THE CYCLIC HOMOLOGY OF CROSSED PRODUCT ALGEBRAS, II. TOPOLOGICAL ALGEBRAS

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This article is a sequel to [6], in which we constructed a spectral sequence for the cyclic homology of a crossed product algebra $A \rtimes G$, where A is an algebra and G is a discrete group acting on A. In this article, we will show how similar results hold when A is a topological algebra, and G is a Lie group acting differentiably on A. We assume the notation and results of [6], which we will refer to as Part I.

All of the topological vector spaces considered in this article will be locally convex, complete and Hausdorff.

A **topological algebra** is a topological vector space A with an associative product $m : A \times A \longrightarrow A$ which is separately continuous. This definition motivates the introduction of Grothendieck's inductive tensor product $V_1 \bar{\otimes} V_2$ of two topological vector spaces, which is the completion of the algebraic tensor product $V_1 \otimes V_2$ with respect to the finest *compatible* tensor product topology, in the sense of Grothendieck ([7], page 89).

Recall some of the properties of the inductive tensor product:

(1) If W is a complete topological vector space, the space $L(V_1 \bar{\otimes} V_2, W)$ of continuous linear transformations from $V_1 \bar{\otimes} V_2$ to W is isomorphic to the space of separately continuous bilinear maps from $V_1 \times V_2$ to W; in particular, the dual $(V_1 \bar{\otimes} V_2)'$ is isomorphic to the space of separately continuous bilinear forms on $V_1 \times V_2$.

(2) If V_1 and V_2 are Fréchet spaces, separately continuous bilinear forms are jointly continuous, so $V_1 \bar{\otimes} V_2$ is equal to the projective tensor product $V_1 \hat{\otimes} V_2$.

(3) If $U = \varinjlim_i U_i$ and $V = \varprojlim_j V_j$ are inductive limits, endowed with the inductive limit topology, then

$$U \bar{\otimes} V = \varinjlim_{i,j} U_i \bar{\otimes} V_j$$

also has the inductive limit topology.

(4) If M and N are smooth manifolds and $\mathcal{D}(M)$ and $\mathcal{D}(N)$ are the spaces of compactly supported smooth functions on M and N topologized as the inductive limit of Fréchet spaces, then

$$\mathcal{D}(M) \bar{\otimes} D(N) \cong \mathcal{D}(M \times N).$$

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In terms of the inductive tensor product, we see that a topological algebra is a topological vector space with associative product a continuous linear map $m : A\bar{\otimes}A \longrightarrow A$. In the rest of this paper, the inductive tensor product will be denoted simply by $\otimes$, and we will study the Hochschild and cyclic homology constructed using this tensor product.

The action $\rho : G \times A \longrightarrow A$ of a Lie group G on a topological algebra A is **differentiable** [1] if

(1) ρ is continuous;
(2) for each $x \in A$, the map $g \mapsto \rho(g, x)$ is infinitely differentiable;
(3) the image of any compact set of G in the automorphism group of A is equicontinuous.

The crossed product algebra $A \rtimes G$ is defined for a discrete group G acting on an algebra A: for differentiable actions of a Lie group on a topological algebra, the analogue of $A \rtimes G$ is the space $\mathcal{D}(G, A)$ of smooth functions of compact support on G with values in A, with product defined as follows: for $u, v \in \mathcal{D}(G, A)$,

$$(uv)(g) = \int_G u(h)\,(hv(h^{-1}g))\,dh.$$

The algebra $\mathcal{D}(G, A)$ does not have an identity unless G is discrete. However, $\mathcal{D}(G, A)$ is a bimodule for the algebra A, with respect to the actions

$$(aub)(g) = au(g)(gb), \quad \text{where } a, b \in A \text{ and } u \in \mathcal{D}(G, A),$$

and we may form the semidirect product $\mathcal{D}^+(G, A)$, which fits into the split short-exact sequence

$$0 \longrightarrow \mathcal{D}(G, A) \longrightarrow \mathcal{D}^+(G, A) \longrightarrow A \longrightarrow 0.$$

For example, the algebra $\mathcal{D}^+(G, \mathbb{C}) = \mathcal{D}^+(G)$ is the result of adjoining an identity to the convolution algebra $\mathcal{D}(G, \mathbb{C}) = \mathcal{D}(G)$. We will apply the construction of Part I to the algebra $\mathcal{D}^+(G, A)$. Wodzicki's theory of excision for H-unital algebras [9] will enable us to derive from this a spectral sequence for the cyclic homology of $\mathcal{D}(G, A)$.

If A is a topological algebra and V is a topological vector space, define a complex $B_n(A, V) = A^{(n)} \otimes V$, $n \geq 1$, with differential $b' : B_n(A, V) \longrightarrow B_{n-1}(A, V)$ given by the formula

$$b'(a_1, \ldots, a_n, v) = \sum_{i=1}^{n-1} (-1)^{i-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_n, v).$$

Following Wodzicki ([8], Remark 3, page 402), we say that a topological algebra A is **strongly H-unital** if the complex $(B(A, V), b')$ is acyclic for every topological vector space V. In particular, a unital algebra A is strongly H-unital, since the map

$$s(a_1, \ldots, a_n, v) = (1, a_1, \ldots, a_n, v)$$

provides a contracting homotopy for the complex $B(A, V)$.

The following result is proved in [9].

Lemma 1. *In a $\mathbb{C}$-split short exact sequence of topological algebras*

$$0 \longrightarrow I \longrightarrow B \longrightarrow A \longrightarrow 0,$$

if two of the algebras are strongly H-unital, then so is the third.

We now have the following result (see Appendix A of [3]).

Proposition 2. *If the algebra A is strongly H-unital, then the crossed product algebra $\mathcal{D}(G, A)$ is strongly H-unital.*

Proof. Applying Lemma 1 to the short exact sequence

$$0 \longrightarrow \mathcal{D}(G, A) \longrightarrow \mathcal{D}(G, A^+) \longrightarrow \mathcal{D}(G) \longrightarrow 0,$$

we see that it suffices to consider the case where A is unital.

Fix an element $\varphi \in C_c^\infty(G)$ such that $\int_G \varphi\, dg = 1$, where dg is the left Haar measure on G. Identify the space $B_n(\mathcal{D}(G, A), V)$ with $C_c^\infty(G^n, A^{(n)} \otimes V)$. Define

$$s : B_n(\mathcal{D}(G, A), V) \longrightarrow B_{n+1}(\mathcal{D}(G, A), V), \quad n \geq 1,$$

by the formula

$$(sf)(g_0, \dots, g_n, v) = (1 \otimes g_0^{-1} \otimes 1 \otimes \dots \otimes 1)(\varphi(g_0) \otimes f(g_0 g_1, \dots, g_n), v).$$

Then s satisfies the formula $sb' + b's = 1$, and so is a contracting homotopy. $\square$

If A is an algebra, let $\mathsf{C}(A)$ be its cyclic bar complex: this is the mixed complex

$$\mathsf{C}_k(A) = \begin{cases} A, & k = 0, \\ A^+ \otimes A^{(k)}, & k > 0, \end{cases}$$

with differentials

$$b(a_0, \dots, a_k) = \sum_{i=0}^{k-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_k) + (-1)^k (a_k a_0, a_1, \dots, a_{k-1}),$$

$$B(a_0, \dots, a_k) = \sum_{i=0}^{k} (-1)^{ik} (1, a_i, \dots, a_k, a_0, \dots, a_{i-1}).$$

If A is unital, then $\mathsf{C}(A)$ is quasi-isomorphic to $\mathsf{C}(A^\natural)$, the mixed complex obtained by forming the chain complex of the cyclic vector space $A^\natural$. In general,

$$\mathsf{C}(A) = \ker(\mathsf{N}((A^+)^\natural) \longrightarrow \mathsf{N}(\mathbb{C}^\natural)).$$

If $0 \longrightarrow I \longrightarrow B \longrightarrow A \longrightarrow 0$ is a $\mathbb{C}$-split extension of algebras, let

$$\mathsf{C}(B, A) = \ker(\mathsf{C}(B) \longrightarrow \mathsf{C}(A))$$

be the relative cyclic bar complex. One of the main properties of H-unital algebras is that they satisfy homological excision, as expressed by the following proposition (Wodzicki [8], [9]).

Proposition 3. *If I is strongly H-unital, the map $\mathsf{C}(I) \to \mathsf{C}(B, A)$ of mixed complexes is a quasi-isomorphism.*

Let W be a graded $\mathbb{C}[u]$-module of finite homological dimension, for example, one of the modules $\mathbb{C}[u]$, $\mathbb{C}[u, u^{-1}]$, $\mathbb{C}[u]/u\mathbb{C}[u]$ and $\mathbb{C}[u, u^{-1}]/u\mathbb{C}[u]$. The cyclic homology of A with coefficients in W is the homology $\mathrm{HC}(A; W)$ of the complex

$$\big(\mathsf{C}(A) \boxtimes W, b + uB\big).$$

If I is strongly H-unital, then Proposition 3 shows that $\mathrm{HC}(I; W) \cong \mathrm{HC}(A, B; W)$.

Given a topological algebra A on which a Lie group G acts differentiably, we will define a cylindrical vector space $\mathsf{L}^+(A, G)$, that is, a contravariant functor from Σ to the category of topological vector spaces or, equivalently, a contravariant functor from $\Lambda_\infty \times \Lambda_\infty$ to the category of topological vector spaces which satisfies the condition $\bar{T} = T^{-1}$. This object will have the property that its diagonal $\Delta(\mathsf{L}^+(A, G))$ is naturally isomorphic as a cyclic vector space to $\mathcal{D}^+(G, A^+)^\natural$. The underlying vector spaces of $\mathsf{L}^+(A, G)$ are

$$\mathsf{L}^+(A, G)([p], [q]) = \mathcal{D}^+(G)^{(p+1)} \otimes (A^+)^{(q+1)}.$$

The action of the category Σ on $\mathsf{L}^+(A, G)([p], [q])$ is given by the following formulas: if $\varphi \in \mathcal{D}^+(G)^{(p+1)}$ and $a_i \in A^+$, and $\omega = \varphi \otimes a_0 \otimes \ldots \otimes a_q$, then

$$\begin{aligned}
\bar{d}\omega(g_0, \ldots, g_{p-1}) &= \varphi(g_0, \ldots, g_{p-1}, g_0) \otimes g_p a_0 \otimes \ldots \otimes g_p a_q, \\
\bar{s}\omega(g_0, \ldots, g_{p+1}) &= \varphi(g_1, \ldots, g_{p+1}) \otimes a_0 \otimes \ldots \otimes a_q, \\
d\omega(g_0, \ldots, g_p) &= \varphi(g_0, \ldots, g_p) \otimes (g^{-1} a_q) a_0 \otimes a_1 \otimes \ldots \otimes a_{q-1}, \\
s\omega(g_0, \ldots, g_p) &= \varphi(g_0, \ldots, g_p) \otimes 1 \otimes a_0 \otimes \ldots \otimes a_q,
\end{aligned}$$

where $g = g_0 \ldots g_p$. The operator $T = t^{q+1} = \bar{t}^{-p-1}$ is given by the formula

$$T\omega = \varphi(g_0, \ldots, g_p) \otimes g^{-1} a_0 \otimes \ldots \otimes g^{-1} a_q.$$

The definition of $\mathsf{L}^+(A, G)$ is the natural extension to the topological setting of the definition of $A \natural G$ in Part I.

Lemma 4. *There is a quasi-isomorphism of mixed complexes*

$$\mathrm{Tot}(\mathsf{N}(\mathsf{L}^+(A, G))) \to \mathsf{C}(\mathcal{D}^+(G, A^+)).$$

Proof. This follows from the identification of the cyclic module $\Delta(\mathsf{L}^+(A, G))$ with $\mathcal{D}^+(G, A^+)^\natural$, combined with Theorem 3.1 of Part I. $\square$

We will now use excision to remove the augmentations which were needed in the proof of Lemma 4.

By naturality, there is a map of cylindrical modules

$$\mathsf{L}^+(A, G) \to \mathsf{L}^+(0, G) \cong \mathcal{D}^+(G)^\natural.$$

Now suppose that A, and hence $\mathcal{D}^+(G,A)$, is strongly H-unital. By Lemma 4, we see that

$$\operatorname{Tot}(\mathsf{N}\ker(\mathsf{L}^+(A,G) \to \mathcal{D}^+(G)^\natural))$$

and $\mathsf{C}(\mathcal{D}^+(G,A^+),\mathcal{D}^+(G))$ are quasi-isomorphic, and hence by Proposition 3, quasi-isomorphic to $\mathsf{C}(\mathcal{D}^+(G,A))$.

Similarly, there is a map of cylindrical modules

$$\mathsf{L}^+(A,G) \to \mathsf{L}^+(A,1) \cong (A^+)^\natural,$$

and we see that

$$\operatorname{Tot}(\mathsf{N}\ker(\mathsf{L}^+(A,G) \to \mathcal{D}^+(G)^\natural \oplus_{\mathbb{C}^\natural} (A^+)^\natural))$$

is quasi-isomorphic to $\mathsf{C}(\mathcal{D}^+(G,A),A)$, and hence by Proposition 3, quasi-isomorphic to $\mathsf{C}(\mathcal{D}(G,A))$. Thus, we see that there is a cylindrical module

$$\mathsf{L}(A,G) = \ker(\mathsf{L}^+(A,G) \to \mathcal{D}^+(G)^\natural \oplus_{\mathbb{C}^\natural} (A^+)^\natural)$$

and quasi-isomorphisms of mixed complexes

$$\operatorname{Tot}(\mathsf{N}(\mathsf{L}(A,G))) \simeq \mathsf{C}(\mathcal{D}(G,A)).$$

This proves the following theorem.

Theorem 5. *Suppose A is an strongly H-unital topological algebra on which the Lie group G acts differentiably. Let W be a one of the $\mathbb{C}[u]$ modules listed above. Then there is a canonical isomorphism*

$$\mathrm{HC}(\mathcal{D}(G,A);W) \cong \mathrm{HC}(\operatorname{Tot}(\mathsf{N}(\mathsf{L}(A,G)));W).$$

Let us use this theorem to derive a spectral sequence for $\mathrm{HC}(\mathcal{D}(G,A);W)$. The normalization $\mathsf{N}(\mathsf{L}(A,G))$ of $\mathsf{L}(A,G)$ has underlying vector spaces

$$\mathsf{N}_{pq}(\mathsf{L}(A,G)) = \begin{cases} \mathcal{D}(G) \otimes A, & p=0, q=0, \\ \mathcal{D}(G) \otimes A^+ \otimes A^{(q)}, & p=0, q>0, \\ \mathcal{D}^+(G) \otimes \mathcal{D}(G)^{(p)} \otimes A, & p>0, q=0, \\ \mathcal{D}^+(G) \otimes \mathcal{D}(G)^{(p)} \otimes A^+ \otimes A^{(q)}, & p>0, q>0. \end{cases}$$

Now, $\mathrm{HC}(\operatorname{Tot}(\mathsf{N}(\mathsf{L}(A,G)));W)$ is the homology of the complex

$$\big(\operatorname{Tot}(\mathsf{N}(\mathsf{L}(A,G))) \boxtimes W, \bar{b} + (b+uB) + uT\bar{B}\big).$$

We filter this as follows:

$$F_i \operatorname{Tot}_\bullet(\mathsf{N}(\mathsf{L}(A,G))) \boxtimes W = \sum_{q \le i} \mathsf{N}_{\bullet q}(\mathsf{L}(A,G)) \boxtimes W.$$

Denote the resulting spectral sequence by E^r_{pq}. In particular, E^0_{pq} is isomorphic to $\mathsf{N}_{pq}(\mathsf{L}(A,G))$ as a bigraded vector space, with differential $\bar{b}$.

Let $A^\natural_G$ be the paracyclic vector space defined by

$$A^\natural_G([n]) = \mathsf{L}(A,G)([0],[n]).$$

There is a differentiable action of the group G on $A^\natural_G$ compatible with the paracyclic structure, and hence an action of the topological algebra $\mathcal{D}(G)$. The following lemma is straightforward.

Lemma 6. *For each $q \geq 0$, there is a short exact sequence of complexes*

$$0 \to C_p\big(\mathcal{D}(G), \mathsf{N}_q(A^\natural_G) \boxtimes W\big) \to E^0_{pq} \to B_p(\mathcal{D}(G)) \otimes \mathsf{C}_q(A) \boxtimes W \to 0,$$

with respective differentials the Hochschild boundary δ, the differential induced on E^0_{pq} by $\bar{b}$, and the differential b' on $B(\mathcal{D}(G))$. Thus, since $\mathcal{D}(G)$ is strongly H-unital, so that $B_p(\mathcal{D}(G)) \otimes \mathsf{C}_q(A) \boxtimes W$ is acyclic, we see that

$$E^1_{pq} \cong H_p(\mathcal{D}(G), \mathsf{N}_q(A^\natural_G) \boxtimes W).$$

By the work of Blanc [1], we may identify the Hochschild homology group $H_\bullet(\mathcal{D}(G), M)$ with the differentiable homology group $H_\bullet(G, M)$. Thus, we obtain our final result.

Corollary 7. *There is a spectral sequence with E^1_{pq}-term*

$$H_p(G, \mathsf{N}_q(A^\natural_G) \boxtimes W) = \begin{cases} H_p(G, A \boxtimes W), & q = 0, \\ H_p(G, A^+ \otimes A^{(q)} \boxtimes W), & q > 0, \end{cases}$$

and differential $b + uB$, converging to the cyclic homology group $\mathrm{HC}(\mathcal{D}(G, A); W)$.

In particular, if G is compact, $E^1_{pq} = 0$ for $p > 0$, so this spectral sequence collapses, and we see that $\mathrm{HC}_\bullet(\mathcal{D}^+(G, A); W) = H_\bullet(\mathsf{C}^G_\bullet(A) \otimes W, b + uB)$ may be calculated by means of the **equivariant cyclic bar complex**

$$\mathsf{C}^G_n(A) = \begin{cases} H_0(G, A \boxtimes W), & n = 0, \\ H_0(G, A^+ \otimes A^{(n)} \boxtimes W), & n > 0. \end{cases}$$

This result generalizes those of Block [2] and Brylinski [4], [BrylinskiKoszul].

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