $\mathbf{R}^M$, defined in terms of the operator $d_V = e^{-V} \cdot d \cdot e^V$ as follows: D is the operator $d_V + d_V^*$, and the Hamiltonian is D^2, which equals

$$d_V^* d_V + d_V d_V^* = -\Delta + |\nabla V|^2 + \rho(\nabla^2 V), \tag{0.9}$$

where $\rho(A_{ij})$ is the linear operator on $\Lambda^* \mathbf{R}^M$ defined by the formula

$$\rho(A_{ij}) = \sum_{ij} A_{ij}(a_i^* a_j - a_i a_j^*). \tag{0.10}$$

Here, a_i^* is the creation operator on $\Lambda^*\mathbf{R}^M$ obtained by taking the exterior product with e_i, and a_i is the annihilation operator, obtained by contracting with e_i.

If we calculate $A^*\, d\kappa$ formally by using the ordinary change of variables formula from finite dimensions, we obtain

$$\begin{aligned} A^*\, d\kappa &= \det(\nabla_f A)\, e^{-\int |A(f)|^2/2}\, df \\ &= \det(d/dt + \nabla_f^2 V)\, e^{-\int |\nabla V|^2/2}\, d\mu. \end{aligned} \tag{0.11}$$

Here, $\nabla_f^2 V$ is the linear operator from C^α to $C^{\alpha-1}$ such that

$$(\nabla_f^2 V \cdot h)(t) = (\nabla_{f(t)}^2 V)(h(t)). \tag{0.12}$$

This calculation is the intuitive justification for stating that $A^*\, d\kappa$ is the path integral for the Hamiltonian H defined above. Indeed, $\int |A(f)|^2$ is the bosonic part of the action corresponding to H, while the determinant gives the result of integrating over the fermion fields, whose action is quadratic in the fermions.

In fact, we will are able to prove the following result.

THEOREM D. *If $f_i \in C_0^\infty(\mathbf{R}^M)$, $1 \leqslant i \leqslant m$, and $0 < t_1 < \cdots < t_m < 1$, then*

$$\begin{aligned} &\int_{C^\alpha} f_1(\omega(t_1)) \cdots f_m(\omega(t_m))\, A^*\, d\kappa(\omega) \\ &\qquad = \mathrm{Str}[e^{-t_1 D^2/2} \cdot f_1 \cdot e^{-(t_2-t_1)D^2/2} \cdots f_m \cdot e^{-(1-t_m)D^2/2}], \end{aligned}$$

where Str *is the supertrace, that is, the trace on even L^2 forms minus the trace on odd L^2 forms.*

COROLLARY. *The index of the operator $d_V + d_V^*$ equals* $\deg(d/dt + \nabla V)$.

Proof. From Theorem D, we see that $\int_{C^\alpha} 1 A^*\, d\kappa = \mathrm{Str}\, e^{-D^2/2}$, which by the McKean-Singer formula is the index of D. The result now follows from the definition of $\deg(A)$. Q.E.D.

Of course, this corollary follows from Bott Periodicity, which shows directly that the index of D equals $\deg(\nabla V)$; but the proof we have given is perhaps more direct.

It may be possible to extend the techniques used to cover the Nicolai map that Parisi and Sourlas [8] discovered for the $N=2$ Wess–Zumino model in 2 space-time dimensions

$$A(f) = \frac{\partial f}{\partial z} + :P(\bar{f}):, \tag{0.13}$$

where f is a complex function on the torus, P is a real polynomial, and $:P:$ denotes Wick ordering. However, it would only be possible to prove the existence of a weak form of the degree, owing to the lack of an analogue of the implicit function theorem.

I am grateful to Arthur Jaffe for being the advisor for my thesis, from which this paper is taken. Part of this article was written while I was visiting the Caltech Department of Mathematics, and I would like to thank Barry Simon for his hospitality. Also, I benefited from some conversations with John Lott, who was working on a related problem at the same time [5].

1. The Nicolai Map as a Wiener Map

In this section, we will prove that the map A of Section 0 is a Wiener map, by establishing the following technical result. Once this is done, Theorem B and its corollary will follow immediately from the results of [0].

Proposition 1.1. *The following function on C^α is bounded*:

$$\exp(\|\nabla F\|^2_{HS(L^{2,1})} - \nabla^* F - \|F\|^2_{L^{1,2}}/2.$$

In showing that the Nicolai map is a Wiener map, the main fact needed will be Fernique's inequality: there is a positive constant c such that

$$\int_{C^\alpha} e^{c\|f\|^2_{C^\alpha}}\, d\lambda < \infty. \tag{1.1}$$

We will make use of the regularizer P_ε given by the Fourier multiplier on $C^\alpha(S, \mathbf{R}^M)$,

$$(P_\varepsilon f)(t) = \sum_{\tau \in \mathbf{Z}} e^{2\pi i \tau t - \varepsilon t^2/2} \hat{f}(\tau), \tag{1.2}$$

where $\hat{f}(\tau)$ is the Fourier transform of f. This regularizer satisfies the bounds

$$\|(1-P_\varepsilon) f\|_{L^\infty} \lesssim \varepsilon^\alpha \|f\|_\alpha \quad \text{and} \quad \|(1-P_\varepsilon) f\|_{L^2} \lesssim \varepsilon^\alpha \|\Lambda^\alpha f\|_{L^2} \quad \text{for } 0<\alpha<1. \tag{1.3}$$

Proposition 1.2. *Let $H(f) = \nabla V(f)\colon C^\alpha(S, \mathbf{R}^m) \to L^2(S, \mathbf{R}^M)$, where V is a C^∞ function on $\mathbf{R}^M$ such that*

$$|\nabla^m V| \lesssim e^{O(|x|)} \qquad \textit{for all } m > 0.$$

Then $H \in W^\infty(C^\alpha, L^2)$ (i.e., $\nabla^m H \in L^p$ for all m and $p<\infty$), and $H_\varepsilon = P_\varepsilon H P_\varepsilon$ converges to H in $W^\infty(C^\alpha, L^2)$ as $\varepsilon \to 0$.

Proof. That $H \in W^0$ follows directly by the exponential bounds and Fernique's inequality. Furthermore, we have

$$\begin{aligned}
&\|\nabla_f^m H - \nabla_f^m H_\varepsilon\|_{HS(L^{2,1}, S^M L^2)} \\
&\quad = \|(\nabla_f^m H - P_\varepsilon \circ \nabla_{P_\varepsilon f}^m H \circ P_\varepsilon)\, \Lambda^{-1}\|_{HS(L^2, S^m L^2)} \\
&\quad \leqslant \|\Lambda^{\alpha-1}\|_{HS(L^2)} \cdot \|(\nabla_f^m H - P_\varepsilon \circ \nabla_{P_\varepsilon f}^m H \circ P_\varepsilon)\, \Lambda^{-\alpha}\|_{B(L^2, S^m L^2)} \\
&\quad \lesssim \|\nabla_f^m H - \nabla_{P_\varepsilon f}^m H\|_{B(L^2, S^m L^2)} \\
&\qquad + \|(1-P_\varepsilon)\circ \Lambda^{-\alpha}\|_{B(L^2)}^m \cdot \|\Lambda^\alpha \circ \nabla_{P_\varepsilon f}^m H \circ \Lambda^{-\alpha}\|_{B(L^2, S^m L^2)} \\
&\qquad + \|P_\varepsilon \circ \nabla_{P_\varepsilon f}^m H\|_{B(L^2, S^m L^2)} \cdot \|(1-P_\varepsilon)\circ \Lambda^{-\alpha}\|_{B(L^2)} \\
&\quad \lesssim \varepsilon^\alpha e^{O(\|f\|_2)}.
\end{aligned}$$

The right-hand side is in L^p for all $p < \infty$ by Fernique's inequality, which proves that $H_\varepsilon \to H$ in W^m for all $m \leqslant \infty$. Q.E.D.

To apply this result to the Nicolai map A, we recall that we can write A as

$$A(f) = \Lambda f + (\nabla V - 1)(f) = \Lambda \circ (1 + F)(f), \tag{1.5}$$

where $F(f) = \Lambda^{-1} \circ (\nabla V - 1)(f)$. By Proposition 1.2, F lies in $W^\infty(C^\alpha, L^{2,1})$, and so A is a Wiener map. Using the approximation of F by F_ε, we can calculate $\nabla^* F$, as the limit of $\nabla^* F_\varepsilon$ when $\varepsilon \to 0$.

LEMMA 1.3. *Let $C = \sum_{n \in \mathbf{Z}} (n^2+1)^{-1}$. Then the divergence of F is given by the formula*

$$\nabla^* F(f) = (f, \nabla V(f))_{L^2} - (f, f)_{L^2} - C \int_S \Delta V(f(t))\, dt.$$

Proof. The divergence of F_ε is the sum of two pieces

$$\nabla^* F_\varepsilon(f) = (f, F_\varepsilon(f))_{L^{2,1}} - \operatorname{Tr}|_{L^{2,1}} \nabla f F_\varepsilon.$$

We will calculate the a.e. limit of these two terms separately. The first term gives

$$\begin{aligned}
&(\Lambda \circ P_\varepsilon f, P_\varepsilon(\nabla V(P_\varepsilon f) - P_\varepsilon f))_{L^2} \\
&\quad = \int_S \frac{d}{dt} [P_\varepsilon V(P_\varepsilon f(t)) - P_\varepsilon \mid P_\varepsilon f(t)|/2]\, dt \\
&\qquad + (P_{2\varepsilon} f, \nabla V(P_\varepsilon f) - P_\varepsilon f)_{L^2},
\end{aligned}$$

which converges to $(f, \nabla V(f) - f)_{L^2}$ a.e. as $\varepsilon \to 0$, since the first term equals zero.

The other contribution to $\nabla^* F_\varepsilon$ is

$$\begin{aligned}
-\mathrm{Tr}\,|_{L^{2,1}} \nabla_f F_\varepsilon &= -\mathrm{Tr}\,|_{L^2} P_\varepsilon \circ \nabla^2_{P_\varepsilon f} V \circ P_\varepsilon \circ \Lambda^{-1} \\
&= -\mathrm{Tr}\,|_{L^2} P_{2\varepsilon} \circ \Lambda^{-1} \circ \nabla^2_{P_\varepsilon f} V \\
&= -\sum_{n \in \mathbf{Z}} e^{-\varepsilon\tau^2} (i\tau + 1)^{-1} \int_S \Delta V(P_\varepsilon f(t))\, dt \\
&= -\sum_{n \in \mathbf{Z}} e^{-\varepsilon\tau^2} (\tau^2 + 1)^{-1} \int_S \Delta V(P_\varepsilon f(t))\, dt.
\end{aligned}$$

As $\varepsilon \to 0$, this converges to $-C \cdot \int_S \Delta V(P_\varepsilon f(t))\, dt$ a.e. Q.E.D.

From this proposition, it follows that

$$\frac{\Lambda^* d\kappa}{d\mu} = Z^{-M} \det{}_2(\Lambda^{-1}(d/dt + \nabla^2_f V))\, e^{-C \cdot \int \Delta V(f) - \|\nabla V(f)\|_{L^2}/2}. \tag{1.6}$$

We will calculate the right-hand side of this formula more explicitly later—for the moment, we content ourselves with proving Proposition 1.1. We see that

$$\begin{aligned}
\|\nabla_f F\|^2_{HS(L^{2,1})} &= \mathrm{Tr}\,|_{L^{2,1}} \Lambda^{-1} \circ \nabla^2_f V \circ \nabla^2_f V \circ \Lambda^{*-1} \\
&= \mathrm{Tr}\,|_{L^2} \nabla^2_f V \circ \nabla^2_f V \circ (\Lambda^* \Lambda)^{-1} \\
&= C \int_S \|\nabla^2_{f(t)} V\|^2_2\, dt.
\end{aligned}$$

Thus the exponent in Proposition 1.1 is bounded by

$$\int_S [O(|\nabla^2 V(f(t))|^2) - |\nabla V(f(t))|^2/2]\, dt,$$

which is bounded above by a fixed constant, by the assumption that for any positive constant c, and large enough $|x|$, $|\nabla^2 V(x)| \leqslant c\, |\nabla V(x)|$.

2. Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics is best thought of as a Z_2-graded version of ordinary quantum mechanics. Its axioms are as follows:

(I) $\mathbf{H}$ is a Hilbert space with a $\mathbf{Z}_2$-grading (i.e., a superspace)

$$\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-; \tag{2.1}$$

(II) D is an odd self-adjoint operator,

$$D: \ \mathbf{H}_{\pm} \to \mathbf{H}_{\mp}.$$

The Hamiltonian of the theory is D^2, which is thus a positive self-adjoint operator.

We will be interested in the following class of examples, first considered by Witten [9]. The Hilbert space $\mathbf{H}$ is taken to be the space of L^2 differential forms on $\mathbf{R}^M$, with the grading under which the forms of even degree lie in $\mathbf{H}_+$ and those of odd degree lie in $\mathbf{H}_-$.

If V is a smooth real function $\mathbf{R}^M$, called the superpotential, then the operator D is defined to be

$$D = e^{-V} \circ d \circ e^{V} + e^{V} \circ d^* \circ e^{-V}. \tag{2.2}$$

Obviously, D is self adjoint.

It follows that D^2 is given by the formula

$$D^2 = -\Delta + |\nabla V|^2 + \rho(\nabla^2 V), \tag{2.3}$$

where $\rho(\nabla^2 V)$ was defined in (0.10).

Here are some more or less physical examples of the above class of operators.

EXAMPLE 2.1. We take $M = 1$. Thus, our Hilbert space is $L^2(R) \oplus L^2(R)$, and D and D^2 are given by the matrices

$$D = \begin{bmatrix} 0 & -d/dx + V'(x) \\ d/dx + V'(x) & 0 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} -d^2/dx^2 + V'(x)^2 - V''(x) & 0 \\ 0 & -d^2/dx^2 + V'(x)^2 + V''(x) \end{bmatrix}.$$

This is the supersymmetric anharmonic oscillator.

EXAMPLE 2.2. In this example, we will let our vector space be $\mathbf{C}^M$, considered as a real vector space. The model is a lattice regularization of Wess and Zumino's $N = 2$ supersymmetric field theory in two space-time dimensions. The superpotential is given by

$$V(z, \bar{z}) = \operatorname{Re} \sum_{i=1}^{M} \left[\frac{1}{iM} \bar{z}_i z_{i-1} + p(z_i) \right],$$

where p is a real polynomial in one variable; the i variable is cyclic, so that we can think of the points $\{1, \ldots, M\}$ as lying on the circle of unit circumference.

It follows that ∇V is given by

$$\nabla V = \operatorname{Re} \sum_i \left[\frac{1}{iM} (\bar{z}_{i+1} - \bar{z}_{i-1}) + p'(z_i) \right] dz_i,$$

and consequently that

$$|\nabla V|^2 = \frac{1}{2} \sum_i \left[\frac{1}{M^2} |z_{i+1} - z_{i-1}|^2 + |p'(z_i)|^2 \right] + \frac{1}{M} \operatorname{Im} \sum_i (z_{i+1} - z_{i-1})\, p'(z_i).$$

To recover the continuum quantum theory, we let $M \to \infty$, and define the complex scalar field $\phi: S \to C$ by

$$\phi(j/M) = z_j.$$

In this limit, $|\nabla V|^2$ converges (formally) to

$$\tfrac{1}{2} \int_S (|\phi'|^2 + |p'(\phi)|^2),$$

which is the potential energy for the Wess–Zumino $N = 2$ Hamiltonian. If we denote the corresponding complex fermion field by ψ, then $\rho(\nabla^2 V)$ converges as $M \to \infty$ to

$$\frac{1}{2i} \int_S (\bar{\psi}\psi' - \psi\bar{\psi}' + p''(\phi)\, \bar{\psi}\psi).$$

An important invariant of a supersymmetric quantum theory is its index, defined by

$$\operatorname{index}(D) = \dim \ker D\,|_{\mathbf{H}_+} - \dim \ker D\,|_{\mathbf{H}_-}. \tag{2.4}$$

When e^{-tD^2} is trace class for all $t > 0$, McKean and Singer gave a convenient formula for the index [6]. This will be the case, for the operator of (2.2), if V satisfies the assumptions of (0.3), for example, if V is a polynomial such that $|\nabla V|$ is a proper map from $\mathbf{R}^M$ to itself.

Before stating the McKean-Singer formula, we need to define the supertrace; this is, the linear function on the trace class operators of $\mathbf{H}$ given by

$$\operatorname{Str} A = \operatorname{Tr} A\,|_{\mathbf{H}_+} - \operatorname{Tr} A\,|_{\mathbf{H}_-}. \tag{2.5}$$

Adopting the sign superconvention, whereby the transposition of two $\mathbf{Z}_2$-graded objects engenders an extra minus sign if both objects have odd parity, let us define the supercommutator of two operators on $\mathbf{H}$ to be

$$[A, B] = A \circ B - (-1)^{|A| \cdot |B|} B \circ A, \tag{2.6}$$

where $|A|$ is $+1$ if $A: \mathbf{H}_\pm \to \mathbf{H}_\pm$ and -1 if $A: \mathbf{H}_\pm \to \mathbf{H}_\mp$. With this definition, the supertrace satisfies the formula

$$\mathrm{Str}[A, B] = 0. \tag{2.7}$$

PROPOSITION 2.1. *If e^{-tD^2} is trace class, then*

$$\mathrm{index}(D) = \mathrm{Str}\, e^{-tD^2}.$$

Proof. It is clear that $\mathrm{index}(D) = \lim_{t\to\infty} \mathrm{Str}\, e^{-tD^2}$; it remains to be shown that $\mathrm{Str}\, e^{-tD^2}$ is independent of t. This follows from the calculation of the derivative of $\mathrm{Str}\, e^{-tD^2}$ with respect to t,

$$\begin{aligned} \frac{d}{dt} \mathrm{Str}\, e^{-tD^2} &= -\mathrm{Str}\, D^2 e^{-tD^2} \\ &= -\frac{1}{2} \mathrm{Str}[D, De^{-tD^2}] = 0 \qquad \text{by (2.7).} \end{aligned}$$

Q.E.D.

The index of the operator D can be calculated by using the Atiyah-Singer index theorem. However, Witten found a simpler method to obtain the following result [9].

THEOREM 2.2. *The index of the operator D equals the Leray-Schauder degree of $\nabla V: \mathbf{R}^M \to \mathbf{R}^M$.*

Let us check this result for D given in Example 2.1. The operator D has kernel spanned by

$$f_1 = (e^{-V(x)}, 0) \qquad \text{and} \qquad f_2 = (0, e^{V(x)}). \tag{2.8}$$

There are three cases to be dealt with:

(i) if $V(x) \to \infty$ as $|x| \to \infty$, then $\deg(V': \mathbf{R} \to \mathbf{R}) = 1$, as does $\mathrm{index}(D)$, since f_1 is in L^2 and f_2 is not;

(ii) if $V(x) \to -\infty$ as $|x| \to \infty$, then $\deg(V') = \mathrm{index}(D) = -1$, since only f_2 is in L^2;

(iii) otherwise, neither f_1 nor f_2 are in L^2, so $\mathrm{index}(D)$ is zero, as, obligingly, is $\deg(V')$.

Using Theorem 2.2, we find that D in Example 2.2 has index equal to $\deg(p')^M$, where the degree is now understood to refer to the actual degree of the polynomial p', which is the same thing as its Leray-Schauder degree when considered as a map from C to itself. If $\deg(p') > 1$, so that D^2 leads to an interacting field theory, then ker D increases in dimension wildly as we take the continuum limit. This indicates that the lattice cutoff is somewhat sick for this model.

Instead of proving Theorem 2.2 in the way that Witten did, we will give a heat-kernel proof that makes use of the McKean-Singer formula. We will define a measure on the space of continuous loops on $\mathbf{R}^M$, $C(S, \mathbf{R}^M)$, using the operator D^2. As is well known, it is sufficient to define such a measure on the cylindrical functions, of the form

$$F(\omega)=f_1(\omega(t_1))\cdots f_n(\omega(t_n)), \tag{2.9}$$

where $f_i \in C_0^\infty(R^M)$, and $0<t_1<\cdots<t_n<1$. This presumes, of course, that the resulting linear form is bounded; this is automatic for pure bosonic theories, since the path integral is positive, but for fermions it is a bigger issue.

Define the expectation of the function $F(\omega)$ given by (2.7) to be

$$\int F(\omega)\, d\nu(\omega)=\mathrm{Str}[e^{-t_1 D^2/2} f_1 e^{-(t_2-t_1)D^2/2} \cdots f_n e^{-(1-t_n)D^2/2}]. \tag{2.10}$$

Using the Feynman–Kac formula, we will derive a useful alternative expression for this integral.

Let $d\mu$ be the Brownian measure on $C(S, \mathbf{R}^M)$; it is defined by a formula analogous to (2.10), using the Laplacian Δ instead of D^2, and the trace on functions rather than the supertrace on differential forms

$$\int F(\omega)\, d\mu(\omega)=\mathrm{Tr}[e^{-t_1 \Delta/2} f_1 e^{-(t_2-t_1)\Delta/2} \cdots f_n e^{-(1-t_n)\Delta/2}]. \tag{2.11}$$

Recall the definition of the path ordered exponential: if $a(t)$: $[0, 1] \to \mathrm{End}(V)$ then

$$\frac{d}{ds} T\exp \int_0^s a(t)\, dt = a(s) \cdot T\exp \int_0^s a(t)\, dt, \tag{2.12}$$

with the boundary condition that $\lim_{s\to 0} T\exp \int_0^s a(t)\, dt=1$.

PROPOSITION 2.3. *The measure $d\nu$ is finite, and*

$$\frac{d\nu}{d\mu}=\mathrm{Str}[Te^{-\int \rho(\nabla^2 V(\omega(t)))/2\,dt}]\, e^{-\int |\nabla V(\omega(t))|^2/2\,dt}.$$

Proof. The formula for $d\nu/d\mu$ follows from the Feynman–Kac formula.

To show that dv is a finite measure, we observe that $|dv/d\mu| \in L^\infty(C(S, \mathbf{R}^M))$. Indeed

$$\left|\frac{dv}{d\mu}\right| \lesssim |Te^{-\int \rho(\nabla^2 V)/2\,dt}|\ e^{-\int |\nabla V|^2/2\,dt}$$
$$\lesssim e^{\int (|\nabla^2 V| - |\nabla V|^2)/2\,dt}.$$

This is bounded by the assumptions that have been made on V in (0.3).
Q.E.D.

It follows from Proposition 2.3 and the McKean-Singer formula that

$$\operatorname{index}(D) = \int 1\ dv. \tag{2.13}$$

Our plan is to calculate the index of D by finding the weak limit dv_0 of the measures dv_ε as $\varepsilon \to 0$, which are obtained from dv by replacing d_V by $\varepsilon e^{-\varepsilon^{-1}V} \circ d \circ e^{\varepsilon^{-1}V}$ in (2.10). Since the index of D is unchanged by this replacement, the total mass of dv_0 will be the index of D.

THEOREM 2.4. *Suppose V has nondegenerate critical point $x_1, ..., x_n$. Then the weak limit of the measures dv_ε as $\varepsilon \to 0$ is*

$$dv_0 = \sum_{\kappa=1}^{n} \operatorname{sgn}(\nabla^2 V(x_\kappa))\ \delta(\omega(t) - x_\kappa).$$

Proof. Let $d\mu_\varepsilon$ be the measure obtained from the Brownian measure by replacing Δ by $\varepsilon^2 \Delta$ in formula (2.11); that is, $d\mu_\varepsilon(\omega) = d\mu(\varepsilon\omega)$. Since $|dv_\varepsilon / d\mu_\varepsilon|$ decreases to zero exponentially fast outside any neighbourhood of the set of critical points as $\varepsilon \to 0$, it follows that the family dv_ε is precompact in the space of finite measures on $C(S, \mathbf{R}^M)$. From Proposition 2.3, it is clear that any weak limit of the family dv_ε must be supported on the set of stationary loops at the critical points of V, otherwise a factor of $e^{-\varepsilon^{-2}|\nabla V(x)|^2/2}$ can be extracted from $|dv_\varepsilon/d\mu_\varepsilon|$. Thus, we may write

$$dv_0 = \sum_{\kappa=1}^{n} c_\kappa \delta(\omega(t) - x_\kappa),$$

for some real numbers c_κ.

The numbers c_κ are local, in the sense that they can only depend on V in an arbitrarily small neighbourhood of x_κ. Given small enough $\delta > 0$, replace V by

$$V^\delta(x) = \phi(\delta^{-1}\,|x - x_\kappa|)\ V(x)$$
$$+ (1 - \phi(\delta^{-1}\,|x - x_\kappa|))(x - x_k) \cdot \nabla^2 V(x_\kappa) \cdot (x - x_\kappa)/2,$$

where ϕ is a smooth cutoff function equal to 1 near zero and 0 further away. We see that dv_0 is replaced by

$$dv_0^\delta = c_\kappa \delta(\omega(t) - x_\kappa).$$

Thus, c_i is the index of the operator D^δ obtained by replacing V by V^δ in (2.3). Sending $\delta \to 0$ does not change the index, so we obtain

$$c_i = \operatorname{index}(d_{V_\kappa} + d_{V_\kappa}^*),$$

where $V_\kappa = (x - x_\kappa, \nabla^2 V(x_\kappa)(x - x_\kappa))/2$.

We will complete the proof by showing that if V is a homogeneous quadratic polynomial, then

$$\operatorname{index}(d_V + d_V^*) = \operatorname{sgn}(\nabla^2 V).$$

By diagonalizing $\nabla^2 V$, we reduce the calculation to the case in which V is the function $\lambda x^2/2$ on $\mathbf{R}$. We calculated the index of this operator in Example 2.1, where we found that it was indeed equal to the sign of λ. Q.E.D.

Theorem 2.2 is an immediate corollary of this theorem: modifying V if necessary while leaving its index fixed, we can assume that it is has no degenerate critical points. By Theorem 2.4, the index of D is $\sum_{\kappa=1}^n \operatorname{sgn}(\nabla^2 V(x_\kappa))$, which is the definition of the Leray-Schauder degree of ∇V.

3. The Relationship between the Nicolai Map A and dv

In this section, we will prove Theorem D of Section 0, which relates the Nicolai map $A = d/dt + \nabla V$ to the path integral dv of the last section, by showing that $A^* d\kappa = dv$. By comparing formula (1.6) for $A^* d\kappa$, obtained from the change of variables formula for Wiener maps, to Proposition 2.3, obtained by applying the Feynman–Kac formula to the path integral for the operator D^2, we see that this theorem follows from the formula

$$\operatorname{Str} Te^{-\int_0^1 \rho(\nabla^2 V)\,dt/2} = Z^{-M} \det{}_2(\Lambda^{-1}(d/dt + \nabla_f^2 V))\, e^{-C \cdot \int \Delta V(f(t))\,dt}. \tag{3.1}$$

To show this equality, we will present a list of axioms which both sides satisfy and which characterize them completely.

Let $\mathbf{A}$ be the space of all connections on the trivial complex M-dimensional vector bundle over the circle; that is, an element of $\mathbf{A}$ is an operator

$$\frac{d}{dt} + a(t), \tag{3.2}$$

where $a(t)$ is a map from the circle to the space of $M \times M$ complex matrices.

A function ϕ: $\mathbf{A} \to \mathbf{C}$ will be called a determinant function if it satisfies the following three axioms:

(a) ϕ is holomorphic;

(b) ϕ is gauge invariant – if $u(t)$: $S \to GL(M)$, then

$$\phi(u(t)^{-1} \circ D \circ u(t)) = \phi(D);$$

(c) if $a(t)$ is the constant matrix $\operatorname{diag}(a_1, \dots, a_M)$, then there is some constant such that

$$\phi(D) = c \cdot \prod_{i=1}^{M} \sinh a_i/2.$$

PROPOSITION 3.1. *Any two determinant functions corresponding to the same constant c are equal.*

Proof. Since any determinant function is holomorphic, it suffices to prove equality for connections for which $a(t)$ is skew-adjoint. But for such a connection, there is a gauge transformation transforming $d/dt + a(t)$ to a constant connection:

Let $U = Te^{\int_0^1 a(t)\,dt}\; U(M)$ be the holonomy if the connection, and let

$$u(t) = T \exp \int_0^t [\log U - a(s)]\, ds.$$

This is a periodic map from to $GL(M)$, and

$$u(t)^{-1} \left[\frac{d}{dt} + a(t)\right] u(t) = \frac{d}{dt} + \log U.$$

Since any two determinant functions which agree for constant connections are equal, we see that the constant c characterizes the determinant function. Q.E.D.

Incidentally, from this proof, we see that a determinant function as defined above has the most important property that one would want from a determinant—namely, if a connection D is not invertible, then its determinant vanishes.

We now give three examples of determinant functions.

(I) The zeta-function determinant: recall that this is defined for D invertibly by analytically continuing the zeta-function of D, $\zeta_D(s) = \operatorname{Tr} D^{-s}$, from the domain in which it is initially defined, $\operatorname{Re} s > 1$, to the half-plane $\operatorname{Re} s > -\varepsilon$; the zeta-function determinant is

$$\det(D) = e^{-\zeta_D'(0)}. \tag{3.3}$$

It is a standard fact that this is a gauge-invariant, holomorphic function of D. To show that it is actually a determinant function on $\mathbf{A}$, it only remains to calculate $\det(d/dt + a)$ when a is the matrix $\mathrm{diag}(a_1,..., a_M)$, $a_i\mathbf{C}$.

Proposition 3.2. *If $a = \mathrm{diag}(a_1,..., a_M)$, where $a_i \in \mathbf{C}$, then the zeta-function determinant of $d/dt + a$ is proportional to $\prod_{i=1}^{M} \sinh a_i/2$.*

Proof. It is easy to see that $d/dt + a$ has eigenvalues $2\pi in + a_i$, $n \in \mathbf{Z}$. Thus the zeta-function of $d/dt + a$ is $\zeta_D(s) = \sum_{i=1}^{M} \zeta_{a_i}(s)$, where $\zeta_\lambda(s)$ is the zeta-function

$$\zeta_\lambda(s) = \sum_{n \in \mathbf{Z}} (2\pi in + \lambda)^{-s}.$$

To calculate $\zeta'_\lambda(0)$, we differentiate $\zeta'_\lambda(s)$ with respect to λ for $\mathrm{Re}\, s$ large and analytically continue to $s = 0$. We obtain

$$\frac{d}{d\lambda} \zeta'_\lambda(s) = \sum_{n \in \mathbf{Z}} (2\pi in + \lambda)^{-s-1} + \sum_{n \in \mathbf{Z}} s \cdot In(2\pi in + \lambda) \cdot (n - \lambda)^{-s-1}.$$

Both of these sums are convergent for $\mathrm{Re}\, s > 0$, so we can calculate the limit of the right-hand side as $s \to 0$, taking advantage of the continuity of the left-hand side at $s = 0$. The second of the two sums behaves like $s \cdot \log s$ as $s \to 0$, do does not contribute to the limit. The first sum may be rewritten as

$$\lambda^{-s-1} + \sum_{n=1}^{\infty} [(2\pi in + \lambda)^{-s-1} + (-2\pi in + \lambda)^{-s-1}],$$

and this equals

$$\lambda^{-s-1} + \sum_{n=1}^{\infty} \left(\frac{(-2\pi in + \lambda)^{s+1} + (2\pi in + \lambda)^{s+1}}{(4\pi^2 n^2 + \lambda^2)^{s+1}} \right) \Bigg] + O(s \cdot \log s).$$

Taking the limit $s \to 0$ gives

$$\frac{d}{d\lambda} \zeta'_\lambda(0) = 2^{-1} \coth \lambda/2.$$

The solution of this differential equation is

$$\zeta'_\lambda(0) = \log \sinh \lambda/2 + \text{constant}.$$

Inserting this into the formula for the zeta-function determinant of $d/dt + a$ gives

$$\det(d/dt + a) = e^{-\zeta'_D(0)} = \prod_{i=1}^{M} c \cdot \sinh a_i/2,$$

where c is a universal constant independent of M. Q.E.D.

(II) The function $\phi(d/dt + a(t)) = \operatorname{Str} Te^{-\int_0^1 \rho(a(t))/2\,dt}$: it is obvious that ϕ is gauge invariant and holomorphic, so once more, the nontrivial part is to calculate $\phi(D)$ for $a(t) = \operatorname{diag}(a_1, \ldots, a_M)$. The matrix $e^{-\int \rho(a)/2\,dt}$ has eigenvectors $\Lambda_{i \in I} v_i \in \Lambda^* R^M$ with eigenvalues $\prod_{i \in I} e^{-a_i} \prod_{i \notin I} e^{a_i}$, where I is a subset of $\{1, \ldots, M\}$. Adding these up with the correct signs gives

$$\operatorname{Str}[Te^{-\int \rho(a(t))/2)\,dt}] = \prod_{i=1}^{M} 2 \sinh a_i/2.$$

(III) The function $\psi(D) = \det_2(\Lambda^{-1} D)\, e^{\cdot\, -C(\cdot \int \operatorname{tr} a(t)\,dt)}$: the easiest way to show that this function is a determinant function is to relate it to the zeta-function determinant.

PROPOSITION 3.2. *If $a(t)$ is a continuous map from the circle to $M \times M$ complex matrices, then the zeta-function determinant of $d/dt + a(t)$ is proportional to $\psi(d/dt + a(t))$.*

Proof. It is sufficient to prove the lemma for $a(t)$ real symmetric and close to 1, since both quantities under consideration are holomorphic in $a(t)$. We will use the following facts about the zeta-function determinant (see, e.g., Forman [3]):

(a) replacing D by the regularized operator $\Lambda + P_\varepsilon(a(t) - 1) P_\varepsilon$ produces a continuous perturbation of the zeta-function determinant as $\varepsilon \to 0$;

(b) if A is a trace-class pseudodifferential operator, then

$$\det(D(1 + A)) = \det(D) \cdot \det(1 + A),$$

where $\det(1 + A)$ is the Fredholm determinant of A.

It follows that

$$\begin{aligned}
\det(d/dt + a(t)) &= \lim_{\varepsilon \to 0} \det(\Lambda + P_\varepsilon(a(t) - 1) P_\varepsilon) \\
&= \lim_{\varepsilon \to 0} \det(\Lambda) \det(1 + \Lambda^{-1}(P_\varepsilon(a(t) - 1) P_\varepsilon)) \\
&= \det(\Lambda) \lim_{\varepsilon \to 0} \det{}_2(1 + \Lambda^{-1}(P_\varepsilon(a(t) - 1) P_\varepsilon)) \\
&\quad \times \lim_{\varepsilon \to 0} \exp(-\operatorname{Tr}|_{L^{2,1}} \Lambda^{-1} P_\varepsilon(a(t) - 1) P_\varepsilon).
\end{aligned}$$

It is now easy to take the limit $\varepsilon \to 0$. Since $\Lambda^{-1}(P_\varepsilon(a(t) - 1) P_\varepsilon)$ converges to $\Lambda^{-1}(a(t) - 1)$ in $HS(L^{2,1})$ as $\varepsilon \to 0$, it follows that

$\det_2(1+\Lambda^{-1}(P_\varepsilon(a(t)-1)P_\varepsilon))$ converges to $\det_2(\Lambda^{-1}(d/dt+a(t)))$. The trace in the exponent equals

$$\operatorname{Tr}|_{L^{2,1}} \Lambda^{-1}P_\varepsilon(a(t)-1)P_\varepsilon = \operatorname{Tr}|_{L^2}(a(t)-1)P_{2\varepsilon}^2\Lambda^{-1}$$
$$= \sum_{n\in Z}(\tau^2+1)^{-1}e^{-\varepsilon\tau^2}\int_S(\operatorname{tr} a(t)-N)\,dt,$$

which converges to $C\cdot\int_S \operatorname{tr} a(t)\,dt - C\cdot M$. Q.E.D.

From the above discussion, we have learned that formula (3.1) is true up to a constant which is independent of V, and hence that dv and $A^*\,d\kappa$ differ by this universal constant. Of course, if we wished to, we could calculate this constant explicitly, but it must clearly be equal to 1, since when $V=|x|^2/2$, the Nicolai map A equals Λ, so that $A^*\,d\kappa$ equals the Ornstein-Uhlenbeck measure $d\lambda$, while dv defined in Section 2 equals $d\lambda$ essentially by definition. In this way, we have proved Theorem D.

One piece of unfinished business remains: in Section 2, we calculated the index of the operator D by calculating the weak limit of a family of measures dv_ε as $\varepsilon\to 0$. In fact, this is precisely the same calculation which is required to establish Theorem C, the calculation of the weak limit of $A^*\,d\kappa_\varepsilon$, since in fact $dv_\varepsilon = A^*\,d\kappa_\varepsilon$, as follows from the results of this section.

References

0. E. Getzler, Degree theory for Wiener maps, *J. Funct. Anal.* **68** (1986), 388–403.
1. M. Atiyah, Circular symmetry and the Atiyah–Singer index theorem, Colloquium in honour of Laurent Schwarz, *Astérisque* **131** (1985).
2. S. Cecotti and L. Girardello, Functional measure, topology, and dynamical supersymmetry breaking, *Phys. Lett. B* **110** (1982), 39–42.
3. R. Forman, "Zeta-Function Determinants and Differential Geometry," Harvard thesis, 1985.
4. E. Getzler, A short proof of the local Atiyah–Singer index theorem, *Topology* **25** (1986), 111–117.
5. J. Lott, Supersymmetric path integrals, *Commun. Math. Phys.* **108** (1987), 605–629.
6. H. P. McKean, Jr. and I. M. Singer, Curvature and the eigenvalues of the Laplacian, *J. Differential Geom.* **1** (1967), 43–69.
7. H. Nicolai, Supersymmetry and functional integration measures, *Nuc. Phys.* **176** (1980), 419–428.
8. G. Parisi and N. Sourlas, Supersymmetric field theories and stochastic differential equations, *Nucl. Phys. B* **206** (1982), 321–332.
9. E. Witten, Supersymmetry and Morse theory, *J. Differential Geom.* **17** (1982), 661–692.