JOURNAL OF FUNCTIONAL ANALYSIS 68, 388–403 (1986)

Degree Theory for Wiener Maps

EZRA GETZLER*

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138 and Université de Paris-Sud, Orsay, France

Communicated by Paul Malliavin

Received October 1985

Degree theory is developed for a class of Wiener maps, as well as a few interesting tools, such as partitions of unity and Sard's theorem. © 1986 Academic Press, Inc.

The usual theory for the degree of a map for compact manifolds, has two aspects, one geometric, the other coming from the theory of integration. In this paper, we will show how, in certain circumstances, both of these viewpoints can be extended to the theory of Wiener maps. (For a brief outline of the theory of Wiener maps, see the introduction to Malliavin [12].) Thus, we are attempting to implement a piece of the program which was begun in the paper of Eells and Elworthy [4].

Let $f\colon M \to N$ be a smooth map, where M and N are both n-dimensional compact manifolds. In the geometric method of defining the degree of a map, we choose a regular value $n \in N$ of f, which exists by Sard's theorem, and define the degree of f to be

$$\deg(f) = \sum_{f(m)=n} \operatorname{sgn}(\nabla_m f), \tag{0.1}$$

where $\operatorname{sgn}(\nabla_m f)$ is the sign of $\det(\nabla_m f)$. (The sum is finite since $f^{-1}(n)$ is compact and discrete.) It is then proved that this integer is independent of the point n chosen, so long as n is regular. From this point of view, it is clear that the degree is an integer.

If ω is an n-form on N, then we can also define $\deg(f)$ by

$$\int_M f^*\omega = \deg(f) \int_N \omega. \tag{0.2}$$

That $\deg(f)$ defined in this way is independent of ω follows from Stokes' theorem; its equality with the geometrically defined degree comes from

* Alfred P. Sloan Foundation Doctoral Dissertation Fellow. Supported in part by the National Science Foundation under Grant PHY-82-03669.

0022-1236/86 $3.00

Sard's theorem, which tells us that the integral over N may be restricted to the set of regular values of f, which is an open subset of N for which f is a covering map.

As an example of the type of map which we would like to extend degree theory to, consider the infinite dimensional map $Af = df/dt + f^2$, called a Nicolai map. If $0 < \alpha < 1$, then A is a smooth map from $C^\alpha(S)$ to $C^{\alpha-1}(S)$, where $C^\alpha(S)$ is the Hölder space on the circle S. But $C^\alpha(S)$ is certainly not compact, and worse still, A is not proper, so we cannot apply the geometric degree theory of Leray and Schauder which generalizes finite dimensional degree theory. On the other hand, for $\alpha < \frac{1}{2}$, $C^\alpha(S)$ is a Wiener space for the Gaussian measure corresponding to $|f|_H^2 = \int_S |f'|^2 + |f|^2 = \int_S |(d/dt+1)f|^2$. Since $(d/dt+1)^{-1} A: f \to f + (d/dt+1)^{-1}(f^2 - f)$ is a smooth Wiener map on $C^\alpha(S)$, there is some chance of defining the degree of A as the degree of $(d/dt+1)^{-1} A$ obtained by adapting the integration approach to degree—except that we replace n-forms by measures of the form $f\,d\mu$, $f \in L^\infty(C^\alpha(S))$, in the definition. In this paper, we develop an abstract theory which will be applied to the map A, and more general examples, in another paper. The technique used is inspired by de Alfaro *et al.* [2].

The results that we obtain for nonsmooth Wiener maps are an example of a philosophy due to D. Stroock (and, perhaps, to others as well): in studying Wiener maps, it is better to work with weak (integrated) quantities, rather than hoping to make sense of the corresponding strong (pointwise) quantities.

The first section of this paper recalls some results on Wiener spaces that are needed later in the paper. Sections 2 and 3 develop some technical results that are of interest in their own right—the existence of partitions of unity for Wiener spaces, and Sard's theorem for smooth Wiener maps. These results are not completely new, but they do not occur anywhere in the literature in the form that we need them. Section 4 applies the results of these sections to the definition and study of the degree for Wiener maps.

A point on notation—by $A \lesssim B$, we mean that there is a positive constant c such that $A < cB$. Likewise, $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

I would like to thank P. Malliavin for discussions that were helpful in writing Section 2.

1. Wiener Spaces

If B is a separable Banach space, then a Gaussian measure on B is a finite Borel measure $d\mu$ on B that satisfies

$$\int_B e^{i\alpha(x)}\, d\mu(x) = e^{-(\alpha,\alpha)/2}, \qquad \text{where} \quad \alpha \in B^*, \tag{1.1}$$

for some bounded inner product (,) on B^* called the **covariance** of the measure. We say that $(B, d\mu)$ is a **Wiener space**.

A measurable function on B that only depends on a finite number of linear forms $\{\alpha_i \in B^* \mid 1 \leqslant i \leqslant n\}$, so that $f(x) = f(\alpha_1(x), ..., \alpha_n(x))$ for some measurable function f on R^n, is called a **cylindrical function**. If f lies in $C_0^\infty(R^n)$, then by the Fourier inversion formula,

$$\int_B f(\alpha_1(x), ..., \alpha_n(x))\, d\mu(x) = \frac{1}{\sqrt{\det(2\pi A)}} \int f(x)\, e^{-\sum_{ij} A_{ij} x^i x^j/2}\, dx, \quad (1.2)$$

where A_{ij} is the matrix inverse of the $n \times n$ matrix $(A^{-1})_{ij} = (\alpha_i, \alpha_j)$. Since cylindrical functions generate the Borel σ-algebra of B, it follows immediately that the measure $d\mu$ is positive, and thus is a probability measure.

The following proposition shows that there is a plentiful supply of integrable functions on B if it carries a Gaussian measure. The proof may be found in Fernique [5].

PROPOSITION 1.1. *There exists a number $\alpha > 0$ such that*

$$\int_B e^{\alpha \|x\|^2}\, d\mu(x) < \infty.$$

If B is a Wiener space, then the tautological injection of B^* into $L^2(B)$ is an isometry if B^* is given the pre-Hilbert topology defined by the inner product (,):

$$\int_B |\alpha(x)|^2\, d\mu(x) = (\alpha, \alpha). \quad (1.3)$$

We will denote the closure of the image of B^* in $L^2(B)$ by H. The space H is a Hilbert space which may be thought of as a dense subspace of B using the adjoint of the injection of B^* into H. The injection $I: H \to B$ determines the inner product (,) on B^* and hence the Gaussian measure $d\mu$.

If H is a Hilbert space, then its nth symmetric power $S^n H$ is the closed subspace of its nth Hilbert tenor product spanned by the vectors

$$\frac{1}{n!} \sum_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \qquad \text{for} \quad v_i \in H. \quad (1.4)$$

The inner product on $S^n H$ is completely determined by

$$(v^n, w^n) = (v, w)^n \qquad \text{for} \quad v, w \in H. \quad (1.5)$$

It is possible to develop a differential calculus for functions on B. The main difference between this calculus and the finite-dimensional calculus is that derivatives are only taken along tangent vectors in $H \subset B$. Thus, ∇^n will be an unbounded operator

$$\nabla^n\colon L^p(B; G) \to L^p(B; G \otimes S^n H). \tag{1.6}$$

If $f(P_V x)$ is a cylinder function, where $f \in C_0^\infty(V)$ and P_V is the orthogonal projection from B onto a finite-dimensional space $V \subset B^*$, then it is easy to see what $\nabla^n f$ must be:

$$\nabla^n f = (\nabla^n f)(P_V x) \in W^\infty(B; S^n V) \subset W^\infty(B; S^n H). \tag{1.7}$$

We can now define the number (or Ornstein–Uhlenbeck) operator N to be the self-adjoint operator equal to $\nabla^*\nabla$. In finite dimensions, N is the operator $-\partial^2 + x \cdot \partial$, which is unitarily equivalent to the harmonic oscillator acting on $L^2(R^n, dx)$:

$$e^{-|x|^2/4}(-\partial^2 + x \cdot \partial)\, e^{|x|^2/4} = -\partial^2 + |x|^2/4 - n/2. \tag{1.8}$$

The number operator is used to define the Sobolev spaces on B. If $s \geqslant 0$ and $1 < p < \infty$, let $L^{p,s}(B)$ be the domain of $(1+N)^{s/2}\colon L^p(B) \to L^p(B)$. (If $p = 1$ or $p = \infty$, we could define the Sobolev spaces $L^{p,s}(B)$, but they are not useful.) Similarly, if G is a Hilbert space, we can define the Sobolev spaces $L^{p,s}(B; G)$. Malliavin's spaces $W^s(B)$ are the Frechet spaces defined by

$$W^s(B) = \bigcap_{1 < p < \infty} L^{p,s}(B). \tag{1.9}$$

Unlike in finite dimensions, there is no analog of Sobolev's lemma for Wiener spaces; the space $W^\infty(B)$ contains many discontinuous functions.

The fundamental inequalities for the derivative operator are analogs of the singular integral estimates of Euclidean harmonic analysis. They are due to Meyer [13]: if $1 < p < \infty$ and n is an integer, then for some $c(p, n) > 0$,

$$\|N^{n/2} f\|_p \sim \|\nabla^n f\|_p. \tag{1.10}$$

Using this, we see that ∇^n extends to a bounded operator from $L^{p,s+n}(B; G)$ to $L^{p,s}(B; G \otimes S^n H)$, for $s \geqslant 0$.

We will also need the adjoint of the operator ∇, defined by

$$\int_B (\nabla f, g)\, d\mu = \int_B f \cdot \nabla^* g\, d\mu$$

$$\text{for } f \in W^\infty(B) \text{ and } g \in W^\infty(B; H). \tag{1.11}$$

In finite dimensions, ∇^* is the operator $\nabla^*(f_i\,dx_i) = -\partial_i f_i + x_i f_i$. In Krée [8], it is proved that ∇^* is bounded from $L^{p,s+1}(B; G \otimes H)$ to $L^{p,s}(B; G)$.

Using the heat kernel of the Ornstein–Uhlenbeck operator, it is possible to produce W^∞ functions, as in finite dimensions. This technique goes back to Goodman [6].

PROPOSITION 1.2. *If P_t, $t > 0$, denotes the operator e^{-tN} on $L^2(B)$, then P_t is bounded from $L^p(B)$ to $L^{p,\infty}(B)$, for $1 < p < \infty$.*

Proof. It is sufficient to prove that P_t is bounded from $L^p(B)$ to $L^{p,2n}(B)$ for all $n > 0$. By Mayer's inequality, the norm $\|N^n f\|_p$ is a norm defining the topology of $L^{p,2n}(B)$. Since $N^n \circ P_t$ is bounded on $L^p(B)$ for all $t > 0$, it follows that $\|N^n \circ P_t f\|_p \lesssim \|f\|_p$. Q.E.D.

There is an explicit formula for $P_t f$ in terms of integration with respect to $d\mu$:

$$P_t f(x) = \int_B f(e^{-t}x + (1 - e^{-2t})\, y)\, d\mu(y). \tag{1.12}$$

This is proved by checking it for functions of the form $e^{\alpha(x)}$, $\alpha \in B^*$, for which it reduces to a one dimensional integral. This formula has the following important consequence.

PROPOSITION 1.3. *If f is a bounded Lipschitz function on B, then $P_t f$ converges uniformly to f as $t \to 0$.*

Proof. If $x \in B$, we have

$$\begin{aligned} |f(x) - P_t f(x)| &\leqslant \int_B [f(x) - f(e^{-t}x - (1 - e^{-2t})\, y)]\, d\mu(y) \\ &\leqslant \mathrm{Lip}(f) \left[(1 - e^{-t})\, \|x\| + (1 - e^{-2t}) \int_B \|y\|\, d\mu \right] \\ &\leqslant O(t)\, \mathrm{Lip}(f). \end{aligned}$$

Q.E.D.

We now state a martingale convergence theorem in the form in which it will be used later. Let V_n be an increasing flag of subspaces of B^*, that is, $\dim V_n = n$ and the union of the spaces V_n is dense in H. Let P_n be the orthogonal projection from B onto V_n, and let Σ_n be the σ-field obtained by pulling back the Borel field of V_n by P_n. The conditional expectation operator E_n is defined to be the conditional expectation for the Σ-field Σ_n—it is just integration over the finite codimension space orthogonal to V_n. Since $E_n f$ is a martingale for $f \in L^1(B; G)$, we obtain the following useful result:

PROPOSITION 1.4. *If $f \in L^1(B; G)$, then $E_n f$ converges to f a.e. as $n \to \infty$.*

2. Capacities and Partitions of Unity

To explain the purpose of capacities, recall Egorov's theorem: If f is an integrable function on R^n, then for each $\varepsilon > 0$, there is a set A with Lebesgue measure less than ε such that f is continuous on A^c.

Thus, Lebesgue measure gives a gauge of the size of the set on which f is irregular in the sense of not being continuous. If f is in the Sobolev space $L^{p,k}(R^n)$, then we expect that there should be some finer measure of the size of the set of singularities of f, called (p, k)-capacity; for example, if $k > n/p$, then f is continuous, so even a single point will have positive (p, k)-capacity.

In studying W^∞-functions on Wiener spaces, the use of capacities is a vital tool, since W^∞-functions are not continuous, unlike in finite dimensions. The use of capacities was first introduced in this context by Malliavin [12].

The capacity of an open set U with respect to the Banach space $L^{p,k}(B)$, $k \geqslant 0$, is

$$\operatorname{cap}_{p,k}(U) = \inf\{\|f\|_{p,k} \mid f \in W^\infty(B), f > 1 \text{ a.e. on } U \text{ and } f > 0 \text{ a.e.}\}. \tag{2.1}$$

It is immediate from this definition that $\operatorname{cap}_{p,k}$ is subadditive:

$$\operatorname{cap}_{p,k}(U \cup V) \leqslant \operatorname{cap}_{p,k}(U) + \operatorname{cap}_{p,k}(V). \tag{2.2}$$

The definition of $\operatorname{cap}_{p,k}$ is extended to all subsets of B by

$$\operatorname{cap}_{p,k}(A) = \inf_{U \supset A} \operatorname{cap}_{p,k}(U). \tag{2.3}$$

Proposition 2.1. *If $f \in L^{p,k}(B)$, there is an open set A with $\operatorname{cap}_{p/2,k}(A) \leqslant \varepsilon$ such that f is continuous on A^c.*

Proof. Let f_n be a sequence of smooth cylinder functions converging in $L^{p,k}(B)$ to f, such that $\|f_n - f_{n+1}\|_{p,k} \leqslant 8^{-n}$. If $U_n = \{|f_n - f_{n+1}| > 2^{-n}\}$, then

$$\begin{aligned} \operatorname{cap}_{p/2,k}(U_n) &\leqslant 4^n \,\|(f_n - f_{n+1})^2\|_{p/2,k} \\ &\lesssim 4^n \,\|f_n - f_{n+1}\|_{p,k} \\ &\lesssim 2^{-n}. \end{aligned}$$

Setting $A_N = \bigcup_{n > N} U_n$, we see that $\operatorname{cap}_{p/2,k}(A_N) \lesssim 2^{-N}$, and it is clear that f is continuous on $(A_N)^c$, since the sequence f_n converges uniformly on this set. Q.E.D.

The following result often enables us to treat B as if it were locally compact. It is a generalization of a result of Kusuoka [10].

PROPOSITION 2.2. *Given p, k and $c > 0$, there is a compact subset K of B such that* $\operatorname{cap}_{p,k}(K) \leqslant \varepsilon$.

Proof. We need a result of Gross [7] (see also Kusuoka [10]), that for every Wiener space B, there is a Banach space E and a compact inclusion of E in B, such that the measure of $B \backslash E$ is zero. The space thus constructed has the property that $\nabla^k(\|x\|_E)$ is bounded on the set $\|x\|_E \geqslant 1$, for all $k \geqslant 1$.

Let ϕ be a smooth decreasing function on R_+ such that $\phi(t) = 0$ for $t \leqslant \frac{1}{2}$ and $\phi(0) = 1$ for $t \geqslant 1$. If we let $f_n(x) = \phi(n^{-1} \|x\|_E)$, then $f_n \geqslant 1$ on E_n, the ball of radius n in E, which is compact in B, so

$$\operatorname{cap}_{p,k}(E_n) \leqslant \| f_n \|_{p,k} .$$

It is easy to show, by repeated application of Leibniz's rule, that $\| f_n \|_{p,k}$ converges to zero as $n \to \infty$. Q.E.D.

Although $\operatorname{cap}_{p,k}$ is not σ-additive, it has the following important continuity property.

PROPOSITION 2.3. *If $F_n \subset B$ is a decreasing sequence of closed sets, then*

$$\operatorname{cap}_{p,k}\left(\bigcap_n F_n\right) = \lim_{n \to \infty} \operatorname{cap}_{p,k}(F_n).$$

Proof. By Proposition 2.2, it is sufficient to prove this result when all of the sets F_n are compact. Let the intersection of the sets F_n be called F, and choose $f \in W^\infty(B)$ in such a way that

$$f \geqslant 1 \text{ a.e. on } F, \qquad f \geqslant 0 \text{ a.e.,} \qquad \text{and} \qquad \| f \|_{p,k} \leqslant \operatorname{cap}_{p,k}(F) + \varepsilon.$$

Let A be an open set with $\operatorname{cap}_{p,k}(A) \leqslant \varepsilon$ such that f is continuous on A^c. Since the sets F_n are compact, it follows that

$$\lim_{n \to \infty} \inf\{ f(x) \mid x \in F_n \backslash A \} = 1.$$

By the definition of capacity, it follows that

$$\begin{aligned} \lim_{n \to \infty} \operatorname{cap}_{p,k}(F_n) &\leqslant \lim_{n \to \infty} \operatorname{cap}_{p,k}(F_n \backslash A) + \varepsilon \\ &\leqslant \lim_{n \to \infty} \sup\{ f(x)^{-1} \mid x \in F_n \backslash A \} \| f \|_{p,k} + \varepsilon \\ &= \operatorname{cap}_{p,k}(F) + 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, this proves the result. Q.E.D.

The following technical result is very useful.

PROPOSITION 2.4. *For any pair (p, k), there is a constant c such that if A is a subset of B, there is a W^∞-function f such that*

(a) $0 \leqslant f \leqslant 1$,

(b) $f=0$ on A,

(c) $\|1-f\|_{p,k} \leqslant c\, \mathrm{cap}_{kp,k}(A)$.

Proof. Let g be a positive W^∞-function such that $g \geqslant 1$ on A and $\|g\|_{kp,k} \leqslant 2 \cdot \mathrm{cap}_{kp,k}(A)$. Then $f=\phi(g)$ is the sought after function, where ϕ is the function used in the proof of Proposition 2.2. Q.E.D.

In differential geometry, a major tool is the existence of partitions of unity. Unfortunately, since Wiener spaces are not locally compact, W^∞ partitions of unity do not exist in general; we must settle for a weak form of the partition of unity. Although this result will not be used elsewhere in this paper, it is included since it is of interest in its own right.

PROPOSITION 2.5. *If $\{U_\alpha\}$ is an open cover of B by balls, then for any p, k and $\varepsilon>0$, there is a finite set of positive W^∞-functions $\{f_1,\ldots, f_n\}$ subordinate to the cover such that*

$$\|1-\Sigma f_i\|_{p,k} \leqslant \varepsilon.$$

Proof. By Propositions 2.2, there is a compact set K such that $\mathrm{cap}_{p,k}(K^c) \leqslant \varepsilon$. Since K is compact, it is covered by a finite number of balls $\{U_1,\ldots, U_n\}$.

Choose a refinement $\{V_i\}$ of the cover $\{U_i\}$ of K such that $d(V_i, (U_i)^c)>0$. Let g_i be a Lipschitz function on U_i such that $0 \leqslant g_i \leqslant 1$ and $g_i[V_i]=1$. By Propositions 1.4 and 1.5, the function $P_t g_i$ is W^∞ for any $t>0$, and $P_t g_i$ converges uniformly to g_i as $t \to 0$. Since $\bar{V}_i \cap K$ and $(U_i)^c \cap K$ are compact, they are separated by $P_t g_i$ for t small enough: there exist $s_1<s_2$ such that

$$P_t g_i \leqslant s_1 \quad \text{on} \quad (U_i)^c \cap K$$

and

$$P_t g_i \geqslant s_2 \quad \text{on} \quad \bar{V}_i \cap K.$$

Now choose a smooth increasing function ϕ on R_+ such that $\phi=0$ on $[0, s_1]$ and $\phi=1$ on $[s_2, \infty)$. It is easy to see that the function $h_i=\phi(P_t g_i) \in W^\infty(B)$ is supported on U_i, equals 1 on $\bar{V}_i$, and lies between 0 and 1.

By Proposition 2.4, there exists a W^{∞}-function q such that $0 \leqslant q \leqslant 1$, $q=0$ on K^c, and $\|q\|_{p,k} \leqslant c \cdot \varepsilon$. The function f_i of the partition of unity is defined to be

$$f_i = h_i(1-q)(\Sigma h_i)^{-1}. \qquad \text{Q.E.D.}$$

3. Sard's Theorem and the Pullback of Measures

To generalize the notion of a differentiable map to Wiener spaces, one works with Wiener maps. These are maps from B to itself of the form $1+F$, where $F \in W^1(B; H)$. As we saw in the last section, Wiener maps need not be continuous. On the other hand, Wiener maps have many nice properties. The space $H \otimes H$ is isometric with $HS(H)$, the space of Hilbert–Schmidt operators on H, by the map sending $v \otimes w$ to $v(w, \cdot)$. The tangent map in the H directions of a Wiener map $1+F$ has the form $1+\nabla_x F$ at a point x, so lies in $1+HS(H)$. In particular, it is a Fredholm operator; thus, the theory of Wiener maps bears some analogies to Smale's theory of Fredholm maps (Smale [15]).

We say that $1+F$ is a smooth Wiener map from B to H if F is in $C^{\infty}(B; H)$. The following result (of which a proof may be found in Gross [7]) shows that F is automatically in $W^1_{\text{loc}}(B; H)$. Thus $1+F$ really is a Wiener map, at least locally.

Proposition 3.1. *There is a bounded map from $L(B, H)$ to $HS(H)$, the space of Hilbert–Schmidt operators on H, given by restriction to H, and if $A \in L(B, H)$, then the following inequality holds*:

$$\|A\|_2 \leqslant \left[\int_B \|x\|_B^2 \, d\mu(x)\right]^{1/2} \|A\|_{B,H}.$$

The main impediment to a general theory of Wiener maps is that there is no substitute for the implicit function theorem when F is not assumed to be continuously differentiable (although for certain Wiener maps, such as the solutions to certain stochastic differential equations, one may construct inverse maps explicitly.) Thus, although we will define the degree of a Wiener map in a rather general setting in the next section, we can only give it a point-by-point geometrical interpretation if F is smooth using the following theorem.

Theorem 3.2 (Sard's theorem). *Let $1+F$ be a smooth Wiener map. The set of critical values of $1+F$, that is, the set of points $y \in B$ such that there is* *a solution of* $x + Fx = y$ *such that* $1 + \nabla_x F$ *is not invertible, has measure zero.*

As in the finite-dimensional theory, this result is used to define the pullback of the Gausian measure $d\mu$ by a smooth Wiener map $1 + F$, by the following formula (f is a bounded continuous function on B):

$$\int_B f(x)\, d(1+F)^* \mu = \int_B \sum_{x + Fx = y} \operatorname{sgn}(1 + \nabla_x F) \cdot f(x)\, d\mu(y). \tag{3.1}$$

Sard's theorem shows that the integral on the right may be taken over the complement of the singular values of $1 + F$. Whether or not a particular f will be integrable with respect to $d(1+F)^* \mu$ is something which depends on $1 + F$, but certainly if f is supported on a set on which $1 + F$ is invertible, then it will be integrable.

We will now state a formula for $d(1+F)^* \mu/d\mu$ due to Ramer [14]. Consider first the finite-dimensional case. By the usual change of variables formula, we have

$$\frac{d(1+F)^* \mu}{d\mu} = \det(1 + \nabla F) \cdot e^{-(x,Fx) \quad |Fx|^2/2}. \tag{3.2}$$

To generalize this to infinite dimensions, we have to rearrange the factors in the formula such that each makes sense in infinite dimensions. First of all, instead of the determinant, we use the function $\det_2$, which is a continuous function

$$\det_2 \colon 1 + HS(H) \to \mathbf{R} \tag{3.3}$$

defined for operators of the form $1 + A$, A of finite rank, by

$$\begin{aligned} \det_2(1 + A) &= \det(1 + A) \cdot e^{-\operatorname{Tr} A} \\ &= \exp\left[\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{Tr} A^n\right] \end{aligned} \tag{3.4}$$

and extended by continuity to all of $1 + HS(H)$ (Dunford and Schwartz [3]). It is useful to observe that $\det_2(1 + A) = 0$ if and only if $1 + A$ is not invertible. If $1 + A$ is invertible, we will denote the sign of $\det_2(1 + A)$ by $\operatorname{sgn}(1 + A)$.

The following determinant inequalities (Dunford and Schwartz [3]) will be useful later:

$$\det_2(1 + A) \leqslant e^{\|A\|_2^2/2}$$

and (3.5)

$$\|(1 + A)^{-1}\| \det_2(1 + A) \leqslant e^{\|A\|_2^2 + 1}.$$

Substituting $\det_2$ for det in the finite dimensional change of variables formula gives

$$\frac{d(1+F)^*\mu}{d\mu} = \det_2(1+\nabla F)\cdot e^{-((x,Fx)-\mathrm{Tr}(\nabla F))-|Fx|^2/2}. \tag{3.6}$$

The term $(x, Fx) - \mathrm{Tr}(\nabla F)$ has a natural generalization to infinite dimensions—it is just $\nabla^* F$. If $F \in W^1(B; H)$, then by Meyer's inequality, $\nabla^* F \in W^0(B)$. (But beware that $\nabla^* F$ need not be continuous, even if F is smooth.)

Motivated by this discussion, we can state the change of variables formula.

THEOREM 3.3. *If $1+F$ is a smooth Wiener map, then*

$$\frac{d(1+F)^*\mu}{d\mu} = \delta(F), \qquad \textit{where} \quad \delta(F) = \det_2(1+\nabla F)\, e^{-\nabla^* F - |Fx|^2/2}.$$

In particular, $d(1+F)^*\mu$ is absolutely continuous with respect to $d\mu$.

This theorem strongly suggests that one should define the pullback of $d\mu$ for a general Wiener map to be $\delta(F)\, d\mu$. This no longer has any geometric significance, unless $1+F$ satisfies some kind of inverse function theorem.

We now turn to the proofs of Theorems 3.2 and 3.3. Theorem 3.3 was proved for invertible smooth Wiener maps by Ramer in his important paper (Ramer [14]), extending an earlier result of Kuo [9]; this was later generalized in Kusuoka [11]. Once Sard's theorem is proved, it follows that $d(1+F)^*\mu$ gives measure zero to the singular set $\{\delta(F)=0\}$. Since on the regular set $1+F$ is locally invertible (here we use that it is smooth), Theorem 3.3 follows from Ramer's result and Theorem 3.2.

The proof of Sard's theorem that we will give is modelled on Smale's proof of an analogous result for Fredholm maps: the set of critical values of a Fredholm map is residual. First we need an auxilliary result.

LEMMA 3.4. *If $1+F$ is a smooth Wiener map and $x \in B$, then there is a neighbourhood U of x, and an invertible smooth Wiener map $\phi\colon U \to B$, such that $1+F = (1+K)\circ\phi$; here, V is a finite dimensional subspace of B^* and $K\colon \phi[U] \to V$ is smooth.*

Proof. Let W be the image of $1+\nabla_x F$ considered as a bounded linear operator on B. Since $1+\nabla_x F$ is Fredholm, W is a closed, finite codimension, subspace of B. Let $V \subset B^*$ be the orthogonal complement of W, and let P_W be orthogonal projection from B to W.

The composition $P_W \circ (1+F)$ is a submersion in a small neighborhood of x, so by the implicit function theorem, there is a neighborhood U of x and a change of variables $\phi\colon U \to B$ such that

$$(1+F)\circ\phi^{-1}\colon (v, w)\in V\times W \to (v+K(v, w), w).$$

It is clear that ϕ is a Wiener map, since if $y \in U$, then

$$\begin{aligned}\phi(y)-y &= P_W(\phi(y)-y)+(1-P_W)(\phi(y)-y)\\ &= P_W F(y)+(1-P_W)(\phi(y)-y)\in H.\end{aligned}$$

Q.E.D.

Armed with this result, it is simple to prove Theorem 3.2 from the finite dimensional Sard's theorem. Since B is Lindelöf, it can be covered by a countable number of open sets U on which $1+F$ has the representation $(1+K)\circ\phi$. Thus the critical values of $1+F$ restricted to the set U are the same as the set of critical values of $1+K$ on the set $\phi[U]$, which equals

$$S=\{(v+K(v, w), w) \mid 1+\nabla K(\cdot, w) \text{ is not invertible when restricted to } V\}.$$

The Gaussian measure $d\mu$ equals the product of the Gaussian measures $d\mu_V \times d\mu_{V^\perp}$, and for each $w \in W$, the critical values of $1+K(\cdot, w)$ have $d\mu_V$-measure zero by Sard's theorem. It follows from Fubini's theorem that S has measure zero.

The set of critical values of $1+F$ is the countable union of sets of the form S, so has measure zero by countable additivity. This completes the proof of Theorem 3.2.

4. Degree Theory for Wiener Maps

Motivated by the theory of degree in the compact finite-dimensional case, we define the degree of a Wiener map $1+F$ to be

$$\deg(1+F)=\int_B 1\, d(1+F)^*\,\mu. \tag{4.1}$$

In order to prove anything about the degree of F, we will assume that F is a member of $W^2(B; H)$ satisfying

$$\delta(F),\ \|(1+\nabla F)^{-1}\|\ \delta(F)\in L^{1+\varepsilon}(B) \qquad \text{for some} \quad \varepsilon>0. \tag{4.2}$$

There is a straightforward condition that implies that (4.2) is true:

$$e^{\|\nabla F\|_2^2-\nabla^* F-|Fx|^2/2}\in L^{1+\varepsilon}(B). \tag{4.3}$$

LEMMA 4.1. *Given $v \in B^*$, define the operator T_v by*

$$T_v f = (v, (1+\nabla F)^{-1} \nabla f).$$

If $1+F$ is a Wiener map satisfying (4.2) *and $f \in W^\infty(B)$, then*

$$\int_B T_v f \, d(1+F)^* \mu = \int_B f(x)(v, x+Fx) \, d(1+F)^* \mu.$$

Proof. The idea of the proof is to approximate F by smooth cylinder maps, using the results on conditional expectations proved in Section 1.

Let F_n be the function $P_n E_n F$. The proof of Proposition 2.1 shows that given $p < \infty$ and $\varepsilon > 0$, there is a set A with $\mathrm{cap}_{p,1}(A) < \varepsilon$ such that F_n, ∇F_n and $\nabla^* F_n$ converge uniformly on A^c to the p, 1-functions F, ∇F, and $\nabla^* F$, respectively. In particular, on A^c, the functions F, ∇F, and $\nabla^* F$ are bounded, and $\delta(F_n)$ converges to $\delta(F)$ uniformly.

By Proposition 2.4, there exists a W^∞-function ϕ which is zero on A and such that increasing ε a little,

$$\mathrm{cap}_{p,1}(\mathrm{supp}(1-\phi)) < \varepsilon \qquad \text{and} \qquad \|1-\phi\|_{p,1} < \varepsilon.$$

Multiplying f by ϕ, we can assume that F_n, ∇F_n, and $\nabla^* F_n$ converge uniformly on the support of f. Once established for such a function f, the result follows for all f by letting $\varepsilon \to 0$.

All that remains to be done is an integration by parts:

$$\begin{aligned}
&\int_B T_v f \, d(1+F)^* \mu \\
&\qquad = \lim_{n \to \infty} \int_B (v, (1+\nabla F_n)^{-1} \nabla F_n) \det(1+\nabla F_n) \, e^{-|F_n x|^2/2 - (x, F_n x)} \, d\mu \\
&\qquad = \lim_{n \to \infty} \int_B f \cdot \nabla^* \delta(F_n)((1+\nabla F_n)^{-1})^* v \, d\mu.
\end{aligned}$$

The divergence of $\delta(F_n)((1+\nabla F_n)^{-1})^* v$ is easily calculated, since F_n has finite rank and is smooth, and we obtain

$$\begin{aligned}
&\nabla^* \delta(F_n)((1+\nabla F_n)^{-1})^* v \\
&\qquad = (-(v, (1+\nabla F_n)^{-1} \nabla \ln \delta(F_n)) + (v, (1+\nabla F_n)^{-1} x) \\
&\qquad\quad - \mathrm{Tr}(\nabla_v (1+\nabla F_n)^{-1})) \, \delta(F_n) \\
&\qquad = (v, x + F_n x) \, \delta(F_n).
\end{aligned}$$

Q.E.D.

This lemma gives us everything that we need to show that the degree satisfies the analog of (0.2).

THEOREM 4.2. *For all $f \in L^\infty(B)$,*

$$\int_B (1+F)^* F \, d(1+F)^* \mu = \deg(1+F) \int_B f \, d\mu.$$

Proof. We start by proving the result for the exponential function $f_t(x) = e^{it(v,x)}$, $v \in B^*$. Let $g_t(x) = (1+F)^* f_t(x) = e^{it(v,x+Fx)}$. Applying T_v to $g_t(x)$, we have

$$\begin{aligned} T_v \, g_t(x) &= (v, (1+\nabla F)^{-1} \nabla e^{it(v,x+Fx)}) \\ &= it \, |v|^2 \, g_t(x) \end{aligned}$$

so that Lemma 4.1 gives

$$\frac{d}{dt} \int_B g_t(x) \, d(1+F)^* \mu = it \, |v|^2 \int_B g_t(x) \, d(1+F)^* \mu.$$

It follows that

$$\begin{aligned} \int_B g_t(x) \, d(1+F)^* \mu &= e^{-t^2|v|^2/2} \int_B g_0(x) \, d(1+F)^* \mu \\ &= \int_B e^{it(v,x)} \, d\mu \cdot \deg(1+F). \end{aligned}$$

We can now extend this formula to any smooth cylinder function $f(P_V x)$, $f \in C^\infty(V)$, where V is any finite dimensional subspace of B^*, by using the Fourier representation of f:

$$\begin{aligned} &\int_B (1+F)^* f \, d(1+F)^* \mu \\ &\quad = (2\pi)^{-n/2} \int_V \hat{f}(v) \int_B (1+F)^* e^{i(v,x)} d(1+F)^* \mu \, dv \\ &\quad = (2\pi)^{-n/2} \int_V \hat{f}(v) \, e^{-|v|^2/2} \, dv \cdot \deg(1+F) \\ &\quad = \int_B f(P_V x) \, d\mu \cdot \deg(1+F). \end{aligned}$$

To extend the formula to all $f \in L^\infty(B)$, we approximate f by a sequence of uniformly bounded smooth cylinder functions converging pointwise a.e. to f. The result follows by the dominated convergence theorem. Q.E.D.

For a smooth Wiener map, the pullback has a geometric meaning expressed in (3.1). Thus we may interpret the degree geometrically, generalizing the definition of degree for compact manifolds given in formula (0.1).

THEOREM 4.3. *If F is a smooth Wiener map satisfying* (4.2), *then for a.e.* $y \in B$,

$$\deg(1+F) = \sum_{x+Fx=y} \operatorname{sgn}(1+\nabla_x F).$$

In particular, $\deg(1+F)$ *is an integer.*

Proof. We partition B into disjoint measurable sets:

B_{ij}, $i, j \in N$, is the set of $y \in B$ which are regular values of $1+F$, for which $x+Fx=y$ has i solutions with $\operatorname{sgn}(1+\nabla_x F) = +1$, and j solutions with $\operatorname{sgn}(1+\nabla_x F) = -1$;

B_∞ is the set of $y \in B$ such that y is a singular value of $1+F$ or $x+Fx=y$ has an infinite number of solutions.

Since $d(1+F)^* \mu$ has finite mass, the set B_∞ must have zero measure. We will show that B_{ij} has zero measure unless $(i-j) = \deg(1+F)$.

Let χ_{ij} be the characteristic function of the set B_{ij}. By formula (4.1), we have

$$\int_B (1+F)^* \chi_{ij}\, d(1+F)^* \mu = \int_{B_{ij}} (i-j)\, d\mu.$$

But by Theorem 4.2, the left-hand side equals

$$\deg(1+F) \cdot \int_B \chi_{ij}\, d\mu = \deg(1+F) \cdot \int_{B_{ij}} 1\, d\mu.$$

Equating these two expressions gives the result. Q.E.D.

We would now like to show that the degree is an integer for nonsmooth Wiener maps. We will have to content ourselves with the following result, which is obtained by a method remeniscent of Cruzeiro [1]. (I do not know if the degree is an integer in general.)

Consider the class of Wiener maps $1+F$ satisfying

$$e^{\|\nabla F\|_2^2 + |\nabla^* F|} \in L^{1+\varepsilon} \qquad \text{for some} \quad \varepsilon > 0. \tag{4.4}$$

From the determinant inequality (3.5), we see that F must satisfy (4.2), and we can define the degree of $1+F$ by (4.1).

THEOREM 4.4. *Under assumption* (4.4), $\deg(1+F)$ *is an integer.*

Proof. Jensen's inequality tells us that the sequence F_n defined in the proof of Lemma 4.1 satisfies (4.4) uniformly in n, so that $\delta(F_n)$ is uniformly bounded in $L^{1+\varepsilon}(B)$. By the martingale convergence theorem, the martingales $\nabla F_n = E_n(\nabla F)$ and $\nabla^* F_n = E_n(\nabla^* F)$ converge a.e. to ∇F and $\nabla^* F$. It follows that $\delta(F_n)$ converges a.e. to $\delta(F)$.

Since $\delta(F_n)$ is uniformly bounded in $L^{1+\varepsilon}(B)$, we can conclude that

$$\int_B \delta(F)\, d\mu = \lim_{n \to \infty} \int_B \delta(F_n)\, d\mu$$

$$= \lim_{n \to \infty} \deg(1 + F_n) \int_B 1\, d\mu.$$

We see that $\deg(1+F)$ is the limit of the sequence of integers $\deg(1+F_n)$. Thus $\deg(1+F) = \deg(1+F_n)$ for large enough n, proving the theorem.

Q.E.D.

References

1. A. B. Cruzeiro, Equations différentielles sur l'espace de Wiener et formules de Carneron–Martin non-linéaires, *J. Funct. Anal.* **54** (1983), 206–227.
2. V. de Alfaro *et al.*, Stochastic identities in supersymmetric field theories, *Phys. Lett. B* **142** (1984), 399–402.
3. N. Dunford and J. T. Schwartz, "Linear Operators, Part II," Interscience, New York, 1963.
4. J. Eells and K. D. Elworthy, Wiener integration on certain manifolds. Problems in nonlinear analysis, *in* "Proceedings CIME," IV Ciclo, Edizioni Cremonese, Rome, 1971.
5. M. X. Fernique, Intégrabilité des vecteurs Gaussiens, *C. R. Acad. Sci. Paris Ser. A* **270** (1970), 1698–1699.
6. V. Goodman, Quasi-differentiable functions on Banach spaces, *Proc. Amer. Math. Soc.* **30** (1971), 367–370.
7. L. Gross, Abstract Wiener spaces, *in* "Proceedings, 5th Berkeley Sympos. Math. Statist. Probab. Vol. II, Part 1," pp. 31–42, Univ. of California Press, Berkeley, 1967.
8. P. Krée, Continuité de la divergence dans les espaces de Wiener, 1982, *C. R. Acad. Sci. Paris Ser. A* **296** (1983), 833–836.
9. H.-H. Kuo, Integration theory on infinite dimensional manifolds, *Trans. Amer. Math. Soc.* **159** (1971), 57–78.
10. S. Kusuoka, Dirichlet forms and diffusion processes on Banach spaces, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29** (1982), 79–95.
11. S. Kusuoka, The nonlinear transformation of Gaussian measure on Banach space and absolute continuity, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29** (1982), 567–596.
12. P. Malliavin, Implicit functions in finite corank in Wiener space, *in* "Stochastic Analysis" (K. Itô, Ed.), North-Holland, Amsterdam, 1984.
13. P. Meyer, Note sur les processus d'Ornstein-Uhlenbeck, *in* "Seminar on Probability XVI," pp. 95–133, Lecture Notes in Math. Vol. 920, Springer-Verlag, Berlin/New York, 1982.
14. R. Ramer, On nonlinear transformations of Gaussian measures, *J. Funct. Anal.* **15** (1974), 166–187.
15. S. Smale, An infinite dimensional version of Sard's theorem, *Amer. J. Math.* **87** (1965), 861–866.