THE ODD CHERN CHARACTER IN CYCLIC HOMOLOGY AND SPECTRAL FLOW

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If $\mathcal{A}$ is a Banach $*$-algebra, an odd theta-summable Fredholm module over $\mathcal{A}$ consists of the following data: a Hilbert space $\mathcal{H}$, a continuous $*$-representation ρ of $\mathcal{A}$ on $\mathcal{H}$, and a self-adjoint operator D on $\mathcal{H}$ such that

(1) there is a constant C such that $\|[\mathsf{D},\rho(a)]\| \le C\|a\|$ for all $a \in \mathcal{A}$, and
(2) if $t > 0$, then the operator $e^{-t\mathsf{D}^2}$ is trace class.

Such a Fredholm module determines a class $[\mathsf{D}]$ in the group $K^{-1}(\mathcal{A})$, which might be thought of as an odd-dimensional chain on the non-commutative space with function algebra $\mathcal{A}$.

Dually, unitary matrices $g \in \mathrm{U}_N(\mathcal{A})$ with entries in $\mathcal{A}$ represent elements of the group $K_1(\mathcal{A})$.

If A_0 and A_1 are self-adjoint operators on $\mathcal{H}$ with the same spectrum (including multiplicities), the spectral flow $\mathrm{sf}(A_0, A_1)$, introduced by Atiyah-Patodi-Singer [1], is the integer which counts the number of eigenvalues crossing zero from below minus the number crossing zero from above, as we travel along the family $A_u = (1-u)A_0 + uA_1$ in the direction of increasing u.

If D is an odd theta-summable Fredholm module over D and $g \in \mathrm{U}_N(\mathcal{A})$ is a unitary matrix with entries in $\mathcal{A}$, the spectral flow defines a pairing

$$\langle \mathsf{D}, g\rangle = \mathrm{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$$

between $K^{-1}(\mathcal{A})$ and $K_1(\mathcal{A})$, with values in the integers. This pairing is a homotopy invariant function of D and g. In Section 2, we derive the following formula:

$$\mathrm{sf}(\mathsf{D}, g^{-1}\mathsf{D}g) = \frac{1}{\pi^{1/2}} \int_0^1 \mathrm{Tr}\big(\dot{\mathsf{D}}_u e^{-\mathsf{D}_u^2}\big)\, du,$$

where $\mathsf{D}_u = (1-u)\mathsf{D} + ug^{-1}\mathsf{D}g$ and $\dot{\mathsf{D}}_u = g^{-1}[\mathsf{D}, g]$.

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An elementary example is the operator

$$\mathsf{D} = \frac{1}{2\pi i}\frac{d}{dt}$$

on the Hilbert space $L^2(S^1)$, where S^1 is the circle parametrized by $t \in [0,1]$. If $g : S^1 \longrightarrow S^1 \subset \mathbb{C}$ is a differentiable map from S^1 to the unit circle in the complex plane, then the spectral flow $\mathrm{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$ equals the degree $\deg(g)$ of g. Indeed, this is clear if $g(t) = e^{2\pi i n t}$, since

$$\mathsf{D}_u = \frac{1}{2\pi i}\frac{d}{dt} + \nu,$$

and follows for general g since $\mathrm{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$ is invariant under homotopies of g.

Our goal in this article is to express $\langle \mathsf{D}, g\rangle$ in terms of the Chern character $\mathrm{Ch}^*(\mathsf{D})$ of D in entire cyclic cohomology introduced by Jaffe-Lesniewski-Osterwalder [11] and the Chern character $\mathrm{Ch}_*(g)$ of g in entire cyclic homology, which we discuss in Section 3. In Section 4, we will prove the following formula:

$$\langle \mathsf{D}, g\rangle = (\mathrm{Ch}^*(\mathsf{D}), \mathrm{Ch}_*(g)).$$

This is the analogue for odd K-theory of the formula of Connes and Getzler-Szenes [7] for the index pairing in even K-theory: if D is an even theta-summable Fredholm module over $\mathcal{A}$, and $p \in M_N(\mathcal{A})$ is an idempotent matrix with values in $\mathcal{A}$, then

$$\langle \mathsf{D}, p\rangle = (\mathrm{Ch}^*(\mathsf{D}), \mathrm{Ch}_*(p)),$$

where $\mathrm{Ch}_*(p)$ is the Chern character of the idempotent p, introduced by Hood and Jones [10].

Since the odd Chern character is less well known than the even Chern character, we have included, by way of an introduction, an expository Section 1, in which we discuss the odd Chern character in differential geometry. The main result of this section, that two different formulas for this Chern character are equal, is a model for the proof of the formula for $\langle \mathsf{D}, g\rangle$.

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Notation

We will denote by $C(q)$ the complex Clifford algebra with odd generators c_i, $1 \le i \le q$, and relations

$$c_i c_j + c_j c_i = 2\delta_{ij}.$$

This is a $\mathbb{Z}/2$-graded $*$-algebra, with $c_i^* = c_i$.

If $\mathcal{E}$ is a graded module for $C(q)$, denote by $\operatorname{End}_{C(q)}(\mathcal{E})$ the algebra of endomorphisms of $\mathcal{E}$ which (graded) commute with the action of $C(q)$ or, equivalently, with the generators c_i. The Clifford supertrace of an operator $A \in \mathcal{L}^1_{C(q)}(\mathcal{E})$ is defined by the formula (see Getzler [5] for the justification)

$$\operatorname{Str}_{C(q)}(A) = (4\pi)^{-q/2} \operatorname{Str}(c_1 \dots c_q A).$$

It is easily verified that this is a supertrace on $\operatorname{End}_{C(q)}(\mathcal{E})$.

We will denote by Δ^n the n-simplex

$$\Delta^n = \{(\sigma_0, \dots, \sigma_n) \in [0,1]^n \mid \sigma_0 + \dots + \sigma_n = 1\}$$

with measure $d\sigma = d\sigma_1 \dots d\sigma_n$.

1. The odd Chern character in differential geometry

If M is a smooth manifold, an element of $K^{-1}(M)$ may be represented by a differentiable map from M to the general linear group $\mathrm{GL}_N(\mathbb{C})$, for N sufficiently large. In this section, we will represent the Chern character of such an element of $K^{-1}(M)$ by a closed differential form; this is the odd analogue of the Chern-Weil formula for the Chern character of a vector bundle in terms of a connection.

If ∇_0 and ∇_1 are two connections on a vector bundle $\mathcal{E}$, their Chern-Simons form is the differential form of odd degree

$$\operatorname{cs}(\nabla_0, \nabla_1) = \int_0^1 \operatorname{Tr}\big(\dot{\nabla}_u e^{\nabla_u^2}\big)\, du,$$

where $\nabla_u = (1-u)\nabla_0 + u\nabla_1$ and $\dot{\nabla}_u = \nabla_1 - \nabla_0$. This is the integral of the Chern character form $\operatorname{Ch}(\tilde{\nabla}) = \operatorname{Tr}(e^{\tilde{\nabla}^2})$ over the fibres of the projection $[0,1] \times M \longrightarrow M$, where $\tilde{\nabla}$ is the connection, on the trivial bundle of rank N over $[0,1] \times M$, given by the formula

$$\tilde{\nabla} = du \frac{\partial}{\partial u} + \nabla_u.$$

The differential form $\operatorname{cs}(d, d+\omega)$ satisfies the **transgression formula**

$$d \operatorname{cs}(\nabla_0, \nabla_1) = \operatorname{Ch}(\nabla_1) - \operatorname{Ch}(\nabla_0).$$

To define the Chern character of an element of $K^{-1}(M)$ represented as a differentiable map $g : M \longrightarrow \mathrm{GL}_N(\mathbb{C})$, we introduce a family of connections $\nabla_u = d + u\omega$ on the trivial bundle $M \times \mathbb{C}^N$, where $\omega = g^{-1}(dg) \in \Omega^1(M, \mathrm{gl}_N(\mathbb{C}))$. Note that $\nabla_0 = d$, while $\nabla_1 = g^{-1} dg$ is gauge equivalent to d, and hence both connections have vanishing curvature. This shows that $d \operatorname{cs}(d, d+\omega) = 0$.

Definition 1.1. *The Chern character* $\mathrm{Ch}(g)$ *of a differentiable map* $g : M \to \mathrm{GL}_N(\mathbb{C})$ *is the odd differential form* $\mathrm{cs}(d, g^{-1}dg)$*.*

Since $d\omega + \omega^2 = 0$, the curvature of the connection ∇_u is given by the formula

$$\nabla_u^2 = -u(1-u)\omega^2.$$

Note that $\mathrm{Tr}(\omega^{2k}) = \frac{1}{2}\mathrm{Tr}[\omega, \omega^{2k-1}] = 0$ for $k > 0$, so that the integrand in the Chern-Simons class of the family of connections ∇_u is given by the formula

$$\mathrm{Tr}\big(\omega\, e^{\nabla_u^2}\big) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} u^k (1-u)^k \,\mathrm{Tr}(\omega^{2k+1}).$$

The explicit formula for $\mathrm{Ch}(g)$ follows from the definition of the beta-function: if $s, t > 0$, then

$$B(s,t) = \frac{\Gamma(s)\,\Gamma(t)}{\Gamma(s+t)} = \int_0^1 u^{s-1}(1-u)^{t-1}\,du.$$

In this way, we obtain the following result.

Proposition 1.2. *The odd Chern character is a closed differential form of odd degree, given by the formula*

$$\mathrm{Ch}(g) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \,\mathrm{Tr}(\omega^{2k+1}).$$

It follows from this formula that $\mathrm{Ch}(g) + \mathrm{Ch}(g^{-1}) = 0$.

Let g_t, $t \in [0,1]$, be a differentiable family of maps from M to $\mathrm{GL}_N(\mathbb{C})$. This may be thought of as a differentiable map $\tilde{g}$ from $[0,1] \times M$ to $\mathrm{GL}_N(\mathbb{C})$, and as such, it has a Chern character $\mathrm{Ch}(\tilde{g})$. This may be decomposed

$$\mathrm{Ch}(\tilde{g}) = \mathrm{Ch}(g_t) + dt \wedge \widetilde{\mathrm{Ch}}(g_t),$$

where $\mathrm{Ch}(g_t)$ and $\widetilde{\mathrm{Ch}}(g_t)$ are independent of dt.

Proposition 1.3. *The secondary Chern character* $\widetilde{\mathrm{Ch}}(g_t)$ *is given by the formula*

$$\widetilde{\mathrm{Ch}}(g_t) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k)!} \,\mathrm{Tr}\big(g_t^{-1}\dot{g}_t \wedge \omega_t^{2k}\big),$$

and satisfies the transgression formula

$$\frac{\partial}{\partial t}\,\mathrm{Ch}(g_t) = d\,\widetilde{\mathrm{Ch}}(g_t).$$

Proof. Observe that

$$\tilde{\omega} = \omega_t + g_t^{-1}\dot{g}_t\,dt.$$

From this, we see that

$$\tilde{\omega}^{2k+1} = \omega_t^{2k+1} + \sum_{i=0}^{2k+1} \omega_t^i \wedge g_t^{-1}\dot{g}_t\,dt \wedge \omega_t^{2k-i}.$$

Taking the trace of both sides gives

$$\operatorname{Tr}(\tilde{\omega}^{2k+1}) = \operatorname{Tr}(\omega_t^{2k+1}) + (2k+1)dt \wedge \operatorname{Tr}(g_t^{-1}\dot{g}_t \wedge \omega_t^{2k}),$$

from which the formula for $\widetilde{\operatorname{Ch}}(g_t)$ follows.

The transgression formula is a consequence of the fact that $\operatorname{Ch}(\tilde{g})$ is closed:

$$\left(dt \wedge \frac{\partial}{\partial t} + d\right)\operatorname{Ch}(\tilde{g}) = d\operatorname{Ch}(g_t) + dt \wedge \left(\frac{\partial}{\partial t}\operatorname{Ch}(g_t) - d\,\widetilde{\operatorname{Ch}}(g_t)\right). \quad \square$$

This proposition shows that the cohomology class of $\operatorname{Ch}(g)$ depends only on the homotopy class of the map $g : M \longrightarrow \operatorname{GL}_N(\mathbb{C})$. Since $\operatorname{Ch}(g \oplus h) = \operatorname{Ch}(g) + \operatorname{Ch}(h)$, it follows that the odd Chern character defines a homomorphism

$$\operatorname{Ch} : K^{-1}(M) \longrightarrow \bigoplus_{i=0}^{\infty} H^{2i+1}(M, \mathbb{C}).$$

The Atiyah-Hirzebruch spectral sequence shows that this is an isomorphism after tensoring with $\mathbb{C}$.

In the special case where M is the circle S^1, the Chern character $\operatorname{Ch}(g)$ is given by the formula

$$\operatorname{Ch}(g) = \operatorname{Tr}(g^{-1}(dg)) = d\log\det(g),$$

so that the Chern character is related to the degree by the formula

$$\frac{1}{2\pi i}\int_{S^1} \operatorname{Ch}(g) = \deg(\det(g) : S^1 \longrightarrow \operatorname{GL}_1(\mathbb{C})).$$

This formula has a generalization for any odd-dimensional sphere.

Proposition 1.4. *If $g : S^{2k-1} \longrightarrow \operatorname{GL}_N(\mathbb{C})$ is a differentiable map, then there is an integer such that*

$$\frac{1}{(-2\pi i)^k}\int_{S^{2k-1}} \operatorname{Ch}(g) = -\deg(g).$$

Proof. Let $C(2k)$ be the Clifford algebra of $\mathbb{R}^{2k}$, and let $S^{\pm}$ be the associated half spinor representations, of rank 2^{k-1}. Let x_i be the basis of $\mathbb{R}^{2k}$, and denote $c(x_i)$ by c_i. By Bott periodicity, the map from S^{2k-1} to $\mathrm{GL}_{2^{k-1}}(\mathbb{C})$ given by the formula

$$g(x) = c_{2k}c(x) \in \mathrm{GL}_{2^{k-1}}(\mathbb{C}) \subset \mathrm{End}(S^+)$$

has degree 1, and generates $\pi_{2k-1}(\mathrm{GL}_N(\mathbb{C}))$ for N large. Thus, it suffices to prove the theorem for this map g.

The formula for ω^{2k-1} over all of S^{2k-1} follows from its value at $y = (0, \dots, 0, 1)$, since ω is equivariant under the action of $\mathrm{SO}(2k)$. But

$$\omega(y) = -\sum_{i=1}^{2k} c_{2k}c_i \, dx^i,$$

and it follows that, modulo terms proportional dx^{2k},

$$\begin{aligned} \omega(y)^{2k-1} &= -\sum_{\pi\in S_{2k-1}} c_{2k}c_{\pi(1)}dx^{\pi(1)} \dots c_{2k}c_{\pi(2k-1)}dx^{\pi(2k-1)} \\ &= \sum_{\pi\in S_{2k-1}} dx^{\pi(1)} \wedge \dots \wedge dx^{\pi(2k-1)} \, c_{\pi(1)} \dots c_{\pi(2k-1)}c_{2k} \\ &= (-1)^k(2k-1)! \, dx^1 \wedge \dots \wedge dx^{2k-1} \, c_1 \dots c_{2k}. \end{aligned}$$

Since $i^k c_1 \dots c_{2k}$ acts by ± 1 on $S^{\pm}$, we see that $\mathrm{Tr}_{S^+}(c_1 \dots c_{2k}) = (-i)^k 2^{k-1}$, so that

$$\begin{aligned} \mathrm{Ch}(g)_{[2k-1]} &= (-1)^{k-1}\frac{(k-1)!}{(2k-1)!}\,\mathrm{Tr}(\omega^{2k-1}) \\ &= -\frac{1}{2}(-2i)^k(k-1)!\,\mathrm{vol}, \end{aligned}$$

where vol is the Riemannian volume form on S^{2k-1}. But the volume of the sphere S^{2k-1} equals $2\pi^k/(k-1)!$, and we see that

$$\frac{1}{(-2\pi i)^k}\int_{S^{2k-1}} \mathrm{Ch}(g) = -1,$$

as was to be shown. $\square$

There is another formula for the odd Chern character $\mathrm{Ch}(g)$, obtained by associating to g a superconnection on the trivial $\mathbb{Z}/2$-graded vector bundle $\mathcal{E}$ over $[0, \infty) \times M$ with fibre $\mathbb{C}^{N|N} = \mathbb{C}^N \oplus \mathbb{C}^N$. Consider the odd endomorphism of $\mathcal{E}$,

$$p = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix},$$

satisfying $p^2 = -1$. Denoting the coordinate of $[0,\infty)$ by s, we introduce the superconnection

$$\mathbb{A} = d + sp$$

on $\mathcal{E}$, with curvature $\mathbb{A}^2 = s\,dp - s^2 + ds\,p$. The Chern form $\mathrm{Ch}(\mathcal{A})$ of the superconnection $\mathcal{A}$ is the even differential form $\mathrm{Str}(e^{\mathbb{A}^2})$ on $[0,\infty)\times M$. If α is a differential form on $[0,\infty)\times M$, denote by $\int_0^\infty \alpha$ the integral of α along the fibres of the projection of $[0,\infty)\times M$ to M.

Proposition 1.5.

$$\int_0^\infty \mathrm{Ch}(\mathcal{A}) = \mathrm{Ch}(g)$$

Proof. Since $\mathrm{Str}\big((dp)^{2k}\big) = 0$, we see that

$$\mathrm{Str}(e^{\mathbb{A}^2}) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \mathrm{Str}\big(p(dp)^{2k+1}\big)\, s^{2k+1} e^{-s^2}\, ds,$$

and hence

$$\int_0^\infty \mathrm{Str}(e^{\mathbb{A}^2}) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \mathrm{Str}\big(p(dp)^{2k+1}\big) \int_0^\infty s^{2k+1} e^{-s^2}\, ds.$$

The change of variables $t = s^2$ gives

$$\int_0^\infty s^{2k+1} e^{-s^2}\, ds = \frac{1}{2}\int_0^\infty t^k e^{-t}\, dt = \frac{k!}{2}.$$

Denote $g^{-1}(dg)$ by ω, and $(dg)g^{-1} = g\omega g^{-1}$ by $\tilde{\omega}$. Taking care to observe the sign rule for graded tensor products, we see that

$$\begin{aligned}
(dp)^2 &= \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\tilde{\omega}^2 \end{pmatrix}, \text{ and} \\
p(dp) &= \begin{pmatrix} \omega & 0 \\ 0 & -\tilde{\omega} \end{pmatrix},
\end{aligned}$$

from which we see that

$$\begin{aligned}
\mathrm{Str}\big(p(dp)^{2k+1}\big) &= (-1)^k\, \mathrm{Str}\begin{pmatrix} \omega^{2k+1} & 0 \\ 0 & -\tilde{\omega}^{2k+1} \end{pmatrix} \\
&= 2(-1)^k\, \mathrm{Str}(\omega^{2k+1}).
\end{aligned}$$

The result follows. $\square$

The coincidence between the two formulas for the odd Chern character may be partially explained by extending the superconnection $\mathbb{A}$ to the trivial $\mathbb{Z}/2$-graded bundle with fibre $\mathbb{C}^{N|N}$ over the product $[0,1]\times[0,\infty)\times M$. This superconnection, which we again denote $\mathbb{A}$, is given by the formula

$$\mathbb{A} = d + up\,dp + sp = \begin{pmatrix} d+u\omega & -sg^{-1} \\ sg & d-u\tilde{\omega} \end{pmatrix},$$

and has curvature

$$\mathbb{A}^2 = u(1-u)(dp)^2 + du\wedge p\,dp + s(1-2u)\,dp + ds\,p - s^2.$$

If $s>0$, denote by $\int_{\gamma_s}\mathrm{Str}(e^{\mathbb{A}^2})\in\Omega^*(M)$ the integral of the differential form $\mathrm{Str}(e^{\mathbb{A}^2})$, over the fibres of the projection $[0,1]\times\{s\}\times M\longrightarrow M$, oriented in the the direction of increasing u. We see from the explicit formula for $\mathbb{A}^2$ that

$$\int_{\gamma_s}\mathrm{Str}(e^{\mathbb{A}^2}) = O(e^{-s^2/2})$$

as $s\longrightarrow\infty$, while

$$\int_{\gamma_0}\mathrm{Str}(e^{\mathbb{A}^2}) = \mathrm{Ch}(g) - \mathrm{Ch}(g^{-1}) = 2\,\mathrm{Ch}(g).$$

Similarly, if $u\in[0,1]$, denote by $\int_{\Gamma_u}\mathrm{Str}(e^{\mathbb{A}^2})\in\Omega^*(M)$ the integral of $\mathrm{Str}(e^{\mathbb{A}^2})$, over the fibres of the projection $\{u\}\times[0,\infty)\times M\longrightarrow M$, oriented in the the direction of increasing s. Examination of the formula for $\mathbb{A}^2$ shows that

$$\int_{\Gamma_0}\mathrm{Str}(e^{\mathbb{A}^2}) = -\int_{\Gamma_1}\mathrm{Str}(e^{\mathbb{A}^2}) = \int_0^\infty \mathrm{Ch}(\mathbb{A}).$$

Finally, denote by $\int_{[0,1]\times[0,\infty)}\mathrm{Str}(e^{\mathbb{A}^2})$ the integral of $\mathrm{Str}(e^{\mathbb{A}^2})$ over the fibres of the projection $[0,1]\times[0,\infty)\times M\longrightarrow M$. Then Stokes's theorem shows that

$$\int_{\Gamma_0}\mathrm{Str}(e^{\mathbb{A}^2}) - \int_{\Gamma_1}\mathrm{Str}(e^{\mathbb{A}^2}) \\ = \int_{\gamma_0}\mathrm{Str}(e^{\mathbb{A}^2}) - \lim_{s\longrightarrow\infty}\int_{\gamma_s}\mathrm{Str}(e^{\mathbb{A}^2}) + d\int_{[0,1]\times[0,\infty)}\mathrm{Str}(e^{\mathbb{A}^2}).$$

Proposition 1.5 shows that the exact differential form $d\int_{[0,1]\times[0,\infty)}\mathrm{Str}(e^{\mathbb{A}^2})$ is in fact equal to zero.

The goal of this paper is to "quantize" the calculations of this section: we replace the manifold by a Banach algebra $\mathcal{A}$, the differential forms by entire cyclic chains $\mathsf{C}^\omega(\mathcal{A})$, and the exterior differential by a theta-summable Fredholm module D over $\mathcal{A}$. The analogue of the Chern-Simons class $\mathrm{cs}(d, g^{-1}dg)$ will be the spectral flow $\mathrm{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$ between the operators D and $g^{-1}\mathsf{D}g = \mathsf{D} + g^{-1}[\mathsf{D}, g]$, while the analogue of Proposition 1.5 will give the desired formula for this integer as a pairing in entire cyclic homology.

2. Spectral flow

Let D_0 be a theta-summable Fredholm operator on a Hilbert space $\mathcal{H}$, that is, a self-adjoint operator such that $e^{-t\mathsf{D}^2}$ is trace class for all $t > 0$. In this section, we will study the affine Banach manifold Φ modelled on $\mathcal{L}_*(\mathcal{H})$, the real Banach space of self-adjoint bounded linear operators on $\mathcal{H}$,

$$\Phi = \{\mathsf{D} = \mathsf{D}_0 + A \mid A \in \mathcal{L}_*(\mathcal{H})\},$$

which is a space of "connections." Let $\mathcal{G}$ be the "gauge group"

$$\mathcal{G} = \{g \in \mathrm{U}(\mathcal{H}) \mid \|[\mathsf{D}_0, g]\| < \infty\}.$$

This group acts on Φ by conjugation:

$$g \cdot (\mathsf{D}_0 + A) = g^{-1}(\mathsf{D}_0 + A)g = \mathsf{D}_0 + g^{-1}[\mathsf{D}_0, g] + g^{-1}Ag.$$

There is an evident stratification of Φ by $\mathcal{G}$-invariant sets

$$\Phi_k = \{\mathsf{D} \mid \dim\ker(\mathsf{D}) = k\}.$$

Denote by $\mathrm{Gr}_k(\mathcal{H})$ the Banach manifold of rank k subspaces of the Hilbert space $\mathcal{H}$, and by $\mathrm{Gr}^k(\mathcal{H})$ the Banach manifold of corank k subspaces of $\mathcal{H}$.

Theorem 2.1. *The stratification $\{\Phi_k\}$ of Φ has the following properties.*

(1) *Each Φ_k is a submanifold of Φ of finite codimension $k(k+1)/2$, and the closure of Φ_k equals $\Phi_k \cup \Phi_{k+1} \cup \dots$.*
(2) *(local finiteness) Each point in Φ has a neighbourhood meeting finitely many strata Φ_k.*
(3) *(Whitney's Condition B) Let $k < \ell$. If $(x_i) \subset \Phi_k$ and $(y_i) \subset \Phi_\ell$ are sequences converging to $y \in \Phi_\ell$, such that the tangent lines $\overline{x_i y_i}$ converge to $L \in \mathrm{Gr}_1(\mathcal{H})$, and the tangent spaces $T_{x_i}\Phi_k$ converge to $T \in \mathrm{Gr}^k(\mathcal{H})$, then $L \subset T$.*

Furthermore, the submanifolds Φ_k are cooriented, that is, the normal bundle of the embedding $\Phi_k \hookrightarrow \Phi$ is an orientable vector bundle.

Proof. We follow closely the proof of Lemme 1 on page 295 of Ruget [13]. If $\mathsf{D}_1 \in \Phi_k$, let $\mathcal{H}_0 \in \mathrm{Gr}_k(\mathcal{H})$ be the kernel of D_1, and let $\mathcal{H}_1$ be its orthogonal complement. For $\mathsf{D} \in \Phi$, we write, with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$,

$$\mathsf{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a \in \mathcal{L}_*(\mathcal{H}_0)$, $b \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$, $c = b^*$, and $d \in \mathcal{L}_*(\mathcal{H}_1)$. Let $U \subset \Phi$ be the neighbourhood of D_1 on which d is invertible. Consider the map

$$F(\mathsf{D}) = a - bd^{-1}c : U \longrightarrow \mathcal{L}_*(\mathcal{H}_0),$$

which is clearly a submersion.

The stratification of $\mathcal{L}_*(\mathcal{H}_0)$ by subsets $\{V_i \mid 0 \leq i \leq k\}$, where

$$V_i = \{A \in \mathcal{L}_*(\mathcal{H}_0) \mid \dim \ker(A) = i\},$$

is a Whitney stratification (see Section 1.2 of Gibson et al. [8]). Since $\Phi_i \cap U = F^{-1}(V_i)$, we see that the stratification $\{\Phi_i \cap U \mid 0 \leq i \leq k\}$ is a Whitney stratification of U. Since sets of the form U cover Φ, properties (2-3) of the stratification follow.

Since $V_k = \{0\}$, we see that $\Phi_k \cap U = F^{-1}(0)$; thus $\Phi_k \cap U$ is a submanifold of Φ of codimension equal to the dimension of $\mathcal{L}_*(\mathcal{H}_0)$, namely $k(k+1)/2$. This also shows that Φ_k is a cooriented submanifold of Φ, since the normal bundle of the embedding $\{0\} \hookrightarrow \mathcal{L}_*(\mathcal{H}_0)$ is canonically oriented. $\square$

Corollary 2.2. *There is a fundamental class* $[\Phi_1] \in H^1(\Phi, \Phi_0)$.

Proof. By the above theorem, the stratum Φ_1 is a cooriented codimension 1 submanifold of Φ. The boundary of Φ_1 is the union of strata

$$\partial\Phi_1 = \overline{\Phi}_1 \backslash \Phi_1 = \Phi_2 \cup \Phi_3 \cup \dots,$$

and thus has codimension 3 in Φ. Since $\operatorname{codim}(\partial\Phi_1) \geq \operatorname{codim}(\Phi_1)$, Proposition 2.3 of Ruget [13] is applicable, and shows the existence of the fundamental class of Φ_1 in Φ. $\square$

Let $I = [0, 1]$ be the unit interval, with boundary $\partial I = \{0, 1\}$. It follows from Corollary 2.2 that if $\gamma : (I, \partial I) \longrightarrow (\Phi, \Phi_0)$ is a path in Φ whose endpoints lie in Φ_0, there is a well-defined intersection number $\#(\gamma, \Phi_1) \in \mathbb{Z}$ between γ and Φ_1, defined by integrating the pull-back $\gamma^*[\Phi_1] \in H^1(I, \partial I)$ over the fundamental class $[I] \in H_1(I, \partial I)$ of $(I, \partial I)$.

Definition 2.3. *(Atiyah-Patodi-Singer* [1]*) If* D_0 *and* D_1 *are two elements of* Φ_0*, and* γ *is the path* $\gamma_t = (1-t)\mathsf{D}_0 + t\mathsf{D}_1$ *joining them, then the* **spectral flow** $\operatorname{sf}(\mathsf{D}_0, \mathsf{D}_1) \in \mathbb{Z}$ *is the intersection number* $\#(\gamma, \Phi_1)$ *of* γ *with the submanifold* Φ_1.

Our goal in this section is to prove an analytic formula for $\operatorname{sf}(\mathsf{D}_0, \mathsf{D}_1)$. Our approach is inspired by that of Bismut and Freed [3], who work in the setting of Dirac operators. We start by defining an analogue of the eta-invariant of Atiyah-Patodi-Singer. Recall that if D is a Dirac operator on a compact oriented odd dimensional manifold, the eta-invariant is the number

$$\eta = \frac{1}{\pi^{1/2}} \int_0^\infty \operatorname{Tr}(\mathsf{D}e^{-t\mathsf{D}^2})\, t^{-1/2}\, dt.$$

However, the existence of this eta-invariant relies on special properties of differential operators, so we need to regularize this formula, by truncating the integral near $t = 0$: if $\varepsilon > 0$, let

$$\eta_\varepsilon = \frac{1}{\pi^{1/2}} \int_\varepsilon^\infty \operatorname{Tr}(\mathsf{D}e^{-t\mathsf{D}^2})\, t^{-1/2}\, dt.$$

We also introduce a one-form α_ε on Φ, defined by the formula

$$\alpha_\varepsilon(X) = -(\varepsilon/\pi)^{1/2} \operatorname{Tr}\bigl(X e^{-\varepsilon \mathsf{D}^2}\bigr), \text{ where } X \in T_{\mathsf{D}}\Phi \cong \mathcal{L}_*(\mathcal{H}).$$

Both ε_ε and α_ε are invariant under the action of $\mathcal{G}$.

The following lemma will be used in estimating the derivatives of η_ε.

Lemma 2.4. *If $\mathsf{D} = \mathsf{D}_0 + A \in \Phi$, where D is a theta-summable operator and $A \in \mathcal{L}_*(\mathcal{H})$, then for $t > 0$ and $s \geq 0$,*

$$\operatorname{Tr}(|\mathsf{D}|^s e^{-t\mathsf{D}^2}) \leq \begin{cases} e^{t\|A\|^2} \operatorname{Tr}(e^{-t\mathsf{D}_0^2/4}), & s = 0, \\ \left(\dfrac{s}{et}\right)^{s/2} e^{t\|A\|^2} \operatorname{Tr}(e^{-t\mathsf{D}_0^2/4}), & s > 0. \end{cases}$$

Proof. Since $\mathsf{D}^2 = (\mathsf{D}_0 + A)^2 = (1-\varepsilon^2)\mathsf{D}_0^2 + (\varepsilon\mathsf{D}_0 + \varepsilon^{-1}A)^2 + (1-\varepsilon^{-2})A^2$, we see that

$$\mathsf{D}^2 \geq (1-\varepsilon^2)\mathsf{D}_0^2 - \varepsilon^{-2}\|A\|^2.$$

If $A \geq B$ are self-adjoint, then $e^A \geq e^B$. As in the proof of Theorem C of Getzler-Szenes [7], we see that

$$\operatorname{Tr}(e^{-t\mathsf{D}^2}) \leq e^{\varepsilon^{-2}t\|A\|^2} \operatorname{Tr}(e^{-(1-\varepsilon^2)t\mathsf{D}_0^2}).$$

Set $\varepsilon = 2^{-1/2}$; this proves the lemma for $s = 0$.

If $s > 0$, apply the Hölder inequality

$$\operatorname{Tr}\bigl(|\mathsf{D}|^s e^{-t\mathsf{D}^2}\bigr) \leq \bigl\||\mathsf{D}|^s e^{-t\mathsf{D}^2/2}\bigr\| \operatorname{Tr}(e^{-t\mathsf{D}^2/2}).$$

Since $|\lambda|^s e^{-t\lambda^2/2} \leq (s/et)^{-s/2}$, the result follows. $\square$

Proposition 2.5.

(1) *α_ε has a locally bounded first derivative, and $d\alpha_\varepsilon = 0$.*
(2) *The regularized eta-invariant η_ε has a locally bounded second derivative on Φ_0, and $d\eta_\varepsilon = 2\alpha_\varepsilon$.*

Proof. In the proof that α_ε has locally bounded first derivative, we set $\varepsilon = 1$: the general case follows by replacing D_0 by $\varepsilon^{1/2}\mathsf{D}_0$.

The derivative of α is given by the formula

$$\begin{aligned} \nabla_Y \alpha(X) &= \frac{1}{\pi^{1/2}} \int_0^1 \operatorname{Tr}\bigl(X e^{-\sigma\mathsf{D}^2}(\mathsf{D}Y + Y\mathsf{D})e^{-(1-\sigma)\mathsf{D}^2}\bigr)\, d\sigma \\ &= \frac{1}{\pi^{1/2}}\bigl(a(\mathsf{D}, X, Y) + a(\mathsf{D}, Y, X)\bigr), \end{aligned}$$

where

$$a(\mathsf{D}, X, Y) = \int_0^1 \mathrm{Tr}\big(X\mathsf{D}e^{-\sigma\mathsf{D}^2} Y e^{-(1-\sigma)\mathsf{D}^2}\big)\, d\sigma.$$

We apply the Hölder inequality for the trace: if $\sigma_0 + \cdots + \sigma_k = 1$,

$$\mathrm{Tr}(A_0 \dots A_k) \leq \mathrm{Tr}\big(|A|^{\sigma_0^{-1}}\big)^{\sigma_0} \dots \mathrm{Tr}\big(|A_k|^{\sigma_k^{-1}}\big)^{\sigma_k}.$$

It follows that, with $\mathsf{D} = \mathsf{D}_0 + A$,

$$\begin{aligned} |a(\mathsf{D}, X, Y)| &\leq \|X\|\,\|Y\| \int_0^1 \mathrm{Tr}\big(|\mathsf{D}|^{\sigma^{-1}} e^{-\mathsf{D}^2}\big)^{\sigma}\, \mathrm{Tr}\big(e^{-\mathsf{D}^2}\big)^{1-\sigma}\, d\sigma \\ &\leq \|X\|\,\|Y\|\, e^{\|A\|^2}\ \mathrm{Tr}(e^{-\mathsf{D}_0^2/4}) \int_0^1 (e\sigma)^{-1/2}\, d\sigma, \end{aligned}$$

where on the last line we have applied Lemma 2.4. This shows that the derivative of α_ε is locally bounded.

We now show that $d\eta_\varepsilon = \alpha_\varepsilon$ on Φ_ε. If X is a tangent vector to Φ,

$$\begin{aligned} d\eta_\varepsilon(X) &= \frac{1}{\pi^{1/2}} \int_\varepsilon^\infty \mathrm{Tr}(Xe^{-t\mathsf{D}^2})\, t^{-1/2}\, dt \\ &\quad - \frac{1}{\pi^{1/2}} \int_\varepsilon^\infty \int_0^1 \mathrm{Tr}(\mathsf{D}e^{-st\mathsf{D}^2} t(\mathsf{D}X + X\mathsf{D})e^{-(1-s)t\mathsf{D}^2})\, t^{-1/2}\, ds\, dt \\ &= \frac{1}{\pi^{1/2}} \int_\varepsilon^\infty \mathrm{Tr}(Xe^{-t\mathsf{D}^2})\, t^{-1/2}\, dt - \frac{2}{\pi^{1/2}} \int_\varepsilon^\infty \mathrm{Tr}(X\mathsf{D}^2 e^{-t\mathsf{D}^2})\, t^{1/2}\, dt. \end{aligned}$$

A little manipulation shows that

$$\begin{aligned} \int_\varepsilon^\infty \mathrm{Tr}(X\mathsf{D}^2 e^{-t\mathsf{D}^2})\, t^{1/2}\, dt &= -\int_\varepsilon^\infty \frac{d}{dt} \mathrm{Tr}(Xe^{-t\mathsf{D}^2})\, t^{1/2}\, dt \\ &= \int_\varepsilon^\infty \mathrm{Tr}(Xe^{-t\mathsf{D}^2})\, \frac{d(t^{1/2})}{dt}\, dt - \mathrm{Tr}(Xe^{-t\mathsf{D}^2})\, t^{1/2}\Big|_{t=\varepsilon}^{t=\infty} \\ &= \frac{1}{2} \int_\varepsilon^\infty \mathrm{Tr}(Xe^{-t\mathsf{D}^2})\, t^{-1/2}\, dt + \varepsilon^{1/2}\, \mathrm{Tr}(Xe^{-\varepsilon\mathsf{D}^2}). \end{aligned}$$

In the last step, we have used the fact that D is invertible, and hence that $\mathrm{Tr}(Xe^{-t\mathsf{D}^2})$ decays exponentially as $t \to \infty$: if λ_0 is the lowest eigenvalue of D,

$$|\,\mathrm{Tr}(Xe^{-t\mathsf{D}^2})| \leq \|X\|\, \|e^{-(t-1)\mathsf{D}^2}\|\ \mathrm{Tr}(e^{-\mathsf{D}^2}) = O(e^{-t\lambda_0^2}).$$

The formula $d\eta_\varepsilon = 2\alpha_\varepsilon$ on Φ_0 follows. In particular, since $d\eta_\varepsilon$ has a locally bounded first derivative, it follows that η_ε has a locally bounded second derivative.

It only remains to show that $d\alpha_\varepsilon = 0$. This can be shown explicitly, but it also suffices to observe that α_ε is exact on Φ_0, which is the complement of a set of codimension one, and it has a locally bounded first derivative. □

From this proposition follows an analytic formula for the spectral flow.

Theorem 2.6. *Let $\gamma(u) = \mathsf{D}_u$ be a differentiable family of operators in Φ, such that D_0 and D_1 are invertible. Then*

$$\begin{aligned}\mathrm{sf}(\mathsf{D}_0, \mathsf{D}_1) &= -\int_\gamma \alpha_\varepsilon + \frac{1}{2}\eta_\varepsilon(\mathsf{D}_1) - \frac{1}{2}\eta(\mathsf{D}_0) \\ &= \Big(\frac{\varepsilon}{\pi}\Big)^{1/2} \int_0^1 \mathrm{Tr}\big(\dot{\mathsf{D}}_u e^{-\varepsilon \mathsf{D}_u}\big)\, du + \frac{1}{2}\eta_\varepsilon(\mathsf{D}_1) - \frac{1}{2}\eta(\mathsf{D}_0).\end{aligned}$$

Proof. Since the differential form α_ε is closed, it follows that the integral $\int_\gamma \alpha_\varepsilon$ is invariant under isotopies of the differentiable path γ fixing the endpoints. By Theorem 2.2, there is an isotopy from γ to a path transversal to Φ_1: thus, we may as well suppose that it is γ which is transversal to Φ_1.

Let $0 < u_1 < \cdots < u_k < 1$ be the sequence of times at which $\gamma(u)$ intersects Φ_1. By Proposition 2.5, $\gamma^*\eta_\varepsilon$ is a differentiable function of $u \in [0,1]$, except at the points u_i. Since the function

$$\eta_\varepsilon(\lambda) = \frac{1}{\pi^{1/2}} \int_\varepsilon^\infty \lambda e^{-u\lambda^2} u^{-1/2}\, du$$

satisfies $\eta_\varepsilon(0\pm) = \pm 1$, we see that

$$\begin{aligned}\mathrm{sf}(\mathsf{D}_0, \mathsf{D}_1) &= \frac{1}{2}\sum_{i=1}^k \Big(\lim_{u\longrightarrow u_i+} \eta_\varepsilon(\mathsf{D}_u) - \lim_{u\longrightarrow u_i-} \eta_\varepsilon(\mathsf{D}_u)\Big) \\ &= \frac{1}{2}\Big(-\int_\gamma d\eta_\varepsilon + \eta_\varepsilon(\mathsf{D}_1) - \eta_\varepsilon(\mathsf{D}_0)\Big),\end{aligned}$$

where in the second line we have applied the fundamental theorem of calculus. The theorem follows from the formula $d\eta_\varepsilon = 2\alpha_\varepsilon$. □

Corollary 2.7. *If D is a theta-summable Fredholm operator and $g \in \mathcal{G}$, then for $\varepsilon > 0$,*

$$\mathrm{sf}(\mathsf{D}, g^{-1}\mathsf{D}g) = \Big(\frac{\varepsilon}{\pi}\Big)^{1/2} \int_0^1 \mathrm{Tr}\big(\dot{\mathsf{D}}_u e^{-\varepsilon \mathsf{D}_u^2}\big)\, du,$$

where $\mathsf{D}_u = (1-u)\mathsf{D} + ug^{-1}\mathsf{D}g$.

Proof. Since η_ε is invariant under the action of $\mathcal{G}$, the terms $\eta_\varepsilon(\mathsf{D})$ and $\eta_\varepsilon(g^{-1}\mathsf{D}g)$ cancel in the formula of Theorem 2.6. □

As an application of this result, we outline a calculation of the spectral flow $\mathrm{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$ in the special case where D is the Dirac operator on a compact odd dimensional spin-manifold of dimension $2k+1$, and g is a differentiable map from M to $\mathrm{U}(N)$. The Dirac operator is i^{-1} times the composition

$$\Gamma(M, \mathcal{S}) \xrightarrow{\nabla^{\mathcal{S}}} \Gamma(M, T^*M \otimes \mathcal{S}) \xrightarrow{c} \Gamma(M, \mathcal{S})$$

and is a self-adjoint operator. (Here, $\nabla^{\mathcal{S}}$ is the natural connection on the spinor bundle $\mathcal{S}$ over M induced by the Levi-Civita connection on the tangent bundle TM.) Note that our conventions differ from those of Berline-Getzler-Vergne [2], since we use the Clifford algebra such that $c(v)^2 = |v|^2$.

If S is the spinor module for the Clifford algebra $C(2k+1)$ and Tr denotes the trace over S, and if $1 \le i_1 < \dots < i_m \le 2k+1$, then

$$\operatorname{Tr}(c_{i_1} \dots c_{i_m}) = \begin{cases} (2i)^k, & m = 2k+1, \\ 0, & 0 < m < 2k+1, \\ 2^k, & m = 0. \end{cases}$$

Since the operator $\dot{\mathsf{D}}_u e^{-\varepsilon \mathsf{D}_u^2}$ involves an odd number of Clifford variables, we see that only the coefficient of $c_1 \dots c_{2k+1}$ with respect to any local orthonormal frame of the tangent bundle contributes to the trace $\operatorname{Tr}(\dot{\mathsf{D}}_u e^{-\varepsilon \mathsf{D}_u^2})$. Thus, we may apply the same asymptotic analysis as is used to prove the Atiyah-Singer index theorem in Chapter 4 of [2]. This shows that we may make the following replacements:

$$\begin{array}{ll} \varepsilon^{1/2}\dot{\mathsf{D}}_u = i^{-1}\varepsilon^{1/2}c(\omega) & \text{exterior multiplication by } i^{-1}\omega \\ e^{-\varepsilon \mathsf{D}_u^2} & \hat{A}(M) \wedge e^{(d+u\omega)^2} \\ \operatorname{Tr}(\dots) & \dfrac{(2i)^k}{(4\pi)^{k+1/2}} \displaystyle\int_M \operatorname{Tr}(\dots) = \dfrac{-i\pi^{1/2}}{(-2\pi i)^{k+1}} \displaystyle\int_M \operatorname{Tr}(\dots) \end{array}$$

where

$$\hat{A}(M) = \det{}^{1/2}\Big(\frac{R/2}{\sinh R/2}\Big),$$

and $R \in \Omega^2(M, \operatorname{End}(T^*M))$ is the Riemannian curvature. This allows us to replace

$$\Big(\frac{\varepsilon}{\pi}\Big)^{1/2} \int_0^1 \operatorname{Tr}\big(\dot{\mathsf{D}}_u e^{-\varepsilon \mathsf{D}_u^2}\big)\, du$$

by

$$-\frac{1}{(-2\pi i)^{k+1}} \int_0^1 \int_M \hat{A}(M) \wedge \operatorname{Tr}(\omega e^{(d+u\omega)^2})\, du = -\frac{1}{(-2\pi i)^{k+1}} \int_M \hat{A}(M) \wedge \operatorname{Ch}(g).$$

Here, the integral over u is calculated by the same method as in the proof of Proposition 1.2. Thus, we obtain the following result.

Theorem 2.8. *Let M be a compact spin-manifold of dimension $2k+1$, with Dirac operator D. If g is a differentiable map from M to $\mathrm{U}(N)$, then*

$$\operatorname{sf}(\mathsf{D}, g^{-1}\mathsf{D}g) = -\frac{1}{(-2\pi i)^{k+1}} \int_M \hat{A}(M) \wedge \operatorname{Ch}(g).$$

Note that if M is the sphere S^{2k+1}, then $\hat{A}(M) = 1$, and we see by Proposition 1.4 that

$$\operatorname{sf}(\mathsf{D}, g^{-1}\mathsf{D}g) = -\frac{1}{(-2\pi i)^{k+1}} \int_M \operatorname{Ch}(g) = \deg(g),$$

generalizing the well-known result for the circle.

3. The odd Chern character in cyclic homology

If A is an algebra over a ring R, the space of cyclic chains of degree n is defined to be

$$\mathsf{C}_n(A) = A \otimes (A/R)^{\otimes n}.$$

Denote the element $a_0 \otimes \ldots \otimes a_n$ of $\mathsf{C}_n(A)$ by $(a_0, \ldots, a_n)$. Sometimes, we will write this as $(a_0, \ldots, a_n)_n$ in order to make the degree n explicit. The operators $b : \mathsf{C}_n(A) \longrightarrow \mathsf{C}_{n-1}(A)$ and $B : \mathsf{C}_n(A) \longrightarrow \mathsf{C}_{n+1}(A)$ are given by the formulas

$$\begin{aligned} b(a_0, \ldots, a_k) &= \sum_{i=1}^{n} (-1)^i (a_0, \ldots, a_{i+1}a_i, \ldots, a_n), \\ &\quad + (-1)^{n+1} (a_n a_0, a_1, \ldots, a_{n-1}) \\ B(a_0, \ldots, a_n) &= \sum_{i=0}^{n} (-1)^{ni} (1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}). \end{aligned}$$

It is a standard calculation that the operators b and B are well defined, and that $b^2 = B^2 = bB + Bb = 0$.

From the spaces $\mathsf{C}_n(A)$, we may form the graded vector space $\mathsf{C}_*(\mathcal{A})[\![u]\!]$, of which the degree n summand consists of power series

$$c_n + uc_{n+2} + u^2 u_{n+4} + \ldots$$

where $c_i \in \mathsf{C}_i(A)$: thus, we treat u as a formal variable of degree -2. On $\mathsf{C}_*(A)[\![u]\!]$, we have the differential $b + uB$, of total degree -1. The homology of this operator is the negative cyclic homology $\mathrm{HC}^-_*(A)$ of Goodwillie [9] and Jones [12].

In order to define the Chern character of an invertible matrix, we start by considering the universal example, the element x in the group algebra $R[x, x^{-1}]$ of the infinite cyclic group. The universal Chern character is the degree 1 cyclic chain on $R[x, x^{-1}]$ given by the formula

$$\mathrm{Ch}_*(x) = \sum_{k=0}^{\infty} u^k k!\, (x^{-1}, x, \ldots, x^{-1}, x)_{2k+1}.$$

Proposition 3.1.

(1) $(b + uB)\,\mathrm{Ch}_*(x) = 0$

(2) $(b + uB) \sum_{k=0}^{\infty} u^k k!\, (1, x^{-1}, x, \ldots, x^{-1}, x)_{2k+2} + \mathrm{Ch}_*(x) + \mathrm{Ch}_*(x^{-1}) = 0$

Proof. It is easily seen that

$$b(x^{-1}, x, \ldots, x^{-1}, x)_{2k+1} = -(1, x^{-1}, x, \ldots, x^{-1}, x)_{2k} + (1, x, x^{-1}, \ldots, x, x^{-1})_{2k}.$$

On the other hand,

$$\begin{aligned}&B(x^{-1},x,\dots,x^{-1},x)_{2k+1}\\&\qquad=(k+1)(1,x^{-1},x,\dots,x^{-1},x)_{2k+2}-(k+1)(1,x,x^{-1},\dots,x,x^{-1})_{2k+2}.\end{aligned}$$

From this, the formula $(b+uB)\operatorname{Ch}_*(x)=0$ follows immediately.

A similar calculation shows that

$$b(1,x^{-1},x,\dots,x^{-1},x)_{2k+2}=-(x^{-1},x,\dots,x^{-1},x)_{2k+1}+(x,x^{-1},\dots,x,x^{-1})_{2k+1},$$

while $B(1,x^{-1},x,\dots,x^{-1},x)_{2k+2}=0$. From this, part (2) follows. □

Now, suppose that $g\in \mathrm{GL}_N(A)$ is an invertible matrix with entries in A. Then the map $x\mapsto g$ induces a map from $R[x,x^{-1}]$ to the algebra of square matrices $M_N(A)$ with entries in A, and hence a map of mixed complexes $\mathsf{C}_n(R[x,x^{-1}])\to \mathsf{C}_n(M_N(A))$. The trace map

$$\operatorname{Tr}(a_0,\dots,a_n)=\sum_{0\le i_0,\dots,i_n\le N}((a_0)_{i_0i_1},(a_1)_{i_1i_2},\dots,(a_n)_{i_ni_0})$$

induces a map of mixed complexes $\mathsf{C}_n(M_N(A))\to\mathsf{C}_n(A)$. The image of $\operatorname{Ch}_*(x)$ under the composition of these maps is the Chern character of the invertible element $g\in\mathrm{GL}_N(A)$, and is denoted by $\operatorname{Ch}_*(g)$.

It is instructive to analyse this Chern character in the special case that $A=C^\infty(M)$ is the algebra of differentiable functions on M. In this case, there is a map of complexes

$$\alpha:\big(\mathsf{C}_*(C^\infty(M))[\![u]\!],b+uB\big)\to\big(\Omega^*(M)[\![u]\!],d\big),$$

defined by the formula

$$\alpha(a_0,\dots,a_n)=\frac{1}{u^n\,n!}a_0\,da_1\dots da_n.$$

If g is a differentiable map of from M to the Lie group $\mathrm{GL}_N(\mathbb{C})$, it is easily checked that the image of $\operatorname{Ch}_*(g)$ in $\Omega^*(M)[\![u]\!]$ is the Chern character $\operatorname{Ch}(g)$ of Section 1.

The odd Chern character may also be studied in the setting of entire cyclic homology. We now suppose that $\mathcal{A}$ is a Banach algebra, and n-chains are now defined using the projective tensor product rather than the algebraic tensor product. The $\mathbb{Z}/2$-graded toplogical vector space $\mathsf{C}^\omega_\pm(\mathcal{A})$ of entire cyclic chains is the space of series of chains $c_0+c_1+c_2+\dots$ such that the norm

$$\|c_0+c_1+c_2+\dots\|_\lambda=\sup_n\frac{\lambda^n}{\Gamma(n/2)}\|c_n\|$$

is finite for some $\lambda>0$.The operators b and B, defined by the same formulas as before, are bounded on $\mathsf{C}^\omega_\pm(\mathcal{A})$, and the homology of the operator $b+B$ is the entire cyclic homology, denoted $H^\omega_*(\mathcal{A})$.

Proposition 3.2. *If $g \in \mathrm{GL}_N(\mathcal{A})$, then*

$$\mathrm{Ch}_*(g) = \sum_{k=0}^{\infty} k! \, \mathrm{Tr}(g^{-1}, g, \dots, g^{-1}, g)_{2k+1}$$

is a closed element of $\mathsf{C}^{\omega}_{-}(\mathcal{A})$.

Proof. Indeed, the norm of $\mathrm{Ch}_{2k+1}(g)$ is bounded by $k!\|g^{-1}\|^{k+1}\|g\|^{k+1}$, so that $\|\mathrm{Ch}_*(g)\|_\lambda$ is finite for $\lambda < (4\|g^{-1}\| \, \|g\|)^{-1}$. $\square$

The following result, analogous to Proposition 1.3, shows that the odd Chern character in cyclic homology is homotopy invariant. We omit the proof, which is straightforward.

Proposition 3.3. *Let $\mathcal{A}$ be a Banach algebra. If $g_t : [0,1] \to\in \mathrm{GL}_N(\mathcal{A})$ is a differentiable family of invertible matrices with entries in $\mathcal{A}$, denote by $\widetilde{\mathrm{Ch}}_*(g_t)$ the entire cyclic chain*

$$\widetilde{\mathrm{Ch}}_*(g_t) = \sum_{k=0}^{\infty} k! \sum_{i=0}^{k} \mathrm{Tr}(g^{-1}, g, \dots, g^{-1}, g, g^{-1}\dot{g}, g^{-1}, g, \dots, g^{-1}, g)_{2k+2}$$

Then we have the formula

$$\frac{d}{dt} \mathrm{Ch}_*(g_t) = (b+B)\widetilde{\mathrm{Ch}}_*(g_t).$$

Thus, the homology class of $\mathrm{Ch}_*(g)$ is homotopy invariant in the entire cyclic homology group $H^{\omega}_{-}(\mathcal{A})$.

Appendix. Another approach to the odd Chern character

There is an alternative approach to the universal Chern character $\mathrm{Ch}_*(x)$ which makes use of the product on the cyclic bar complex of a graded commutative algebra which was constructed in Getzler-Jones [6]. We will show that the universal Chern character is the analogue of the Maurer-Cartan form. This calculation is related to Fedosov's construction of the Chern character of an idempotent [4].

Recall that in the article [6], we constructed multilinear operators

$$B_k : \mathsf{C}_*(A)^{\otimes k} \longrightarrow \mathsf{C}_*(A),$$

such that B_1 is the just the usual operator B, and the higher B_k share with B the property that their image lies in the space of chains of the form $(1, a_1, \dots, a_n)$, and they vanish on chains of this form. As a simple example,

$$B_2((x),(y)) = (1, x, y).$$

If $\mathcal{A}$ is commutative, we define a product on $\mathsf{C}_*(\mathcal{A})[\![u]\!]$ by the formula

$$\alpha_1 \circ \alpha_2 = \alpha_1 * \alpha_2 - (-1)^{|\alpha_1|} u B_2(\alpha_1, \alpha_2),$$

where $\alpha_1 * \alpha_2$ is the shuffle product.

Lemma 3.4. *The right inverse of the chain* $(x) \in \mathsf{C}_*(\mathbb{C}[x, x^{-1}])[\![u]\!]$ *for the product* $\circ$ *is given by the formula*

$$(x)^{-1} = \sum_{k=0}^{\infty} u^k k!\,(x^{-1}, x, \dots, x, x^{-1})_{2k}.$$

Proof. Observe that $(x) \circ (x^{-1}) = (1) - u(1, x, x^{-1})$. It follows that the right inverse of (x) is given by the series

$$\sum_{k=0}^{\infty} u^k (x^{-1}) \circ (1, x, x^{-1})^k.$$

An easy induction shows that

$$(1, x, x^{-1})^k = k!\,(1, x, x^{-1}, \dots, x, x^{-1})_{2k},$$

and the formula follows. $\square$

Corollary 3.5. $u\,\mathrm{Ch}_*(x) = (b + uB)(x) \circ (x)^{-1}$

Proof. Note that $u^{-1}(b + uB)(x) = (1, x)$. From the lemma, we see that

$$\begin{aligned}(1, x) \circ (x)^{-1} &= \sum_{k=0}^{\infty} u^k k!\,(1, x) \circ (x^{-1}, x, \dots, x, x^{-1})_{2k} \\ &= \sum_{k=0}^{\infty} u^k k!\,(x^{-1}, x, \dots, x^{-1}, x)_{2k+1} \\ &= \mathrm{Ch}_*(x). \quad \square\end{aligned}$$

4. Theta-summable Fredholm modules

In this section, we will give a formula for the spectral flow in terms of entire cyclic homology. Recall from the introduction that an odd theta-summable Fredholm module over a Banach $*$-algebra $\mathcal{A}$ consists of a $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ together with a theta-summable Fredholm operator D on $\mathcal{H}$ such that $\|[\mathsf{D}, \rho(a)]\| \le C\|a\|$ for all $a \in \mathcal{A}$, for some constant C.

Let us introduce the more general notion of a degree q theta-sumable Fredholm module: this is a theta-summable Fredholm module such that the underlying Hilbert space $\mathcal{E}$ is a Hilbert module for $C(q)$, and the Fredholm operator $\mathcal{D}$ and representation ρ of $\mathcal{A}$ commute with the action of $C(q)$. In fact, we prefer to change our conventions slightly by taking the Fredholm operator $\mathcal{D}$ to be skew-adjoint.

Given an odd theta-summable Fredholm module, we may replace the Hilbert space $\mathcal{H}$ by a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^{1|1}$, such that $\mathcal{E}^+ \cong \mathcal{E}^- \cong \mathcal{H}$. This is a module over the Clifford algebra $C(1)$, with generator given by the matrix

$$c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Elements $a \in \mathcal{A}$ act diagonally on $\mathcal{E}$. Define the operator $\mathcal{D}$ to be the matrix $\mathcal{D} = \begin{pmatrix} 0 & -\mathsf{D} \\ \mathsf{D} & 0 \end{pmatrix}$; it is skew-adjoint, odd, and (graded) commutes with the action of $C(1)$. Thus, $\mathcal{D}$ defines a degree 1 theta-summable Fredholm module for $\mathcal{A}$.

Suppose $\mathcal{D}$ is a degree q theta-summable Fredholm module. Define a multilinear form on $\mathcal{L}_{C(q)}(\mathcal{E})$ by the formula

$$\langle A_0, \dots, A_n \rangle = \int_{\Delta^n} \operatorname{Str}_{C(q)} \big(A_0 e^{\sigma_0 \mathcal{D}^2} \dots A_n e^{\sigma_n \mathcal{D}^2} \big)\, d\sigma;$$

recall that $\operatorname{Str}_{C(q)}(A) = (4\pi)^{-q/2} \operatorname{Str}(c_1 \dots c_q A)$.

Definition 4.1. *The Chern character of a degree q theta-summable Fredholm module is the entire cyclic cocycle on $\mathcal{A}$ given by the formula*

$$\operatorname{Ch}^n(\mathcal{D})(a_0, \dots, a_n) = \langle a_0, [\mathcal{D}, a_1], \dots, [\mathcal{D}, a_n] \rangle.$$

Note that $\operatorname{Ch}^*(\mathcal{D})$ is even (odd) if q is even (odd). In the special case that $\mathcal{D}$ is the degree 1 Fredholm module constructed as above from an odd Fredholm module, we define $\operatorname{Ch}^*(\mathsf{D})$ to equal $\operatorname{Ch}^*(\mathcal{D})$.The following formula is an easy exercise.

Proposition 4.2. *The Chern character of an odd theta-summable Fredholm module $(\mathsf{D}, \mathcal{H})$ is the odd entrire cyclic cochain given by the formula*

$$\begin{aligned} &\operatorname{Ch}^{2k+1}(\mathsf{D})(a_0, \dots, a_{2k+1}) \\ &\qquad = \frac{(-1)^k}{\pi^{1/2}} \int_{\Delta^{2k+1}} \operatorname{Tr}\big(a_0 e^{-\sigma_0 \mathsf{D}^2} [\mathsf{D}, a_1] e^{-\sigma_1 \mathsf{D}^2} \dots [\mathsf{D}, a_{2k+1}] e^{-\sigma_{2k+1} \mathsf{D}^2} \big)\, d\sigma. \end{aligned}$$

The main result of this article is the following formula.

Theorem 4.3. *Let D be an odd theta-summable Fredholm module over the Banach $*$-algebra $\mathcal{A}$. If $g \in \mathrm{U}_N(\mathcal{A})$ is a unitary matrix with values in $\mathcal{A}$, then*

$$\operatorname{sf}(\mathsf{D}, g^{-1} \mathsf{D} g) = (\operatorname{Ch}^*(\mathsf{D}), \operatorname{Ch}_*(g)).$$

Proof. The proof is quite similar to the construction at the end of Section 1. Let $\mathcal{E} \otimes \mathbb{C}^{N|N}$ be the Hilbert space obtained by tensoring $\mathcal{E}$ with the graded vector space $\mathbb{C}^{N|N}$. Introduce the odd element $p \in \mathcal{A} \otimes \operatorname{End}(\mathbb{C}^{N|N})$, represented by the matrix

$$p = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix},$$

which satisfies $p^2 = -1$.

Lemma 4.4.

$$\Big(\mathrm{Ch}^*(\mathcal{D}), \sum_{k=0}^{\infty} k!\ \mathrm{Str}(p, \dots, p)_{2k+1}\Big) = 2(\mathrm{Ch}^*(\mathcal{D}), \mathrm{Ch}_*(g))$$

Proof. Since

$$[\mathcal{D}, p] = \begin{pmatrix} 0 & [\mathcal{D}, g^{-1}] \\ -[\mathcal{D}, g] \end{pmatrix}$$

and (paying careful attention to the sign rule)

$$[\mathcal{D}, p]e^{\sigma\mathcal{D}^2}[\mathcal{D}, p]e^{\tau\mathcal{D}^2} = \begin{pmatrix} [\mathcal{D}, g^{-1}]e^{\sigma\mathcal{D}^2}[\mathcal{D}, g]e^{\tau\mathcal{D}^2} & 0 \\ 0 & [\mathcal{D}, g]e^{\sigma\mathcal{D}^2}[\mathcal{D}, g^{-1}]e^{\tau\mathcal{D}^2} \end{pmatrix},$$

we see that

$$\begin{aligned} &\mathrm{Str}\langle p, [\mathcal{D}, p], \dots, [\mathcal{D}, p]\rangle \\ &\quad = \mathrm{Tr}\langle g^{-1}, [\mathcal{D}, g], \dots, [\mathcal{D}, g^{-1}], [\mathcal{D}, g]\rangle - \mathrm{Tr}\langle g, [\mathcal{D}, g^{-1}], \dots, [\mathcal{D}, g], [\mathcal{D}, g^{-1}]\rangle. \end{aligned}$$

This shows that

$$\sum_{k=0}^{\infty} k!\ \big(\mathrm{Ch}^*(\mathcal{D}), \mathrm{Str}(p, \dots, p)_{2k+1}\big) = (\mathrm{Ch}^*(\mathcal{D}), \mathrm{Ch}_*(g) - \mathrm{Ch}_*(g^{-1})).$$

By Proposition 2.1 (2), $\mathrm{Ch}_*(g^{-1})$ is cohomologous to $\mathrm{Ch}_*(g)$. Since $\mathrm{Ch}^*(\mathcal{D})$ is closed, we see that $(\mathrm{Ch}^*(\mathcal{D}), \mathrm{Ch}_*(g^{-1})) = -(\mathrm{Ch}^*(\mathcal{D}), \mathrm{Ch}_*(g)$, completing the proof of the lemma. □

Let $\mathcal{D}_u$ be the family of degree 1 theta-summable Fredholm operators acting on the $\mathbb{Z}/2$-graded Hilbert module $\mathcal{E} \otimes \mathbb{C}^{N|N}$, parametrized by $u \in [0, 1]$, and given by the formula

$$\mathcal{D}_u = (1 - u)\mathcal{D} + up\mathcal{D}p.$$

Let $\mathcal{D}_{u,s}$ be the family of degree 1 theta-summable Fredholm operators acting on the $\mathbb{Z}/2$-graded Hilbert module $\mathcal{E} \otimes \mathbb{C}^{N|N}$, parametrized by (u, s) in the strip $[0, 1] \times [0, \infty)$, and given by the formula

$$\mathcal{D}_{u,s} = \mathcal{D}_u + sp = \begin{pmatrix} \mathcal{D} + ug^{-1}[\mathcal{D}, g] & -sg^{-1} \\ sg & \mathcal{D} - ug[\mathcal{D}, g^{-1}] \end{pmatrix}.$$

We introduce the superconnection $\mathbb{A} = d + \mathcal{D}_{u,s}$ acting on the trivial degree 1 Clifford module over $[0, 1] \times [0, \infty)$ with fibre $\mathcal{E} \otimes \mathbb{C}^{N|N}$. The curvature of $\mathbb{A}$ is given by the formula

$$\mathbb{A}^2 = \mathcal{D}_u^2 + du\, \dot{\mathcal{D}}_u + s(1 - 2u)[\mathcal{D}, p] + ds\, p - s^2.$$

The Chern character of $\mathbb{A}$ is the one-form on $[0,1]\times[0,\infty)$ given by the formula

$$\begin{aligned}\operatorname{Ch}(\mathbb{A}) &= \operatorname{Str}_{C(1)}(e^{\mathbb{A}^2}) \\ &= du\ \operatorname{Str}_{C(1)}(\dot{\mathcal{D}}_u\, e^{\mathcal{D}_u^2+s(1-2u)[\mathcal{D},p]-s^2}) + ds\ \operatorname{Str}_{C(1)}(p\, e^{\mathcal{D}_u^2+s(1-2u)[\mathcal{D},p]-s^2}).\end{aligned}$$

The same method of proof as that of Proposition 2.5 shows that

$$|\operatorname{Ch}(\mathbb{A})| + |\partial_u \operatorname{Ch}(\mathbb{A})| + |\partial_s \operatorname{Ch}(\mathbb{A})| = O(e^{-s^2/2}),$$

and that $\operatorname{Ch}(\mathbb{A})$ is closed.

Let Γ_u be the contour $\{u\}\times[0,\infty)$, oriented in the direction of increasing s, and let γ_s be the contour $[0,1]\times\{s\}$, oriented in the direction of increasing u. Since $\operatorname{Ch}(\mathbb{A})$ is closed, it follows that

$$\int_{\Gamma_0}\operatorname{Ch}(\mathbb{A}) - \int_{\Gamma_1}\operatorname{Ch}(\mathbb{A}) = \int_{\gamma_0}\operatorname{Ch}(\mathbb{A}) - \lim_{s\longrightarrow\infty}\int_{\gamma_s}\operatorname{Ch}(\mathbb{A}) = \int_{\gamma_0}\operatorname{Ch}(\mathbb{A}).$$

Lemma 4.5.

$$\int_{\gamma_0}\operatorname{Ch}(\mathbb{A}) = 2\operatorname{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$$

Proof. Observe that

$$\begin{aligned}\mathcal{D}_u &= \begin{pmatrix}\mathcal{D}+ug^{-1}[\mathcal{D},g] & 0 \\ 0 & \mathcal{D}-ug[\mathcal{D},g^{-1}]\end{pmatrix} \\ &= \begin{pmatrix} 0 & -\mathsf{D}-ug^{-1}[\mathsf{D},g] & 0 & 0 \\ \mathsf{D}+ug^{-1}[\mathsf{D},g] & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathsf{D}+ug[\mathsf{D},g^{-1}] \\ 0 & 0 & \mathsf{D}-ug[\mathsf{D},g^{-1}] & 0\end{pmatrix}\end{aligned}$$

From this formula and the definition of $\operatorname{Str}_{C(1)}$, it follows that

$$\begin{aligned}\int_{\gamma_0}\operatorname{Ch}(\mathbb{A}) &= \int_0^1 \operatorname{Tr}(g^{-1}[\mathsf{D},g]e^{-(\mathsf{D}+ug^{-1}[\mathsf{D},g])^2})\,du \\ &\quad - \int_0^1 \operatorname{Tr}(g[\mathsf{D},g^{-1}]e^{-(\mathsf{D}-ug[\mathsf{D},g^{-1}])^2})\,du.\end{aligned}$$

Corollary 2.7 shows that the first term equals $\operatorname{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$ and the second equals $\operatorname{sf}(g\mathsf{D}g^{-1},\mathsf{D})$. But invariance of spectral flow under conjugation shows that

$$\operatorname{sf}(\mathsf{D}, g^{-1}\mathsf{D}g) = \operatorname{sf}(g\mathsf{D}g^{-1},\mathsf{D}),$$

proving the lemma. $\square$

In order to prove the theorem, it suffices to prove that

$$\int_{\Gamma_0} \mathrm{Ch}(\mathbb{A}) = \int_{\Gamma_1} \mathrm{Ch}(\mathbb{A}) = \frac{1}{2}(\mathrm{Ch}^*(\mathsf{D}), \mathrm{Ch}_*(p)).$$

Restricted to the contour Γ_0, the curvature $\mathbb{A}^2$ takes the form

$$\mathbb{A}^2 = \mathcal{D}^2 - s^2 + s[\mathcal{D}, p] + ds\, p.$$

The exponential of the curvature may be expanded in a series by means of Duhamel's formula, giving

$$\begin{aligned}\mathrm{Ch}(\mathbb{A}) &= \sum_{k=0}^{\infty} s^{2k+1} e^{-s^2}\, ds \sum_{i=0}^{2k+1} \langle 1, \underbrace{[\mathcal{D}, p], \dots, [\mathcal{D}, p]}_{i \text{ times}}, p, \underbrace{[\mathcal{D}, p], \dots, [\mathcal{D}, p]}_{2k+1-i \text{ times}} \rangle \\ &= \sum_{k=0}^{\infty} s^{2k+1} e^{-s^2}\, ds\, \langle p, \underbrace{[\mathcal{D}, p], \dots, [\mathcal{D}, p]}_{2k+1 \text{ times}} \rangle.\end{aligned}$$

Thus, we see that

$$\int_{\Gamma_0} \mathrm{Ch}(\mathbb{A}) = \sum_{k=0}^{\infty} \int_0^{\infty} s^{2k+1} e^{-s^2}\, ds \sum_{i=0}^{2k+1} \langle p, \underbrace{[\mathcal{D}, p], \dots, [\mathcal{D}, p]}_{2k+1 \text{ times}} \rangle.$$

The change of variables $t = s^2$ gives

$$\int_0^{\infty} s^{2k+1} e^{-s^2}\, ds = \frac{1}{2} \int_0^{\infty} t^k e^{-t}\, dt = \frac{k!}{2},$$

and we see that

$$\int_{\Gamma_0} \mathrm{Ch}(\mathbb{A}) = \frac{1}{2}\Big(\mathrm{Ch}^*(\mathcal{D}), \sum_{k=0}^{\infty} k!\ \mathrm{Str}(p, \dots, p)_{2k+1})\Big).$$

It is easy to see that the integral over Γ_1 is the negative of that over Γ_0, since the curvature is identical except that the term $[\mathsf{D}, p]$ changes sign. $\square$

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