

THE CYCLIC HOMOLOGY OF CROSSED PRODUCT ALGEBRAS, I.

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INTRODUCTION

In their article [9] on cyclic homology, Feigin and Tsygan have given a spectral sequence for the cyclic homology of a crossed product algebra, generalizing Burghelea's calculation [4] of the cyclic homology of a group algebra. For an analogous spectral sequence for the Hochschild homology of a crossed product algebra, see Brylinski [2], [3].

In this article, we give a new derivation of this spectral sequence, and generalize it to negative and periodic cyclic homology $HC_{\bullet}^{-}(A)$ and $HP_{\bullet}(A)$. The method of proof is itself of interest, since it involves a natural generalization of the notion of a cyclic module, in which the condition that the morphism $\tau \in \Lambda(\mathbf{n}, \mathbf{n})$ is cyclic of order $n + 1$ is relaxed to the condition that it be invertible. We call this category the **paracyclic category**.

Given a paracyclic module P , we can define a chain complex $C(P)$, with differentials b and B , which respectively lower and raise degree. The condition that the module P is paracyclic translates to the condition on $C(P)$ that $1 - (bB + Bb)$ is invertible. Our main result is to show that there is an analogue of the Eilenberg-Zilber theorem for bi-paracyclic modules. It is then easy to obtain a new expression for the cyclic homology of a crossed product algebra which leads immediately to the spectral sequence of Feigin and Tsygan.

If M is a module over a commutative ring \mathbf{k} , we will denote by $M^{(k)}$ the iterated tensor product, defined by $M^{(0)} = \mathbf{k}$ and $M^{(k+1)} = M^{(k)} \otimes M$. If M and N are graded modules, we will denote by $M \otimes N$ their graded tensor product.

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1. PARACYCLIC MODULES AND CROSSED PRODUCT ALGEBRAS

Let A be a unital algebra over a fixed commutative ring \mathbf{k} . Let G be a (discrete) group which acts on A by automorphisms (which we suppose to fix the identity). Recall the definition of the crossed product algebra $A \rtimes G$: the underlying \mathbf{k} -module is $A \otimes \mathbf{k}[G]$, that is, functions from G to A with finite support, and the product is given on elementary tensor products $a \otimes g$ by the formula

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = (a_1(g_1 a_2)) \otimes (g_1 g_2).$$

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It is easy to check that this product is associative and unital. In the special case $\mathbf{k} \rtimes G$, we obtain the group ring $\mathbf{k}[G]$.

A cyclic module $P(\mathbf{n})$ has an underlying simplicial structure, with face morphisms $d_i : P(\mathbf{n} + \mathbf{1}) \rightarrow P(\mathbf{n})$, $0 \leq i \leq n$, and degeneracy morphisms $s_i : P(\mathbf{n} - \mathbf{1}) \rightarrow P(\mathbf{n})$, $0 \leq i \leq n$. In addition, it has morphisms $t : P(\mathbf{n}) \rightarrow P(\mathbf{n})$ for each n , such that $t^{n+1} = 1$, and $t \cdot d_i \cdot t^{-1} = d_{i+1}$, $t \cdot s_i \cdot t^{-1} = s_{i-1}$. We denote the morphism $d_n : P(\mathbf{n}) \rightarrow P(\mathbf{n})$ by d , and the morphism $t \cdot s_0 \cdot t^{-1} : P(\mathbf{n} - \mathbf{1}) \rightarrow P(\mathbf{n})$ by s . The composition $d \cdot s$ is equal to t ; it follows that together, the morphisms d and s generate the action of the cyclic category on P .

Connes defines in [5] a cyclic module B^\natural for any unital algebra B . This cyclic module has as its n -th space $B^\natural(\mathbf{n})$ the \mathbf{k} -module $B^{(n+1)}$, and the actions of d , s and $t = d \cdot s$ are given by the following formulas:

$$\begin{aligned} d(a_0, \dots, a_n) &= (a_n a_0, a_1, \dots, a_{n-1}), \\ s(a_0, \dots, a_n) &= (1, a_0, \dots, a_n), \\ t(a_0, \dots, a_n) &= (a_n, a_0, \dots, a_{n-1}). \end{aligned}$$

The goal of this article is to understand the cyclic module $(A \rtimes G)^\natural$ associated to the crossed product algebra $A \rtimes G$. This is the cyclic module whose n -th space $(A \rtimes G)^\natural(\mathbf{n})$ is $\mathbf{k}[G^{n+1}] \otimes A^{(n+1)}$. Denote the elementary tensor

$$(a_0 \otimes g_0) \otimes \dots \otimes (a_n \otimes g_n) \in (A \rtimes G)^\natural(\mathbf{n})$$

by $(g_0, \dots, g_n | h_0^{-1} a_0, \dots, h_n^{-1} a_n)$, where $h_i = g_i \dots g_n$. This notation is motivated by the fact that in reordering the tensor product so that all of the factors $\mathbf{k}[G]$ occur to the left, we must pass the group elements g_i, \dots, g_n past the algebra element $a_i \in A$.

We have the following formulas for d , s and $t = d \cdot s$ acting on the cyclic \mathbf{k} -module $(A \rtimes G)^\natural$:

$$\begin{aligned} d(g_0, \dots, g_n | a_0, \dots, a_n) &= (g_n g_0, g_1, \dots, g_{n-1} | g_n((g^{-1} a_n) a_0), g_n a_1, \dots, g_n a_{n-1}), \\ s(g_0, \dots, g_n | a_0, \dots, a_n) &= (1, g_0, \dots, g_n | 1, a_0, \dots, a_n), \\ t(g_0, \dots, g_n | a_0, \dots, a_n) &= (g_n, g_0, \dots, g_{n-1} | g_n g^{-1} a_n, g_n a_0, \dots, g_n a_{n-1}), \end{aligned}$$

where $g = g_0 \dots g_n$.

Inspired by the above formulas, we would like to define a bi-cyclic module $A \natural G$ whose diagonal is the above cyclic module. Denote the two sets of generators by $(\bar{d}, \bar{s}, \bar{t})$ and (d, s, t) respectively; the barred and unbarred maps commute with each other. We define $A \natural G(\mathbf{p}, \mathbf{q})$ to be $\mathbf{k}[G^{p+1}] \otimes A^{(q+1)}$, spanned by elementary tensor products which we denote $(g_0, \dots, g_p | a_0, \dots, a_q)$. The two sets of generators for the bi-cyclic structure should be defined in such a way as to factor each of the generators of the cyclic structure on $C(A \rtimes G)$ into two pieces, the barred ones acting within $\mathbf{k}[G^{p+1}]$ and the unbarred ones within

$\mathbf{k}[G^{q+1}]$. The natural formulas for the action of $(\bar{d}, \bar{s}, \bar{t})$ and (d, s, t) are as follows:

$$\begin{aligned}\bar{d}(g_0, \dots, g_p | a_0, \dots, a_q) &= (g_p g_0, g_1, \dots, g_{p-1} | g_p a_0, \dots, g_p a_q), \\ \bar{s}(g_0, \dots, g_p | a_0, \dots, a_q) &= (g_0, \dots, g_p | 1, a_0, \dots, a_q), \\ \bar{t}(g_0, \dots, g_p | a_0, \dots, a_q) &= (g_p, g_0, \dots, g_{p-1} | g_p a_0, \dots, g_p a_q), \\ d(g_0, \dots, g_p | a_0, \dots, a_q) &= (g_0, \dots, g_p | (g^{-1} a_q) a_0, a_1, \dots, a_{q-1}), \\ s(g_0, \dots, g_p | a_0, \dots, a_q) &= (g_0, \dots, g_p | 1, a_0, \dots, a_q), \\ t(g_0, \dots, g_p | a_0, \dots, a_q) &= (g_0, \dots, g_p | g^{-1} a_q, a_0, \dots, a_{q-1}),\end{aligned}$$

where $g = g_0 \dots g_p$. However, it is easy to see that this does not define a bi-cyclic structure: on $A\sharp G(\mathbf{p}, \mathbf{q})$, the operators \bar{t}^{p+1} and t^{q+1} are not equal to the identity, although $\bar{t}^{p+1} = t^{-q-1}$. Let $T = \bar{t}^{p+1} = t^{-q-1}$: it is given by the formula

$$T(g_0, \dots, g_p | a_0, \dots, a_q) = (g_0, \dots, g_p | g a_0, \dots, g a_q).$$

In order to understand the structure of $A\sharp G$, we use a category related to the cyclic category Λ of Connes. This category Λ_∞ , which we call the **paracyclic category** has the same set of objects as the simplicial category Δ , namely the natural numbers \mathbf{n} . Recall that the morphisms $\Delta(\mathbf{n}, \mathbf{m})$ from \mathbf{n} to \mathbf{m} are the monotonically increasing maps from the set $\{0, \dots, n\}$ to the set $\{0, \dots, m\}$. Similarly, the morphisms $\Lambda_\infty(\mathbf{m}, \mathbf{n})$ from \mathbf{m} to \mathbf{n} in the paracyclic category Λ_∞ are monotonically increasing maps f from \mathbb{Z} to itself such that

$$f(i + k(m + 1)) = f(i) + k(n + 1)$$

for all $k \in \mathbb{Z}$. We identify Δ with the subcategory of Λ_∞ such that $f \in \Lambda_\infty(\mathbf{m}, \mathbf{n})$ lies in Δ if and only if f maps $\{0, \dots, m\} \subset \mathbb{Z}$ into $\{0, \dots, n\}$. The paracyclic category has been studied by Fiedorowicz and Loday [8] and Nistor [15]. Dwyer and Kan [7] study the duplicial category, similar to the paracyclic category, except that \mathbb{Z} is replaced by \mathbb{N} . The cyclic category Λ of Connes [5] is the quotient of Λ_∞ by the relation $T = 1$, while the categories Λ_r of Feigin-Tsygan and Bökstedt-Hsiang-Madsen [1] are the quotient of Λ_∞ by the relation $T^r = 1$.

The category Λ_∞ is generated by morphisms $\partial : \mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$,

$$\partial(k) = k + 1 \quad \text{if } 0 \leq k \leq n,$$

and $\sigma : \mathbf{n} \rightarrow \mathbf{n} - \mathbf{1}$,

$$\sigma(k) = k \quad \text{if } 0 \leq k \leq n.$$

The map ∂ is the face map ∂_n in the simplicial category Δ , while σ does not lie in Δ . Denote $\sigma\partial$ by τ ; it corresponds to the map

$$\tau(k) = k + 1 \quad \text{for all } k \in \mathbb{Z}.$$

The face and degeneracy maps of the simplicial category Δ embedded in Λ_∞ are given by the formulas

$$\begin{aligned}\partial_i &= \tau^{-i-1} \cdot \partial \cdot \tau^i : \mathbf{n} - \mathbf{1} \rightarrow \mathbf{n}, \quad 0 \leq i \leq n, \\ \sigma_i &= \tau^i \cdot \sigma \cdot \tau^{-i-1} : \mathbf{n} + \mathbf{1} \rightarrow \mathbf{n}, \quad 0 \leq i \leq n.\end{aligned}$$

Since $\sigma = \tau^{-1} \cdot \sigma_0 \cdot \tau$, we may think of σ as an extra degeneracy σ_{-1} .

Each object \mathbf{n} in the category Λ_∞ has an automorphism $T = \tau^{n+1}$, and it is easily seen that if f is any morphism in Λ_∞ , then $T \cdot f = f \cdot T$. This shows that T induces an invertible automorphism of the category Λ_∞ .

A **paracyclic k -module** P is a contravariant functor from Λ_∞ to the category of k -modules. In particular, a paracyclic module may be considered as a simplicial module, by the inclusion $\Delta \subset \Lambda_\infty$. Denote the actions of ∂ , σ and τ on a paracyclic module P by d , s and t , and of ∂_i and σ_i by d_i and s_i . The category of paracyclic modules has an automorphism T , induced by the automorphism T of the category Λ_∞ .

We now see that $A\sharp G$ is a bi-paracyclic module which satisfies the extra relation $\bar{T} = T^{-1}$; this implies that the diagonal paracyclic module of $A\sharp G$ is cyclic. We call a bi-paracyclic module satisfying this extra relation a **cylindrical** module. The cylindrical category Σ is the quotient of $\Lambda_\infty \times \Lambda_\infty$ by the relation $\bar{T} = T^{-1}$; a cylindrical module is a contravariant functor from Σ to the category of modules.

Later, we will be interested in the paracyclic module $A\sharp G(\mathbf{0}, \mathbf{n})$ which forms the bottom row of the bi-paracyclic module $A\sharp G$. This paracyclic module, which we denote A_G^\sharp , is given explicitly by $\mathbf{n} \mapsto \mathbf{k}[G] \otimes A^{(n+1)}$. The group G acts on A_G^\sharp by the formula

$$h \cdot (g|a_0, \dots, a_n) = (hgh^{-1}|ha_0, \dots, ha_n).$$

Let Λ_∞^n be the paracyclic set $\mathbf{m} \mapsto \Lambda_\infty(\mathbf{m}, \mathbf{n})$, and let $|\Lambda_\infty^n|$ be the geometric realization of the simplicial set underlying Λ_∞^n . In the following proposition, we parametrize the n -simplex Δ^n by

$$\Delta^n = \{0 \leq t_1 \leq \dots \leq t_n \leq 1\}.$$

(This result is similar to Proposition 2.7 of Dwyer-Hopkins-Kan [6] and Theorem 3.4 of Jones [12], and may be proved in the same way.)

Proposition 1.1. *The geometric realization $|\Lambda_\infty^n|$ is homeomorphic to $\mathbb{R} \times \Delta^n$, with non-degenerate $n+1$ -simplices given, for $0 \leq j \leq n$ and $k \in \mathbb{Z}$, by*

$$S_k^j = \{(t|t_1, \dots, t_n) \in \mathbb{R} \times \Delta^n \mid t_j \leq t + k \leq t_{j+1}\}.$$

Each simplex S_k^j is identified with Δ^{n+1} by the map

$$(t_1, \dots, t_{n+1}) \mapsto (t_j + k|t_{j+1}, \dots, t_{n+1}, t_1, \dots, t_{j-1})$$

The maps $\partial : |\Lambda_\infty^n| \rightarrow |\Lambda_\infty^{n+1}|$, $\sigma : |\Lambda_\infty^n| \rightarrow |\Lambda_\infty^{n-1}|$, $\tau : |\Lambda_\infty^n| \rightarrow |\Lambda_\infty^n|$ and $T : |\Lambda_\infty^n| \rightarrow |\Lambda_\infty^n|$ are given by the formulas

$$\begin{aligned} \partial(t|t_1, \dots, t_n) &= (t|t_1, \dots, t_n, t_n), \\ \sigma(t|t_1, \dots, t_n) &= (t + t_1|t_2 - t_1, \dots, t_n - t_1, t_n), \\ \tau(t|t_1, \dots, t_n) &= (t + t_1|t_2 - t_1, \dots, t_n - t_1), \\ T(t|t_1, \dots, t_n) &= (t + 1|t_1, \dots, t_n). \end{aligned}$$

2. PARACHAIN COMPLEXES

The following definition is inspired by the definition of a duchain complex due to Dwyer and Kan [7].

Definition 2.1. *A **parachain complex** is a graded \mathbf{k} -module $(V_i)_{i \in \mathbb{N}}$ with two operators $b : V_i \rightarrow V_{i-1}$ and $B : V_i \rightarrow V_{i+1}$, such that*

- (1) $b^2 = B^2 = 0$, and
- (2) *the operator $T = 1 - (bB + Bb)$ is invertible.*

It may be easily checked that T commutes with both b and B . When T is the identity, the two differentials b and B commute; such a parachain complex is called a **mixed complex**. In the definition of a duchain complex, there is no condition on T : parachain complexes bear the same relationship to paracyclic modules that duchain complexes bear to duplicial modules.

If V_\bullet is a graded vector space, let $V_\bullet[[u]]$ be the graded vector space of formal power series in a variable u of degree -2 with coefficients in V_\bullet . If V_\bullet is a mixed complex, one considers the associated complex $V_\bullet[[u]]$ with differential $b + uB$; this motivates considering the operator $b + uB$ on $V_\bullet[[u]]$ even when V_\bullet is only a paracyclic module.

Definition 2.2. *A morphism between parachain complexes V_\bullet and \tilde{V}_\bullet is a map from $V_\bullet[[u]]$ to $\tilde{V}_\bullet[[u]]$ homogeneous of degree 0,*

$$f = \sum_{k=0}^{\infty} u^k f_k,$$

such that $(\tilde{b} + u\tilde{B}) \cdot f = f \cdot (b + uB)$.

Without introducing the operator $b + uB$, a morphism $f : V_\bullet \rightarrow \tilde{V}_\bullet$ may be defined as a sequence of maps $f_k : V_i \rightarrow \tilde{V}_{i+2k}$, $k \geq 0$, such that

$$\tilde{b} \cdot f_k + \tilde{B} \cdot f_{k-1} = f_k \cdot b + f_{k-1} \cdot B.$$

The composition of two parachain complex maps is a parachain complex map, and a map of parachain complexes f satisfies $\tilde{T} \cdot f_i = f_i \cdot T$. Thus, the operator T defines an action of \mathbb{Z} on the category of parachain complexes.

There is a functor \mathbf{C} from paracyclic modules to parachain complexes, with underlying graded module $\mathbf{C}_n(P) = P(\mathbf{n})$ and operators $b = \sum_{i=0}^n (-1)^i d_i$ and $B = (1 - (-1)^{n+1} t) sN$; here N is the norm operator $N = \sum_{i=0}^n (-1)^i t^i$.

The proof of the following theorem is close to the discussion of Section 1 of [14].

Theorem 2.3. *The functor $P \mapsto \mathbf{C}(P)$ is a \mathbb{Z} -equivariant functor from the category of paracyclic \mathbf{k} -modules to the category of parachain complexes over \mathbf{k} , that is,*

- (1) $b^2 = B^2 = 0$,
- (2) $bB + Bb = 1 - T$, and
- (3) *it intertwines the natural transformations T of these two categories.*

Proof. The proof that $b^2 = 0$ is the same as usual, since it only depends on the underlying simplicial module structure on P .

The operator $B^2 : C_n(P) \rightarrow C_{n+2}(P)$ is given by

$$\begin{aligned} B^2 &= (1 - (-1)^{n+2}t)sN(1 - (-1)^{n+1}t)sN \\ &= (1 - (-1)^{n+2}t)s(1 - T)sN \\ &= (1 - (-1)^{n+2}t)ssN(1 - T) \\ &= (s_n - (-1)^{n+2}s_{n+1})sN(1 - T), \end{aligned}$$

which shows that B^2 is zero in the associated chain complex. Here, we have used the formulas $(1 - (-1)^n t)N = 1 - T$, $ss = s_n s$ and

$$ts = T^{-1}(ts)T = t^{-n-1}st^{n+1} = s_{n+1}.$$

To calculate $bB + Bb$, we introduce the operator $b' = \sum_{i=1}^n (-1)^i d_i$ on $C_n(P)$.

Lemma 2.4. *If P is a paracyclic module, then on $C(P)_n$ we have the formulas*

- (1) $b(1 - (-1)^n t) = (1 - (-1)^{n-1} t)b'$,
- (2) $Nb = b'N$, and
- (3) $sb' + b's = 1$.

Proof. The first formula is proved in the same way as in [14]. The operators b and b' on $C_n(P)$ are given by

$$b = \sum_{i=0}^n (-1)^i t^i dt^{-i-1}, \quad b' = \sum_{i=0}^{n-1} (-1)^i t^i dt^{-i-1}.$$

Thus,

$$\begin{aligned} b(1 - (-1)^n t) &= \sum_{i=0}^n (-1)^{n-i} t^i d(t^{-i-1} - (-1)^n t^{-i}) \\ &= -(-1)^n d + (1 - (-1)^{n-1} t) \sum_{i=0}^{n-1} (-1)^i t^i dt^{-i-1} + (-1)^n t^n dt^{-n-1}. \end{aligned}$$

However, $t^n dt^{-n-1} = d$, and the formula follows.

We leave the proof of the second formula to the reader. To prove the third formula, we use the fact that on \mathbf{n} ,

$$\tau^{-i-1} \partial \tau^i \sigma = \sigma \tau^{-i-2} \partial \tau^{i+1}$$

for $0 \leq i \leq n-1$. Thus, it follows that

$$sb' = \sum_{i=0}^{n-1} (-1)^i st^i dt^{-i-1} = \sum_{i=0}^{p-1} (-1)^i t^{i+1} dt^{-i-2} s = 1 - b's. \quad \square$$

As a corollary of this lemma, we see that on $C_n(P)$,

$$\begin{aligned} bB &= (1 - (-1)^n t) b' s N, \quad \text{and} \\ Bb &= (1 - (-1)^n t) s b' N, \end{aligned}$$

and hence that

$$\begin{aligned} Bb + bB &= (1 - (-1)^n t)(s b' + b' s) N = (1 - (-1)^n t) N \\ &= 1 - t^{n+1} = 1 - T. \end{aligned}$$

This completes the proof of the theorem. \square

A multi-parachain complex is a \mathbb{N}^k -graded module $V_{n_1 \dots n_k}$ with operators

$$\begin{aligned} b_i &: V_{n_1 \dots n_i \dots n_k} \rightarrow V_{n_1 \dots n_i - 1 \dots n_k}, \\ B_i &: V_{n_1 \dots n_i \dots n_k} \rightarrow V_{n_1 \dots n_i + 1 \dots n_k}. \end{aligned}$$

The operators $\{b_i, B_i\}$ and $\{b_j, B_j\}$ are required to (graded) commute if i and j are not equal, while $T_i = 1 - (b_i B_i + B_i b_i)$ is required to be invertible.

There is a functor $V \mapsto \text{Tot}(V)$ from multiparachain complexes to parachain complexes, which we will call the **total parachain complex**. It is formed by setting

$$\text{Tot}_n(V) = \sum_{n_1 + \dots + n_k = n} V_{n_1 \dots n_k},$$

with operators

$$\begin{aligned} \text{Tot}(b) &= \sum_{i=1}^k b_i, \\ \text{Tot}(B) &= \sum_{i=1}^n T_{i+1} \dots T_k B_i. \end{aligned}$$

The definition of $\text{Tot}(B)$ on $\text{Tot}(V)$ may seem a little strange, but is justified by the following lemma, which shows that $\text{Tot}(V)$ is a parachain complex.

Lemma 2.5. *The total T -operator $\text{Tot}(T) = T_1 \dots T_k$.*

Proof. The proof uses the fact that $\{b_i, B_i\}$ and $\{b_j, B_j\}$ commute for $i \neq j$. Thus, we see that

$$\begin{aligned} 1 - (\text{Tot}(b) \text{Tot}(B) + \text{Tot}(B) \text{Tot}(b)) &= 1 - \sum_{i=1}^k [b_i, B_i] T_{i+1} \dots T_k \\ &= 1 - \sum_{i=1}^k (1 - T_i) T_{i+1} \dots T_k, \end{aligned}$$

from which the lemma follows. \square

We are most interested in the special case of biparachain complexes. We will denote b_1 and b_2 by \bar{b} and b , and B_1 and B_2 by \bar{B} and B . When $\bar{T} = T^{-1}$, we call a bi-parachain complex V a **cylindrical complex**; in this case, the above lemma shows that $\text{Tot}(T) = 1$, that is, $\text{Tot}(V)$ is a mixed complex.

Finally, we have the normalized chain functor \mathbf{N} from paracyclic modules to parachain complexes, with underlying graded module

$$\mathbf{N}_n(P) = P(\mathbf{n}) / \sum_{i=0}^n \text{im}(s_i),$$

and operators b, B induced by those on $\mathbf{C}(P)$. It is a standard result that the quotient map $(\mathbf{C}(P), b) \rightarrow (\mathbf{N}(P), b)$ is a quasi-isomorphism of complexes. More generally, if P is a multi-paracyclic module, we denote by $\mathbf{N}(P)$ the multi-paracyclic complex obtained by normalizing successively in all directions.

3. THE EILENBERG-ZILBER THEOREM FOR PARACYCLIC MODULES

Let $P(\mathbf{p}, \mathbf{q})$ be a bi-paracyclic module, and let $\mathbf{C}(P)$ be the biparachain complex obtained by forming the chain complex successively in both directions. Let $\text{Tot}(\mathbf{C}(P))$ be the total parachain complex of $\mathbf{C}(P)$: by the above results, if P is a cylindrical module, $\text{Tot}(\mathbf{C}(P))$ is a mixed complex. Using the diagonal embedding of Λ_∞ into $\Lambda_\infty \times \Lambda_\infty$, we see that the diagonal $\mathbf{n} \mapsto P(\mathbf{n}, \mathbf{n})$ is a paracyclic object, which we will denote by $\Delta P(\mathbf{n})$. The action of Λ_∞ on $\Delta P(\mathbf{n})$ is generated by the maps $\bar{a}d, \bar{s}s$ and $\bar{t}t$.

The shuffle product is a natural map from the total complex $(\text{Tot}(\mathbf{C}(P)), \text{Tot}(b))$ to the chain complex of the diagonal $(\mathbf{C}(\Delta P), b)$, which is an equivalence of complexes; this is proved using the method of acyclic models. This product was extended to a map of mixed complexes by Hood and Jones [11] when P is bi-cyclic; see also our paper [10], where we give explicit formulas for this map. We will give explicit formulas on normalized chains; to extend these results to the unnormalized chains, we may apply the results of Kassel [13], who shows how to construct a homotopy inverse to the normalization map.

Theorem 3.1. *Let P be a bi-paracyclic module. There is a natural quasi-isomorphism $f_0 + uf_1 : \text{Tot}(\mathbf{C}(P)) \rightarrow \mathbf{C}(\Delta P)$ of parachain complexes such that $f_0 : \text{Tot}(\mathbf{C}(P))_\bullet \rightarrow \mathbf{C}(\Delta P)_\bullet$ is the shuffle map.*

Proof. We must construct a map

$$f_1 : \text{Tot}_\bullet(\mathbf{C}(P)) \rightarrow \mathbf{C}_{\bullet+2}(\Delta P)$$

to satisfy the following two formulas:

$$\begin{aligned} b \cdot f_1 &= f_1 \cdot (b + \bar{b}) - B \cdot f_0 + f_0 \cdot (B + \bar{B}), \\ B \cdot f_1 &= f_1 \cdot (B + \bar{B}). \end{aligned}$$

The fact that $f = f_0 + uf_1$ is a quasi-isomorphism then follows by a standard argument from the fact that it is true for the shuffle product f_0 .

FIGURE 3.1

Let ι_n be the non-degenerate n -simplex in Λ_∞^n , corresponding to the identity map on the object $\mathbf{n} \in \Lambda_\infty$. This simplex corresponds to the geometric simplex

$$\{(0|t_1, \dots, t_n) \mid 0 \leq t_1 \leq \dots \leq t_n \leq 1\} \subset |\Lambda_\infty^n|,$$

in the geometric realization of Λ_∞^n , as described in Proposition 1.1. By definition, the non-degenerate simplices of Λ_∞^n are in one-to-one correspondence with the morphisms of Λ_∞ with range \mathbf{n} , and these simplices are obtained by applying the corresponding morphism of the opposite category $\Lambda_\infty^{\text{op}}$ to ι_n .

If X is a paracyclic set, the chains on X with values in \mathbf{k} , written $\mathbf{k}[X]$, form a paracyclic module in an evident way. Similarly, the module of chains on the bi-paracyclic set $\Lambda_\infty^p \times \Lambda_\infty^q$ is a bi-paracyclic module, which we denote by $\mathbf{k}[\Lambda_\infty^p \times \Lambda_\infty^q]$. The following result is the analogue of Lemma 2.1 of Hood and Jones [11].

Lemma 3.3. *If P is a bi-paracyclic module and $x \in P(\mathbf{p}, \mathbf{q})$, there is a unique map of bi-paracyclic modules $i_x : \mathbf{k}[\Lambda_\infty^p \times \Lambda_\infty^q] \rightarrow V$ such that $i_x(\iota_p \times \iota_q) = x$.*

From this lemma and the fact that f_1 is to be natural, we see that it suffices to define f_1 on the elements $\iota_p \times \iota_q \in \mathbf{k}[\Lambda_\infty^p \times \Lambda_\infty^q]$. The following argument may be better understood by reference to Figure 3.1.

The image of $\iota_p \times \iota_q$ under the map B is the chain

$$\{0\} \times [0, 1] \times \Delta^p \times \Delta^q \subset \mathbb{R}^2 \times \Delta^p \times \Delta^q.$$

Similarly, its image under the map $T\bar{B}$ is the chain

$$[0, 1] \times \{1\} \times \Delta_p \times \Delta_q.$$

Finally, $B \cdot f_0(\iota_p \times \iota_q)$ is the chain

$$\{(t, t) \mid t \in [0, 1]\} \times \Delta^p \times \Delta^q.$$

From this, we see that $f_0 \cdot (T\bar{B} + B) - B \cdot f_0$ applied to $\iota_p \times \iota_q$ is the chain

$$\partial K \times \Delta^p \times \Delta^q,$$

where K is the triangle $\{(s, t) \mid 0 \leq t \leq s \leq 1\} \subset \mathbb{R}^2$. It is now obvious that in order for the formula

$$b \cdot f_1 - f_1 \cdot (b + \bar{b}) = f_0 \cdot (T\bar{B} + B) - B \cdot f_0$$

to hold when applied to $\iota_p \times \iota_q$, we must choose $f_1(\iota_p \times \iota_q)$ to equal the simplicial chain corresponding to the geometric chain

$$K \times \Delta^p \times \Delta^q.$$

This may be done uniquely, because we work in the normalized chain complex. An explicit formula for this chain may be given in terms of the cyclic shuffles introduced in [10]; we see from these formulas or by a geometric argument that $f_1 \cdot B = f_1 \cdot \bar{B} = B \cdot f_1 = 0$ modulo degenerate chains. \square

4. APPLICATION TO THE CYCLIC HOMOLOGY OF CROSSED PRODUCT ALGEBRAS

Recall the definition of the cyclic homology of a mixed complex (V, b, B) . Let W be a graded module over the polynomial ring $\mathbf{k}[u]$, where $\deg(u) = -2$; we will always assume that W has finite homological dimension. If C_\bullet is a mixed complex, we denote $C_\bullet \llbracket u \rrbracket \otimes_{\mathbf{k}[u]} W$ by $C_\bullet \boxtimes W$. We define the cyclic homology of the mixed complex C_\bullet with coefficients in W to be

$$\mathrm{HC}(C_\bullet; W) = H_\bullet(C_\bullet \boxtimes W, b + uB).$$

In the particular case where $V = \mathbb{C}(A^\natural)$, we write

$$\mathrm{HC}_\bullet(A; W) = \mathrm{HC}_\bullet(\mathbb{C}(A^\natural); W).$$

If $f : C_\bullet \rightarrow \tilde{C}_\bullet$ is a map of mixed complexes, it induces a map of cyclic homology

$$f : \mathrm{HC}(C_\bullet; W) \rightarrow \mathrm{HC}(\tilde{C}_\bullet; W).$$

We say that f is a **quasi-isomorphism** (and write $f : C_\bullet \simeq \tilde{C}_\bullet$) if f induces an isomorphism of homology

$$f : H_\bullet(C_\bullet, b) \cong H_\bullet(\tilde{C}_\bullet, \tilde{b}).$$

If $f : C_\bullet \simeq \tilde{C}_\bullet$ is a quasi-isomorphism of mixed complexes, and W is a graded $\mathbf{k}[u]$ -module of finite homological dimension, we obtain isomorphisms of cyclic homology

$$f : \mathrm{HC}(C_\bullet; W) \cong \mathrm{HC}(\tilde{C}_\bullet; W).$$

Let us list some examples of cyclic homology with different coefficients W :

- (1) $W = \mathbf{k}[u]$ gives negative cyclic homology $\mathrm{HC}_\bullet^-(A)$;
- (2) $W = \mathbf{k}[u, u^{-1}]$ gives periodic cyclic homology $\mathrm{HP}_\bullet(A)$;
- (3) $W = \mathbf{k}[u, u^{-1}]/u\mathbf{k}[u]$ gives cyclic homology $\mathrm{HC}_\bullet(A)$;
- (4) $W = \mathbf{k}[u]/u\mathbf{k}[u]$ gives the Hochschild homology $\mathrm{HH}_\bullet(A)$.

Using the Eilenberg-Zilber Theorem for paracyclic complexes (Theorem 3.1), we obtain the following theorem.

Theorem 4.1. *Let A be a unital algebra over the commutative ring \mathbf{k} , and let G be a discrete group which acts on A . There is a quasi-isomorphism of mixed complexes*

$$f_0 + uf_1 : \mathrm{Tot}(\mathrm{N}(A \natural G)) \simeq \mathrm{N}((A \rtimes G)^\natural).$$

Thus, we obtain isomorphism of cyclic homology groups

$$\mathrm{HC}_\bullet(A \rtimes G; W) = \mathrm{HC}_\bullet(\mathrm{Tot}(\mathrm{N}(A \natural G)); W).$$

It is also possible to take the unnormalized chain complex $\mathrm{C}(A \natural G)$ in this theorem, since this is quasi-isomorphic to the normalized chain complex. This allows us to restate our result in the following more explicit form.

Corollary 4.2. *There are operators b, \bar{b}, B and \bar{B} on the complex*

$$\mathrm{Tot}_n(\mathrm{C}(A \natural G)) = \sum_{p+q=n} \mathbf{k}[G^{p+1}] \otimes A^{(q+1)}$$

such that the homology of the complex

$$(\mathrm{Tot}(\mathrm{C}(A \natural G)) \boxtimes W, b + \bar{b} + u(B + T\bar{B}))$$

is the cyclic homology $\mathrm{HC}_\bullet(A \rtimes G; W)$.

The above theorem leads to the spectral sequence of Feigin and Tsygan, converging to $\mathrm{HC}_\bullet(A \rtimes G; W)$ (see Appendix 6 of [9]). We filter the complex $\mathrm{C}(A \natural G)$ by subspaces

$$F_{pq}^i \mathrm{Tot}(\mathrm{C}(A \natural G)) \boxtimes W = \sum_{q \leq i} \mathbf{k}[G^{p+1}] \otimes A^{(q+1)} \boxtimes W.$$

Recall the paracyclic module $A_G^\natural(\mathbf{n}) = A \natural G(\mathbf{0}, \mathbf{n}) \cong \mathbf{k}[G] \otimes A^{(n+1)}$ of Section 1. If M is a G -module, let $C_p(G, M) = \mathbf{k}[G^p] \otimes M$ be the space of p -chains on G with values in M , with boundary $\delta : C_p(G, M) \rightarrow C_{p-1}(G, M)$.

Lemma 4.3. *The E^0 -term of the spectral sequence is isomorphic to the complex*

$$E_{pq}^0 = C_p(G, \mathbb{C}_q(A_G^{\natural}) \boxtimes W).$$

Proof. Consider the map β from $\mathbb{C}_{pq}(A_G^{\natural})$ to $C_p(G, \mathbb{C}_q(A_G^{\natural}))$ given by the formula

$$(g_0, \dots, g_p | a_0, \dots, a_q) \mapsto (g_1, \dots, g_p | g | a_0, \dots, a_q),$$

where $g = g_0 \dots g_p$. It is easily seen that

$$\begin{aligned} (\beta \bar{b} \beta^{-1})(g_1, \dots, g_p | g | a_0, \dots, a_q) &= (g_2, \dots, g_p | g | a_0, \dots, a_q) \\ &\quad + \sum_{i=1}^{p-1} (-1)^i (g_1, \dots, g_i g_{i+1}, \dots, g_p | g | a_0, \dots, a_q) \\ &\quad + (-1)^p (g_1, \dots, g_{p-1} | g_p g g_p^{-1} | g_p a_0, \dots, g_p a_q), \end{aligned}$$

which is just the boundary for group homology with coefficients in $\mathbb{C}_q(A_G^{\natural})$. \square

Although we do not use it, let us state the formula for $\beta \bar{B} \beta^{-1}$:

$$\begin{aligned} (\beta \bar{B} \beta^{-1})(g_1, \dots, g_p | g | a_0, \dots, a_q) &= (1, g_1, \dots, g_p | g | a_0, \dots, a_q) \\ &\quad + \sum_{i=1}^p (-1)^{pi} (g_i \dots g_p) \cdot (1, g_{i+1}, \dots, g_p, g(g_1 \dots g_p)^{-1}, g_1, \dots, g_{i-1} | g | a_0, \dots, a_q). \end{aligned}$$

It follows from Lemma 4.3 that the E^1 -term of the spectral sequence is

$$E_{pq}^1 = H_p(G, \mathbb{C}_q(A_G^{\natural}) \boxtimes W).$$

The following lemma enables us to give the differential d^1 a natural interpretation.

Lemma 4.4. *The homology spaces $H_p(G, A_G^{\natural})$ are cyclic modules, with respect to the cyclic structure induced by the maps*

$$\begin{aligned} d(g | a_0, \dots, a_q) &= (g | (g^{-1} a_q) a_0, a_1, \dots, a_{q-1}), \\ s(g | a_0, \dots, a_q) &= (g | 1, a_0, \dots, a_q), \end{aligned}$$

acting on A_G^{\natural} .

Proof. If we apply the chain functor \mathbb{C} along the G -axis of the bi-paracyclic module $A_G^{\natural} G$, we obtain the paracyclic parachain chain complex $(C_p(G, A_G^{\natural}), \bar{b}, \bar{B})$, where \bar{b} is the homology boundary. The operator $T\bar{B}$ gives a chain homotopy of T to the identity, since

$$\bar{b}\bar{B} + \bar{B}\bar{b} = 1 - \bar{T} = 1 - T^{-1},$$

showing that $H_p(G, A_G^{\natural})$ is a cyclic module for each p . \square

We see that the differential d^1 is just the differential $b + uB$ associated to the cyclic module $H_{\bullet}(G, A_G^{\natural})$. This completes the proof of the following theorem.

Theorem 4.5. *By means of the isomorphism*

$$H_p(G, \mathbb{C}_q(A_G^{\natural}) \boxtimes W) \cong \mathbb{C}_q(H_p(G, A_G^{\natural})) \boxtimes W,$$

the E^2 -term of the spectral sequence may be identified with the cyclic homology

$$\mathrm{HC}_q(H_p(G, A_G^{\natural}); W)$$

of the cyclic module $H_{\bullet}(G, A_G^{\natural})$.

To see the relationship between this spectral sequence and that of Feigin and Tsygan, we observe that the G -module A_G^{\natural} decomposes into a direct sum over the conjugacy classes $[g] = \{hgh^{-1} \mid h \in G\}$ of G :

$$A_G^{\natural} = \sum_{[g]} A_{[g]}^{\natural},$$

where $A_{[g]}^{\natural}$ is the paracyclic G -module such that $A_{[g]}^{\natural}(\mathbf{n})$ consists of all functions from the conjugacy class $[g]$ to $A^{(n+1)}$. Choose an arbitrary element $g \in [g]$, and let A_g^{\natural} be the stalk of A_G^{\natural} over g . This paracyclic module is acted on by the centralizer G^g of g , and it is easily seen that

$$A_{[g]}^{\natural} \cong \mathrm{Ind}_{G^g}^G A_g^{\natural}$$

is an induced module. Shapiro's Lemma now shows that

$$H_p(G, A_G^{\natural}) \cong \sum_{[g]} H_p(G^g, A_g^{\natural}),$$

from which Feigin and Tsygan's form of the spectral sequence follows easily.

Now suppose the order $|G|$ of the group G is finite and invertible in \mathbf{k} . It follows that $E_{pq}^2 = 0$ if $p > 0$, and our spectral sequence collapses. The only remaining contribution to E^2 comes from the cyclic module $H_0(G, A_G^{\natural})$ of coinvariants in A_G^{\natural} , introduced by Brylinski [2].

Proposition 4.6. *If G is finite and $|G|$ is invertible in \mathbf{k} , then there is a natural isomorphism of cyclic homology and*

$$\mathrm{HC}_{\bullet}(A \rtimes G; W) = \mathrm{HC}_{\bullet}(H_0(G, A_G^{\natural}); W),$$

where $H_0(G, A_G^{\natural})$ is the cyclic module

$$H_0(G, A_G^{\natural})(\mathbf{n}) = H_0(G, \mathbf{k}[G] \otimes A^{(n+1)}).$$

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