AN EXTENSION OF GROSS'S LOG-SOBOLEV INEQUALITY FOR THE LOOP SPACE OF A COMPACT LIE GROUP

Ezra Getzler

Department of Mathematics, MIT,
Cambridge, Mass. 02139 USA

Let G be a compact Lie group, and for $T > 0$, let L_*G be the space of based loops of length T

$$L_*G = \left\{ \gamma : [0,T] \to G \,\middle|\, \gamma(0) = \gamma(T) = e \right\}.$$

We denote the expectation with respect to the Wiener measure on L_*G (also known as the Brownian bridge) by $\langle F \rangle_*$. There is a regular Dirichlet form on L_*G, which was constructed using the Malliavin calculus in [1] and [2], and may be represented by the formula

$$\mathcal{E}(F,F) = \langle |d_*F|^2 \rangle_*.$$

Consider the function on L_*G given by the Stratonovitch stochastic integral

$$V(\gamma) = T^{-1} \Big| \int_0^T \dot\gamma(t)\gamma(t)^{-1}\,dt \Big|^2 + 1.$$

Gross has proved the following theorem for $T = 1$ in [4], and our goal in this paper is to extend his proof to show that the constant C may be chosen independent of T for T sufficiently small.

Theorem. *There is a constant $C = 1 + O(T)$ such that for all $F \in W^\infty(L_*G)$,*

$$\langle F^2 \log |F| \rangle_* \le C \langle |d_*F|^2 + V|F|^2 \rangle_* + \tfrac{1}{2} \langle F^2 \rangle_* \log \langle F^2 \rangle_*,$$

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uniformly in T *for small* T.

The above result is to be compared with Gross's logarithmic Sobolev inequality for a Wiener space B, proved in [3], that for $F \in W^\infty(B)$,

$$\langle F^2 \log |F| \rangle \leq \langle |dF|^2 \rangle + \tfrac{1}{2} \langle F^2 \rangle \log \langle F^2 \rangle.$$

The proof of the above theorem uses this inequality at a key point, just as in Gross's proof for $T = 1$.

It is tempting to attempt to get rid of the potential V in the inequality above by means of the Hausdorff-Young inequality: for all $x \in \mathbb{R}$ and $y > 0$,

$$xy \leq e^x + y \log y - y.$$

This shows that for $\langle |F|^2 \rangle_* = 1$,

$$(1 - C\varepsilon/2) \langle F^2 \log |F| \rangle_* \leq C \langle |d_* F|^2 \rangle_* + C \langle e^{V/\varepsilon} \rangle_* + C(\log \varepsilon - 1).$$

In order to obtain a logarithmic Sobolev inequality from this, we need to be able to take $\varepsilon < 2/C$. In Proposition 2.10, we will see that this value of ε is marginally too small for $\langle e^{V/\varepsilon} \rangle_*$ to be finite. It is not clear whether some other method will allow the omission of V from the inequality.

In Section 1, we recall some of the results of [2], and prove some abstract lemmas that will be called upon later. In Section 2, we specialize to $L_* G$, where G is a compact Lie group. In Section 3, we discuss the technique of Gross in which he effectively constructs a smooth tubular neighbourhood of $L_* G$ in Wiener space. In Section 4, we give our proof of the main theorem.

Throughout this paper, we will make frequent use of the Hausdorff-Young inequality:

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1. The geometry of Malliavin maps

Let $\mathfrak{g}$ be a real vector space with inner-product, and let B be the classical Wiener space $B = C_*([0,T], \mathfrak{g}) = \{\gamma \in C([0,T], \mathfrak{g}) \mid \gamma(0) = 0\}$. The Wiener measure on B is the unique probability measure such that

$$\int_B \exp\Big(i \int_0^T (\alpha(t), \gamma(t))\, dt \Big)\, d\mu(\gamma) = e^{-C(\alpha,\alpha)/2} \quad \text{for all } \alpha \in L^\infty([0,T], \mathfrak{g}),$$

where

$$C(\alpha, \alpha) = \int_0^T \int_0^T \min(s,t)\, (\alpha(s), \alpha(t))\, ds\, dt.$$

Let $H \subset B$ be the Hilbert space of finite-energy paths $L^{2,1}_*([0,T], \mathfrak{g})$, with inner product

$$|\gamma|^2 = \int_0^T |\dot{\gamma}(t)|^2 \, dt.$$

A **cylinder function** on B is a function of the form

$$F(\gamma(t_1), \dots, \gamma(t_k)),$$

where $0 < t_1 < \ldots < t_k < T$, and $F \in C_c^\infty(\mathfrak{g}^k)$. The space of all cylinder functions, written $C_c^\infty(B)$, is dense in $L^p(B) = L^p(B, d\mu)$ for each $p < \infty$. If $\tau : B \to \mathfrak{g}^k$ denotes the map

$$\tau(\gamma) = (\gamma(t_1), \dots, \gamma(t_k)),$$

then the above cylinder function may be written $\tau^* F$.

We will often make use of the Hilbert tensor product $H_1 \otimes_2 H_2$ of two Hilbert spaces H_1 and H_2; this is completion of the algebraic tensor product with respect to the quadratic form

$$|v \otimes_2 w|^2 = |v|^2 \cdot |w|^2.$$

We also let $\mathrm{HS}(H)$ denote the space of Hilbert-Schmidt operators on H.

If $f = \tau^* F$ is a cylinder function, its gradient, denoted by df, is the element of $C_c^\infty(B) \otimes H$ defined by applying the map

$$\tau^* : C_c^\infty(\mathfrak{g}^k) \otimes \mathfrak{g}^k \longrightarrow C_c^\infty(B) \otimes H$$

to $dF \in C_c^\infty(\mathfrak{g}^k) \otimes \mathfrak{g}^k$. Forming the closure of this operator, we obtain an unbounded operator from $L^p(B)$ to $L^p(B, H)$, which we will also denote by d; its adjoint d^* is then a closed unbounded operator from $L^p(B, H)$ to $L^p(B)$.

The composition of the operators d and d^* acting on the cylinder functions is the Ornstein-Uhlenbeck operator $L = d^* d$, which is essentially self-adjoint with core $C_c^\infty(B)$. If $\mathcal{H}$ is a Hilbert space, the Sobolev space $L^p_s(B, \mathcal{H})$, where $1 < p < \infty$ and $s \in \mathbb{R}$, is the domain of $L^{s/2}$ on $L^p(B, \mathcal{H})$. The space of Malliavin test functions is

$$W^\infty(B, \mathcal{H}) = \bigcap_{p,s<\infty} L^p_s(B, \mathcal{H}).$$

Meyer has proved that d is bounded from $L^p_s(B)$ to $L^p_{s-1}(B, H)$ and that d^* is bounded from $L^p_s(B, H)$ to $L^p_{s-1}(B)$, for all $s \in \mathbb{R}$ and $p < \infty$. It is important to note that W^∞-functions need not be continuous.

We can also define W^∞-maps from a Wiener space B to a compact Riemannian manifold M.

Definition 1.1. *A map $\pi : B \to M$ is in $W^\infty(B, M)$ if it is measurable and if the pull-back map $\pi^* : C^\infty(M) \to W^\infty(B)$ is bounded.*

An equivalent definition can be made by choosing an embedding $\rho : M \longrightarrow \mathbb{R}^n$ of M in a Euclidean space $\mathbb{R}^n$: then $\pi \in W^\infty(B, M)$ if and only if $\rho \circ \pi \in W^\infty(B, \mathbb{R}^n)$.

If $\pi \in W^\infty(B, M)$, the composition

$$C^\infty(M) \xrightarrow{\pi^*} W^\infty(B) \xrightarrow{\mu} \mathbb{R}$$

is a positive linear form on $C^\infty(M)$ which equals 1 on the constant function 1, and hence defines a probability measure on M. We will write this measure $\pi_*\mu$.

The tangent map

$$d : W^\infty(M, B) \longrightarrow W^\infty(B, \mathrm{Hom}(H, \pi^*TM))$$

may be defined by means of an embedding $\rho : M \longrightarrow \mathbb{R}^n$, by the formula

$$d(\rho \circ \pi) = d\rho \circ d\pi \in W^\infty(B, \mathrm{Hom}(H, \mathbb{R}^n)).$$

Definition 1.2. *Let $\pi : B \longrightarrow M$ be in $W^\infty(B, M)$, with differential Π, and form $\Gamma = \Pi\Pi^* \in W^\infty(B, \mathrm{End}(\pi^*TM))$. Then π is a* **Malliavin map** *if the determinant $\det(\Gamma)$ satisfies*

$$\det(\Gamma)^{-1} \in L^p(B) \quad \textit{for } p < \infty.$$

If π is a Malliavin map, then the operator Γ^{-1} is in $W^\infty(B, \mathrm{End}(\pi^*TM))$, and $\det(\Gamma)^k$ is in $W^\infty(B)$ for all $k \in \mathbb{Z}$. The operator $N = \Pi^*\Gamma^{-1}\Pi \in W^\infty(B, \mathrm{HS}(H))$ is a projector in H of rank n for a.e. $x \in B$. We may think of N as the projector on the normal bundle to the fibres of the map π. Thus the projector $P = 1 - N$ orthogonal to N is the projector onto the tangent bundle to the fibres of π.

We say that an operator A on $W^\infty(B)$ acts along the fibres of π if it satisfies the formula

$$A((\pi^*f)F) = (\pi^*f)(AF),$$

for all $f \in C^\infty(M)$ and $F \in W^\infty(B)$. Using the projector P, we may construct the exterior differential along the fibres

$$d_* : W^\infty(B) \longrightarrow W^\infty(B, H),$$

defined by the formula $d_*F = P(dF)$, and the Ornstein-Uhlenbeck operator along the fibres $L_* : W^\infty(B) \longrightarrow W^\infty(B)$, associated to the Dirichlet form $\langle |d_*F|^2 \rangle$, given by the formula $L_*F = d^*PdF$.

The adjoint of the pull-back $\pi^* : C^\infty(M) \to W^\infty(B)$ is a bounded map $(\pi_*)' : W^\infty(B)' \to C^\infty(M)'$. We define a push-forward map $\pi_* : W^\infty(B) \to C^\infty(M)$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} W^\infty(B) & \xrightarrow{\pi_*} & C^\infty(M) \\ {\scriptstyle F\mapsto F\,d\mu}\big\downarrow & & \big\downarrow{\scriptstyle f\mapsto f\,d(\pi_*\mu)} \\ W^\infty(B)' & \xrightarrow{(\pi^*)'} & C^\infty(M)' \end{array}$$

The following basic result, due to Malliavin, shows that the map $\pi_* : W^\infty(B) \to C^\infty(M)$ is well-defined.

Proposition 1.3. *If π is a Malliavin map, integration along the fibres of π defines a bounded map $\pi_* : W^\infty(B) \to C^\infty(M)$.*

Note that $\pi_* \circ \pi^*$ is the identity, and in particular, that $\pi_* 1 = 1$. If $F \in W^\infty(B)$, we define

$$\langle F \rangle = \int_B F\,d\mu;$$

similarly, if $f \in C^\infty(M)$, we define

$$\langle f \rangle = \int_M f\,d(\pi_*\mu).$$

These integrals are related by the formula $\langle F \rangle = \langle \pi_* F \rangle$.

Definition 1.4. *If X is a vector field on M, its* **horizontal lift** *$\tilde{X} \in W^\infty(B, H)$ is the unique vector field on B such that*

(1) *if $f \in C^\infty(M)$, $\tilde{X}(\pi^* f) = \pi^*(X(f))$;*
(2) *$\tilde{X}$ is horizontal, that is, $P\tilde{X} = 0$.*

It may be checked that the vector field $\Pi^*\Gamma^{-1}\pi^*X$ satisfies the above requirements, so that we obtain an explicit formula for the horizontal lift:

$$\tilde{X} = \Pi^*\Gamma^{-1}\pi^*X.$$

We will denote by $\operatorname{div}_{\pi_*\mu} : \Gamma(M, TM) \to C^\infty(M)$ the adjoint of the exterior differential $d : C^\infty(M) \to \Gamma(M, T^*M)$ with respect to the pairings

$$(f, g) = \int_M fg\,d(\pi_*\mu) \quad \text{and} \quad (X, \omega) = \int_M \langle X, \omega \rangle\,d(\pi_*\mu).$$

In terms of the divergence operator div defined with respect to the Riemannian volume form dx, $\operatorname{div}_{\pi_*\mu}$ is given by the formula

$$\operatorname{div}_{\pi_*\mu} X = \operatorname{div} X + X\left(\log\left(\frac{d(\pi_*\mu)}{dx}\right)\right).$$

Definition 1.5. *If X is a vector field on M, we define the W^∞-function $\alpha(X)$ by the formula*

$$\alpha(X) = d^*\tilde{X} - \pi^*(\mathrm{div}_{\pi_*\mu} X).$$

It is easy to see that α satisfies the formula

$$\alpha(fX) = (\pi^* f)\, \alpha(X) \quad \text{for all } f \in C^\infty(M). \tag{1.6}$$

The following proposition explains our reason for introducing $\alpha(X)$.

Proposition 1.7. *If X is a vector field on M and $F \in W^\infty(B)$, then*

$$X\,(\pi_* F) = \tilde{X}(F) - \alpha(X)F.$$

Proof. If $F \in W^\infty(B)$ and $f \in C^\infty(M)$, then

$$\langle fX(\pi_*F)\rangle = \big\langle \big(-X(f) + (\mathrm{div}_{\pi_*\mu} X) f\big)\pi_*F \big\rangle = \big\langle \pi^*\big(-X(f) + (div_{\pi_*\mu} X) f\big) F \big\rangle \,.$$

We now use the fact that $\pi^*(X(f)) = \tilde{X}(\pi^* f)$, which gives

$$\begin{aligned}\langle fX(\pi_*F)\rangle &= \big\langle \big(-\tilde{X}(\pi^* f) + \pi^*(\mathrm{div}_{\pi_*\mu} X)\, \pi^* f\big) F \big\rangle \\ &= \big\langle \pi^* f\, \big(\tilde{X}(F) - d^*\tilde{X}\, F + \pi^*(\mathrm{div}_{\pi_*\mu} X)\, F\big)\big\rangle,\end{aligned}$$

which proves the lemma, since f was arbitrary. $\square$

Corollary 1.8. *If $F \in W^\infty(B)$ satisfies the formula $\tilde{X}(F) = \frac{1}{2}\alpha(X)F$, then*

(1) $X\pi_*\left(|F|^2\right) = 0$;
(2) $X\pi_*\left(|F|^2 \log |F|^2\right) = \pi_*\left(\alpha(X)|F|^2\right)$.

Proof. To prove (1), we observe that $\tilde{X}(|F|^2) = 2F\tilde{X}(F) = \alpha(X)|F|^2$. Since

$$X\left(\pi_*|F|^2\right) = \pi_*\left(\tilde{X}(|F|^2)\right) - \pi_*\left(\alpha(X)|F|^2\right),$$

(1) follows.

To prove (2), let us calculate $\tilde{X}(|F|^2 \log |F|^2)$:

$$\begin{aligned}\tilde{X}(|F|^2 \log |F|^2) &= 2F\,\tilde{X}(F)\, \log |F|^2 + 2F\,\tilde{X}(F) \\ &= \alpha(X)|F|^2 \log |F|^2 + \alpha(X)|F|^2.\end{aligned}$$

Since $X\pi_*\left(|F|^2 \log |F|^2\right) = \pi_*\left(\tilde{X}(|F|^2 \log |F|^2)\right) - \pi_*\left(\alpha(X)|F|^2 \log |F|^2\right)$, the proof of (2) follows. $\square$

2. Based loops in a compact Lie group

In this section, we will discuss a particular case of a Malliavin map, in which M is a compact Lie group G, with Lie algebra $\mathfrak{g}$, and the map π is the so-called path-ordered exponential. This case is very special, since the Malliavin covariance $\Gamma = \Pi^*\Pi$ is equal to a constant multiple $T\,\mathrm{id}$ of the identity operator, which makes many of the calculations easier.

To simplify the formulation of the results, we will assume that the group G is a linear group; of course, this is no restriction, since every compact Lie group has a faithful linear representation. We also suppose chosen an invariant Riemannian metric on G, which induces an inner product on $\mathfrak{g}$ invariant under the adjoint action $\mathrm{Ad}(g)$ for $g \in G$. Denote the dimension of G by n, and the identity of G by e. We will always identify a Lie algebra element $X \in \mathfrak{g}$ with the corresponding left-invariant vector field on G.

Let (B, H) be the classical Wiener space $\big(C_*([0,T],\mathfrak{g}), L^{2,1}_*([0,T],\mathfrak{g})\big)$. If $x \in H$, we solve the ordinary differential equation for $\gamma(t) : [0,T] \to G$ with initial condition $\gamma(0) = e$,

$$\gamma(t)^{-1}\dot\gamma(t) = \dot x(t).$$

The solution of this equation is known as the **path-ordered exponential**, and we will write it as $\gamma[x]$, or simply as γ if x is implicit.

The path-ordered exponential identifies H with the space of finite-energy based paths in $L^{2,1}_*([0,T],G)$. Let π be the map $x \mapsto \gamma[x](T)$; the fibre of π over e can be identified with the space L_*G of finite-energy paths in G which return to the identity at time T, that is, the based loop space of G.

If $F \in W^\infty(B)$, we will denote by $\langle F\rangle_*$ the integral of F over the fibre $\pi^{-1}(e)$, that is,

$$\langle F\rangle_* = \pi_*(F)(e).$$

When we say that two functions F_1 and F_2 are equal on $\pi^{-1}(e)$, we mean that $\langle (F_1 - F_2)^2\rangle_* = 0$.

The map $\gamma(t) : B \to G$ is extended to a family of W^∞-maps from the Wiener space B to G, by introducing a mollifier on B:

$$x_\varepsilon(t) = \varepsilon^{-1} \int_0^1 \lambda(\varepsilon^{-1}(s-t))\, x(s)\, ds,$$

where λ is any positive symmetric function in $C^\infty_c(-1,1)$ such that $\int_{(-1,1)} \lambda\, dt = 1$. The following proposition is a consequence of the theory of Stratanovitch stochastic differential equations.

Proposition 2.1.

(1) *For each $\varepsilon > 0$, the map $\pi_\varepsilon(x) = \pi(x_\varepsilon)$ is a W^∞-map from B to G.*
(2) *As $\varepsilon \to 0$, the maps π_ε converge in $W^\infty(B, G)$ to a map π.*

We now calculate the differential $d\pi$, and the Malliavin covariance matrix $\Gamma = (d\pi)(d\pi)^*$, of the map π explicitly.

Proposition 2.2.

(1) $\Pi = (d\pi)\pi^{-1} \in W^\infty(B, \mathrm{Hom}(H, \mathfrak{g}))$ *is given by the formula*

$$\Pi(h_t) = \int_0^T \mathrm{Ad}(\gamma(t))\dot{h}_t \, dt.$$

(2) *The adjoint* $\Pi^*(X) \in W^\infty(B, \mathrm{Hom}(\mathfrak{g}, H))$ *of* Π *is given by the formula*

$$(\Pi^* X)_t = \int_0^t \mathrm{Ad}(\gamma(s))^{-1} X \, ds.$$

(3) *The Malliavin covariance matrix* $\Gamma = \Pi\Pi^*$ *equals* T *times the identity of* $\mathfrak{g}$*; in particular, the map* π *satisfies the Malliavin condition, since* $\det(\Gamma) = T^n$ *is a constant, and* N *is given by the formula* $N = T^{-1}\Pi^*\Pi$.

Proof. We will calculate $\Pi_\varepsilon = (d\pi_\varepsilon)(\pi_\varepsilon)^{-1}$, and then take the limit $\varepsilon \to 0$. For $\varepsilon > 0$, the map π_ε is smooth, so we can calculate Π_ε path by path.

By du Hamel's formula, $(d\pi_\varepsilon)(\pi_\varepsilon)^{-1}$ equals

$$(d\pi_\varepsilon)(\pi_\varepsilon)^{-1} = (d\gamma_\varepsilon(T))\gamma_\varepsilon(T)^{-1} = \int_0^T \mathrm{Ad}(\gamma_\varepsilon(t))\dot{h}_\varepsilon(t) \, dt,$$

from which (1) follows, by sending $\varepsilon \to 0$.

Since the metric on $\mathfrak{g}$ is invariant, it follows that if $X \in \mathfrak{g}$, then

$$(X, \Pi(h(t))) = \int_0^T \Big(X, \mathrm{Ad}(\gamma(t))\dot{h}(t)\Big) \, dt = \int_0^T \Big(\mathrm{Ad}(\gamma(t))^{-1}X, \dot{h}(t)\Big) \, dt,$$

from which we obtain the formula for $\Pi^*(X)$. It is clear from this that $\Pi\Pi^* = T$. $\square$

Corollary 2.3. *If* $X \in \mathfrak{g}$*, then its horizontal lift* $\tilde{X} \in W^\infty(B, H)$ *is given by the formula*

$$\tilde{X}(t) = T^{-1} \int_0^t \mathrm{Ad}(\gamma(s))^{-1} X \, ds,$$

and $d^*\tilde{X}$ *is given by the formula*

$$d^*\tilde{X} = T^{-1}(d^*\Pi, X),$$

where $d^*\Pi \in W^\infty(B, \mathfrak{g})$ *is the divergence of* Π.

It follows from Proposition 2.2 that if $f \in C^\infty(G)$, then $d(\pi^* f)$ satisfies

$$|d(\pi^* f)| = T^{1/2}|df|. \tag{2.4}$$

The next proposition collects a number of useful formulas.

Proposition 2.5.

(1) *The gradient* $d\Pi \in W^\infty(B, \mathrm{Hom}(H \otimes_2 H, \mathfrak{g}))$ *is given by the formula*

$$d\Pi(a,b) = \int_{0\le s\le t\le T} [\mathrm{Ad}(\gamma(s))\dot a(s), \mathrm{Ad}(\gamma(t))\dot b(t)]\, ds\, dt.$$

(2) *The divergence* $d^*\Pi \in W^\infty(B, \mathfrak{g})$ *is given by the Stratanovitch integral*

$$d^*\Pi = \int_0^T \mathrm{Ad}(\gamma(t))\dot x(t)\, dt.$$

(3) *If* $a \in H$*, then*

$$dd^*\Pi(a) = \int_{0\le t\le T} \mathrm{Ad}(\gamma(t))\dot a(t)\, dt + \int_{0\le s\le t\le T} [\mathrm{Ad}(\gamma(s))\dot a(s), \mathrm{Ad}(\gamma(t))\dot y(t)]\, ds\, dt.$$

Proof. The gradient of Π_ε is given by the formula

$$d\Pi_\varepsilon(a,b) = \int_{0\le s\le t\le T} \big[\mathrm{Ad}(\gamma_\varepsilon(s))\dot a(s), \mathrm{Ad}(\gamma_\varepsilon(t))\dot b(t)\big]\, ds\, dt.$$

The formula for $d\Pi$ follows by taking $\varepsilon \to 0$.

In a finite-dimensional Wiener space V, $d^*\Pi$ would be given by the formula

$$d^*\Pi = -\,\mathrm{Tr}(d\Pi) + \Pi x,$$

where x is the identity map from V to itself. In our infinite-dimensional situation, this formula makes sense if we replace Π by its approximation Π_ε:

$$d^*\Pi_\varepsilon = -\,\mathrm{Tr}_H(d\Pi_\varepsilon) + \Pi_\varepsilon x.$$

Using the above formula for $d\Pi_\varepsilon$, it is easy to see that $\mathrm{Tr}_H(d\Pi_\varepsilon)$ vanishes. On the other hand, $\Pi_\varepsilon x$ equals

$$\Pi_\varepsilon x = \int_0^T \mathrm{Ad}(\gamma_\varepsilon(t))\dot x(t)\, dt,$$

which converges to

$$\int_0^T \mathrm{Ad}(\gamma_\varepsilon(t))\dot x(t)\, dt$$

as $\varepsilon \to 0$. This proves (2). The proof of (3) is similar to the proof of (1). □

Let $y(t)$ be the Stratanovitch stochastic integral

$$y(t) = \int_0^t \mathrm{Ad}(\gamma(s)) dx(s) = \lim_{\varepsilon\to 0} \int_0^t \mathrm{Ad}(\gamma(s))\dot x_\varepsilon(s)\, ds,$$

so that $d^*\Pi = y(T)$. In the rest of this section, we will study the properties of the stochastic process $y(t)$. The following lemma will be basic to this study.

Lemma 2.6. *The stochastic process $t \mapsto y(t)$ is a Wiener process; that is, the map from B to itself given by sending x to y is measure-preserving.*

Proof. Let us denote the Stratonovitch stochastic differential by $dx(t)$, and the Ito stochastic differential by $\delta x(t)$. The relationship between the two differentials shows that $\gamma^{-t}\delta\gamma(t) = \delta x(t)$. It follows that

$$\begin{aligned} y(t) &= \int_0^t \mathrm{Ad}(\gamma(s))\delta x(s) + \tfrac{1}{2}\sum_{ijk} c^i_{jk} \int_0^t \mathrm{Ad}(\gamma(t))X_i \, d\langle x^j, x^k\rangle \\ &= \int_0^t \mathrm{Ad}(\gamma(s))\delta x(s), \end{aligned}$$

since the quadratic variation $\langle x^j, x^k\rangle = t\delta^{jk}$ is symmetric in j and k, while the structure coefficients c^i_{jk} are antisymmetric. Thus, we see that the quadratic variation process $\langle y, y\rangle$ equals t times the inner product on $\mathfrak{g}$, and hence that $y(t)$ is a Wiener process on $\mathfrak{g}$. □

There is a more geometric way to see that $t \mapsto y(t)$ is a Wiener process, for which we will give only the outline. Consider the diagram

$$B \xrightarrow{x\mapsto -x} B \xrightarrow{x\mapsto\gamma[x]} P_*G \xrightarrow{\gamma\mapsto\gamma^{-1}} P_*G \xrightarrow{\gamma\mapsto x} B.$$

It turns out that the composition of these maps is precisely the map $x \mapsto y[x]$. Since each map is measure-preserving, their composition is, proving that y is a Wiener process.

Corollary 2.7.

(1) $\Big\langle \exp\Big(\dfrac{\lambda}{2}|y(T)|^2\Big)\Big\rangle = (1 - T\lambda)^{-n/2}$

(2) $\Big\langle \exp\Big(\dfrac{\lambda}{2}\displaystyle\int_0^T |y(t)|^2\, dt\Big)\Big\rangle = (\cos T\lambda^{1/2})^{-n/2}$

Proof. Since $y(t)$ is a Brownian process, (1) follows from the calculation of the following integral:

$$(2\pi T)^{-n/2}\int_{\mathfrak{g}} e^{-|\xi|^2/2T + \lambda|\xi|^2/2}\, d\xi = (1 - T\lambda)^{-n/2}.$$

By the Feynman-Kac formula, the left-hand side of (2) is given by the integral of the heat-kernel

$$\big\langle \xi \big| e^{-T(\Delta - \lambda|\xi|^2)/2} \big| 0 \big\rangle = (2\pi T)^{-n/2}\left(\frac{T\lambda^{1/2}}{\sin T\lambda^{1/2}}\right)^{n/2} e^{-(\lambda^{1/2}\cot T\lambda^{1/2})|\xi|^2/2},$$

with respect to ξ, which is

$$\left(\frac{T\lambda^{1/2}}{T\lambda^{1/2}\cot T\lambda^{1/2}\,\sin T\lambda^{1/2}}\right)^{n/2} = (\cos T\lambda^{1/2})^{-n/2}. \quad \square$$

Let $\{X, Y\}$ denote the Killing form on $\mathfrak{g}$, given by the formula

$$\{X, Y\} = -\operatorname{Tr}_{\mathfrak{g}}(\operatorname{ad}(X)\operatorname{ad}(Y)),$$

and let $\|X\|^2 = \{X, X\}$.

Lemma 2.8. *The differential $dy(T)$ of $y(T)$ satisfies the estimate*

$$\begin{aligned} |dy(T)|^2 &= T + \frac{1}{2}\int_{0\le s,t\le T} \min(s,t)\{\dot y(s), \dot y(t)\}\, ds\, dt \\ &\le T + T|y(T)|^2 + \int_0^T \|y(t)\|^2\, dt. \end{aligned}$$

Proof. The proof makes use of the same mollification method as in the proof of Proposition 2.5; hence, we will tacitly suppose that $x(t)$ is smooth.

The formula for $|dy(T)|^2$ easily follows from the formula for $dy(T)$ in Proposition 2.5 (3). Pretending that $x(t)$ is smooth, we integrate twice by parts:

$$\begin{aligned} &\int_{0\le s,t\le T} \min(s,t)\,\{\dot y(s), \dot y(t)\}\, ds\, dt \\ &\qquad = T\|y(T)\|^2 - 2\int_0^T \{y(t), y(T)\}\, dt + \int_0^T \|y(t)\|^2\, dt \\ &\qquad \le 2T|y(T)|^2 + 2\int_{0\le t\le T} \|y(t)\|^2\, ds. \end{aligned} \quad \square$$

Ito's formula shows that the measure $\pi_*\mu$ is determined by the formula

$$\frac{d(\pi_*\mu)}{dg} = k(T, g),$$

where $k(T, g) = \langle g|e^{-T\Delta}|e\rangle$ is the heat-kernel for the invariant Laplacian Δ on G. The asymptotic expansion for the heat kernel shows that $k(T, g)$ may be written for small T as

$$k(T, g) = (4\pi T)^{-\dim(G)/2} e^{-\delta(g)^2/4T}\left(\sum_{i<N} T^i a_i(g) + r_N(T, g)\right),$$

where $\delta(g)$ is the Riemannian distance between g and the identity, $a_i \in C^\infty(G)$, and $r_N \in C^\infty((0,\varepsilon] \times G)$ satisfies the estimates

$$\left|\partial_T^k \partial_g^\alpha r_N(T,g)\right| \le C(k,\alpha) T^{N-2k-|\alpha|}$$

for $N \ge 2k - |\alpha|$. It follows that for small T,

$$C_1 T^{-n/2} e^{-\delta(g)^2/4T} \le k(T,g) \le C_2 T^{-n/2} e^{-\delta(g)^2/4T}, \tag{2.9}$$

We close this section with an estimate which differs from Corollary 2.7 in that it estimates an integral over one fibre of π, and not over all of B.

Proposition 2.10.

$$\begin{aligned}\Big\langle \exp\Big(\frac{\lambda}{2}|y(T)|^2\Big)\Big\rangle_* \\ = \frac{\operatorname{vol}(G/T)}{k(T,e)} \left(\frac{2\pi}{\lambda}\right)^{n/2} \int_{\mathfrak{t}} k(T,e^X) e^{(T-\lambda^{-1})|X|^2/2} \det\nolimits_{\mathfrak{g}/\mathfrak{t}}(1+\operatorname{ad}(X))\, dX\end{aligned}$$

Proof. We will use the formula

$$\Big\langle \exp\Big(\frac{\lambda}{2}|y(T)|^2\Big)\Big\rangle_* = \left(\frac{2\pi}{\lambda}\right)^{n/2} \int_{\mathfrak{g}} \big\langle \exp(X, y(T))\big\rangle_* e^{-|X|^2/2\lambda}\, dX$$

This may be rewritten as an integral over the Cartan subalgebra $\mathfrak{t}$ by the change of variables formula

$$\int_{\mathfrak{g}} f(X)\, dX = \int_{G/T} \left(\int_{\mathfrak{t}} f(\operatorname{Ad}(g)X)\, \det\nolimits_{\mathfrak{g}/\mathfrak{t}}(1+\operatorname{ad}(X))\, dX \right) dg,$$

where

$$1 \le \det\nolimits_{\mathfrak{g}/\mathfrak{t}}(1+\operatorname{ad}(X)) \le O(|X|^{\dim(\mathfrak{g}/\mathfrak{t})}).$$

Using the fact that $\langle \exp(X, y(T))\rangle_*$ is invariant under conjugation $X \mapsto \operatorname{Ad}(g)X$, we see that

$$\begin{aligned}\Big\langle \exp\Big(\frac{\lambda}{2}|y(T)|^2\Big)\Big\rangle_* \\ = \operatorname{vol}(G/T) \left(\frac{2\pi}{\lambda}\right)^{n/2} \int_{\mathfrak{t}} \big\langle \exp(X, y(T))\big\rangle_* e^{-|X|^2/2\lambda}\, \det\nolimits_{\mathfrak{g}/\mathfrak{t}}(1+\operatorname{ad}(X))\, dX.\end{aligned}$$

We now apply the result of Lemma 2.6. By the Ito formula, we see that the Ito stochastic differential

$$\begin{aligned}\delta\left\{f(\gamma(t))e^{(X,y(t))}\right\} &= (df(\gamma(t)) + X, \delta x(t))e^{(X,y(t))}\\ &\quad + \left(-\tfrac{1}{2}\Delta f(\gamma(t)) + X(f)(\gamma(t)) + \tfrac{1}{2}|X|^2\right)e^{(X,y(t))}\end{aligned}$$

From this, it follows that $\left\langle e^{(X,y(T))}\right\rangle_*$ is given by the ratio of heat kernels

$$\frac{\left\langle e\right|\exp T\left(-\frac{1}{2}\Delta + X + \frac{1}{2}|X|^2\right)\left|e\right\rangle}{\left\langle e\right|\exp T\left(-\frac{1}{2}\Delta\right)\left|e\right\rangle} = \frac{e^{T|X|^2/2}k(T, e^{TX})}{k(T,e)},$$

since the vector field X commutes with the Laplacian Δ. □

Note that it is an easy consequence of this proposition that

$$\left\langle \exp\left(\frac{\lambda}{2}|y(T)|^2\right)\right\rangle_* < \infty$$

if and only if $\lambda < T^{-1}$.

3. The tubular neighbourhood of a fibre

In this section, we will explain Gross's idea of constructing a tubular neighbourhood in B of the fibre $\pi^{-1}(e)$ of the map π above the identity element of G. Introduce the family of balls

$$B_r = \{\exp(Y) \mid |Y| < T^{1/2}r\} \subset G,$$

where r and T are small. On such a ball, we will use radial coordinates; thus, we will write $h(x)$ instead of $h(\exp x)$ when h is a function on B_r.

Let R be a smooth vector field on G which on B_r equals the radial vector field of $\mathfrak{g}$. We introduce the vector field R because on B_r, its integral curves are the one-parameter semigroups of G.

Define a map $\varphi : \mathfrak{g} \times B \longrightarrow B$ by

$$\varphi(X, x) = x + T^{-1}\int_0^t \mathrm{Ad}(\gamma(s))^{-1}X\,ds.$$

Since $\gamma[\varphi(X,x)] = \exp(tX/T)\gamma(t)$, we see that

$$\pi(\varphi(X,x)) = \pi(x)\exp(X),$$

and hence that the map φ defines a tubular neighbourhood of the fibre $\pi^{-1}(e)$.

Given an element $x \in B$ such that $\pi(x) = \exp(X) \in B_{2r}$, we may form the path in B

$$\sigma \in [0,1] \mapsto x_\sigma = \varphi((\sigma - 1)X, x),$$

which covers the path $\exp(\sigma X)$ in G; in particular, x_0 lies in the fibre $\pi^{-1}(e)$. It is easy to check that x_σ is the integral curve for the vector field $\tilde{R}$.

If $F \in W^\infty(B)$, define the function $\tilde{F}$ to take the value

$$\tilde{F}(x) = \pi^* \psi \, F(x_0) \exp\left(\frac{1}{2} \int_0^1 \alpha(R)(x_\sigma) \, d\sigma\right) \tag{3.1}$$

at the path x, where $\psi \in C_c^\infty(B_{2r})$ is a smooth cut-off function which equals 1 on the ball B_r. It is clear that $\tilde{F}$ and F are equal on $\pi^{-1}(e)$, and that $\tilde{F}$ satisfies the differential equation

$$\tilde{R}(\tilde{F}) = \frac{\alpha(R)}{2} F \tag{3.2}$$

on $\pi^{-1}(B_r)$. In the remainder of this section, we will prove the following result, which expresses the fact that the tubular neighbourhood constructed above has a certain amount of regularity.

Theorem 3.3. *The function $\tilde{F}$ defined above lies in $W^\infty(B)$.*

The first step in the proof that $\tilde{F} \in W^\infty(B)$ is the special case where $F = 1$. If G_σ is a family of measurable functions on B, then by Leibniz's rule,

$$\exp\Big(\int_0^1 G_\sigma \, d\sigma\Big) \in W^\infty(B)$$

if $\exp(G_\sigma)$ is uniformly in $L^p(B)$, and G_σ is uniformly in $W^\infty(B)$, for all $\sigma \in [0,1]$. In our case, $G_\sigma = \frac{1}{2}\alpha(R)(x_\sigma)$. Thus, it suffices to prove the following lemma.

Lemma 3.4. *Let $\psi \in C_c^\infty(B_{2r})$ be such that $|\psi| \le 1$.*

(1) *The functional*
$$\pi^* \psi \, \alpha(R)(x_\sigma)$$
is in $W^\infty(B)$, uniformly in $\sigma \in [0,1]$.

(2) *The functional*
$$\pi^* \psi \, \exp(\alpha(R)(x_\sigma))$$
is in $L^p(B)$ for all $p < \infty$, uniformly in $\sigma \in [0,1]$.

Proof. If $X \in \mathfrak{g}$, then

$$\begin{aligned}\alpha(X) &= d^*\tilde{X} + \pi^*(X(\log k(T))) \\ &= T^{-1}(y(T), \pi^* X) + \pi^*(X(\log k(T)))\end{aligned}$$

and hence

$$\alpha(R)[x_\sigma] = T^{-1}(y_\sigma(T), \pi_\sigma^* R) + \pi_\sigma^*(R(\log k(T))),$$

where $y_\sigma = y[x_\sigma]$ and $\pi_\sigma(x) = \pi(x_\sigma)$. On $\pi^{-1}(B_{2r})$ the maps π_σ are uniformly W^∞, showing that $\pi^*\psi\, \pi_\sigma^*(R(\log k(T)))$ is uniformly in $W^\infty(B)$. To prove (1), we must prove that $y_\sigma(T)$ is uniformly in $W^\infty(B)$.

By the same argument as was used to prove Lemma 2.6 (1), we see that y_σ is given by the Ito integral

$$(*) \qquad y_\sigma(t) = \int_0^t \mathrm{Ad}((\sigma - 1)sX/T)\,\mathrm{Ad}(\gamma(s))\,\delta x_s + \frac{t(\sigma-1)}{T} X.$$

It follows from Theorem 2.19 of Kusuoka and Stroock [5] that $y_\sigma(T)$ is in $W^\infty(B)$, since this theorem shows that stochastic differential equations with smooth data have W^∞ solutions.

Let us now prove (2). If $X \in \mathfrak{g}$, then on inverse image by π of the ball B_{2r},

$$\begin{aligned}|\alpha(X)| &\le |d^*\tilde{X}| + \sup_{g \in B_{2r}} |X(\log k(T))| \\ &\le \frac{|X|}{T}|y(T)| + \frac{Cr}{T^{1/2}}\end{aligned}$$

It follows by (1.6) that on $\pi^{-1}(B_{2r})$,

$$|\alpha(R)[x_\sigma]| \le \frac{Cr}{T^{1/2}}|y_\sigma(T)| + Cr^2.$$

By $(*)$, we see that $y_\sigma(t) - t(\sigma-1)X/T$ is a Wiener process, and hence that

$$\langle e^{p|y_\sigma(T)|}\rangle \le e^{p(1-\sigma)|X|}\,\langle e^{p|x(T)|}\rangle < \infty,$$

proving (2). $\square$

It remains to be proved that $\pi^*\psi\, F(x_0)$ lies in $W^\infty(B)$ for any $\psi \in C_c^\infty(B_{2r})$. Observe that

$$\pi_*\Big(\Big|F(x_0)\,\exp\Big(\frac{1}{p}\int_0^1 \alpha(R)(x_\sigma)\,d\sigma\Big)\Big|^p\Big)$$

is constant on the ball B_{2r}, and is equal to its value at the identity, namely $\langle |F|^p\rangle_*$; this follows by the same method as was used to prove Corollary 1.8. This shows that the function

$$\pi^*\psi\, F(x_0)\, \exp\Big(\frac{1}{p}\int_0^1 \alpha(R)(x_\sigma)\, d\sigma\Big)$$

is in $L^p(B)$. It follows from Lemma 3.4 (2) that $\pi^*\psi(x)\, F(x_0) \in L^p(B)$ for all $p < \infty$. A similar argument shows that $\pi^*\psi\, |d^kF|^2(x_0) \in L^p(B)$ for all $p < \infty$, where $k \in \mathbb{N}$ and $d^kF \in W^\infty(B, H^{\otimes_2 k})$ is the tensor of k-th derivatives of F.

Denote the map $x \mapsto x_0$ by H; restricted to $\pi^{-1}(B_{2r})$, it is a Wiener map, that is, it lies in $I + W^\infty(\pi^{-1}(B_{2r}), H)$. The chain rule now shows that $\pi^*\psi\, |d^kF(x_0)|^2 \in L^p(B)$ for all $p < \infty$, and hence that $F \in W^\infty(B)$. To give an example, the second derivatives of $F(x_\sigma)$ are given by the formula

$$d^2F(x_0) = H^*(d^2F) \circ (dH \otimes_2 dH) + H^*(dF) \circ d^2H.$$

This completes the proof of Theorem 3.3.

4. The rough logarithmic Sobolev inequalities

Our goal in this section is to prove the following logarithmic Sobolev inequality.

Theorem 4.1. *There is a constant C such that for $F \in W^\infty(B)$, uniformly for small T,*

$$\langle F^2 \log F\rangle_* \le C\langle |d_*F|^2 + (T^{-1}|y(T)|^2 + 1)F^2\rangle_* + \tfrac{1}{2}\langle F^2\rangle_* \log\langle F^2\rangle_*.$$

The idea of the proof is as follows. If F is in $W^\infty(B)$, we use Theorem 3.3 to replace it by another W^∞-function $\tilde F$ equal to F on $\pi^{-1}(e)$ but which satisfies the ordinary differential equation

$$\tilde R(F) = \frac{\alpha(R)}{2}\tilde F. \tag{4.2}$$

It follows that $d_*F = d_*\tilde F$ on $\pi^{-1}(e)$, so that we may replace F by $\tilde F$ in proving the theorem.

Along the fibre $\pi^{-1}(e)$, the horizontal part $Nd\tilde F$ of the differential $d\tilde F$ may be identified by (4.2):

$$\begin{aligned}\tilde X \tilde F\big|_{\pi^{-1}(e)} &= \frac{1}{2}\alpha(X)\tilde F\big|_{\pi^{-1}(e)} \\ &= \frac{1}{2T}(y(T), X)F\big|_{\pi^{-1}(e)}.\end{aligned}$$

From this, we see that

$$\langle |d\tilde F|^2\rangle_* = \langle |d_*F|^2\rangle_* + \frac{1}{4T}\langle |y(T)|^2\, F^2\rangle_*.$$

Thus, the proof of Theorem 4.1 is reduced to that of the following result.

Theorem 4.3. *There is a constant C such that for positive $F \in W^\infty(B)$ satisfying* (4.2), *uniformly for small T,*

$$\langle F^2 \log F\rangle_* \le C\langle |dF|^2 + F^2\rangle_* + \tfrac{1}{2}\langle F^2\rangle_* \log\langle F^2\rangle_*.$$

If u is a smooth positive function on the unit ball $\{x \in \mathfrak{g} \mid |x| < 1\}$ such that

$$\int_{\mathfrak{g}} u^2 \, dx = (4\pi)^{n/2},$$

we define u_r to be the rescaled function $u_r(\exp x) = r^{-n/2} u(x/T^{1/2} r)$ on B_r. We show that the logarithmic Sobolev inequality for $(\pi^* u_r)F$ on B implies the logarithmic Sobolev inequality for F on the fibre $\pi^{-1}(e)$, once r is chosen sufficiently small. This is done by using Gronwall's inequality applied to the ordinary differential equation (4.2) to relate the integrals over the fibre $\pi^{-1}(x)$, for $x \in B_r$,

$$\pi_*(F^2 \log F)(x) \text{ and } \pi_*(|dF|^2)(x),$$

to the analogous integrals over the fibre $\pi^{-1}(e)$,

$$\langle F^2 \log F\rangle_* \text{ and } \langle |dF|^2\rangle_*.$$

Lemma 4.4. *Let $F \in W^\infty(B)$ be a positive function satisfying* (4.2) *and such that $\langle F^2\rangle = 1$. Then there is a constant C such that the following inequality holds uniformly for small T and r:*

$$\langle F^2 \log F\rangle_* \le (1 + O(T + r)) \int_G u_r^2 \, \pi_*(F^2 \log F) \, d(\pi_*\mu) + O(1).$$

Proof. Denote by $\varphi(x)$ the function $x^2 \log x + 1$; we introduce the function φ because it is positive on the positive real interval.

Corollary 1.8 combined with (4.2) shows that the radial derivative of $\pi_*(\varphi(F))$ in the direction $x \in \mathfrak{g}$ equals

$$\frac{d}{dt}\pi_*(\varphi(F))(tx) = \frac{|x|}{2}\pi_*(\alpha(\hat{x})F^2)(tx),$$

where $\hat{x} = |x|^{-1}x$. Since $T^{-1/2}|x| \le r$ on B_r, the Hausdorff-Young inequality shows that

$$\frac{d}{dt}\pi_*(\varphi(F))(tx) \ge -r\pi_*(\varphi(F))(tx) - \frac{r}{2}\pi_*\big(e^{T^{1/2}|\alpha(\hat{x})|}\big)(tx)$$

on the set B_r. By Gronwall's inequality,

$$(*) \qquad \pi_*(\varphi(F))(x) \geq e^{-r} \langle \varphi(F) \rangle_* - \frac{r}{2} \int_0^1 \pi_* H(tx)\, dt,$$

where $H = \sup_{\hat{x} \in S^{n-1}} \big(e^{T^{1/2} |\alpha(\hat{x})|} \big)$.

By the asymptotic expansion for the heat-kernel $k(T, g)$ on G for small T,

$$\int_G u_r^2 \, d(\pi_* \mu) = 1 + O(T + r^2).$$

Multiplying $(*)$ by u_r^2 and integrating over G with respect to the measure $\pi_* \mu$, we see that

$$\begin{aligned} &\int_G u_r^2 \, \pi_*(\varphi(F)) \, d(\pi_* \mu) \\ &\qquad \geq (1 + O(T + r)) \, \langle \varphi(F) \rangle_* - O(r^{1-n}) \int_{B_r} \left(\int_0^1 \pi_* H(tx) \, dt \right) d(\pi_* \mu). \end{aligned}$$

The second term on the right-hand side is estimated by replacing the measure $d(\pi_*\mu)$ by the equivalent measure $T^{n/2}\, dx$ (see (2.8)), and then changing variables from x to $y = tx$:

$$\begin{aligned} \int_{B_r} \left(\int_0^1 (\pi_* H)(tx) \, dt \right) d(\pi_* \mu) &\leq C_2 T^{-n/2} \int_{B_r} \left(\int_0^1 (\pi_* H)(tx) \, dt \right) dx \\ &\leq C_2 T^{-n/2} \int_{B_r} \left(\int_{|y|/T^{1/2} r}^1 t^{-n} \, dt \right) (\pi_* H)(y) \, dy \\ &\leq C_2 \, T^{-1/2} \, r^{n-1} \int_{B_r} |y|^{1-n} (\pi_* H)(y) \, dy. \end{aligned}$$

Hölder's inequality with respect to the measure dy on B_r now shows that if $s > n$,

$$\int_{B_r} |y|^{1-n} \pi_* H \, dy \leq C(n, s) \, (T^{1/2} r)^{1-n/s} \left(\int_{B_r} (\pi_* H)^s \, dy \right)^{1/s}.$$

Applying Hölder's inequality along the fibres of π shows that

$$\begin{aligned} \int_{B_r} (\pi_* H)^s \, dy &\leq C_1^{-1} T^{n/2} \int_{B_r} (\pi_* H)^s \, d(\pi_* \mu) \\ &\leq C_1^{-1} T^{n/2} \int_{B_r} \pi_*(H^s) \, d(\pi_* \mu) = C_1^{-1} T^{n/2} \| H \|_s^s. \end{aligned}$$

Combining all of this, we see that

$$\int_{B_r} \left(\int_0^1 \pi_* H(tx)\,dt \right) d(\pi_*\mu) \le C r^{1-n/s} \|H\|_s,$$

where C is a constant depending on C_1, C_2, s and $C(n,s)$, but not on T. We may as well choose $s = 2n$, but any real number greater than n will do equally well.

It remains to prove that $\|H\|_s < \infty$. First of all, note that H may be bounded using an orthonormal basis x_i of $\mathfrak{g}$, as follows:

$$H = \sup_{\hat{x}\in S^{n-1}} \left(e^{T^{1/2}|\alpha(\hat{x})|} \right) \le e^{T^{1/2}(|\alpha(x_1)|+\cdots+|\alpha(x_n)|)}.$$

It follows that

$$\begin{aligned} \|H\|_s^s &= \int_B \sup_{\hat{x}\in S^{n-1}} e^{sT^{1/2}|\alpha(\hat{x})|}\,d\mu \le \int_B e^{sT^{1/2}(|\alpha(x_1)|+\cdots+|\alpha(x_n)|)}\,d\mu \\ &\le \left(\prod_{i=1}^n \int_B e^{nsT^{1/2}|\alpha(x_i)|}\,d\mu \right)^{1/n}. \end{aligned}$$

If $X \in \mathfrak{g}$, then

$$\begin{aligned} \alpha(X) &= d^*\tilde{X} - \pi^*(\operatorname{div}_{\pi_*\mu} X) \\ &= T^{-1}(y(T), X) - \pi^*(X(\log k(T))). \end{aligned}$$

We see by (2.8) that for small $T > 0$,

$$\begin{aligned} T^{1/2}|\alpha(X)| &\le \frac{C\|X\|_0}{T^{1/2}}(|y(T)| + \delta(\gamma(T))) \\ &\le \frac{\varepsilon}{4T}(|y(T)|^2 + \delta(\gamma(T))^2) + \varepsilon^{-1}C^2\|X\|_0^2, \end{aligned}$$

for some constant C depending only on G; here, ε is an arbitrary positive constant. It now follows by the estimates of Corollary 2.7 that the integral $\left\langle e^{T^{1/2}|\alpha(X)|} \right\rangle$ is uniformly bounded for small T, proving that $\|H\|_s < \infty$.

In this way, we have proved that

$$\begin{aligned} \int_G u_r^2\,\pi_*(F^2 \log F)\,d(\pi_*\mu) &= \int_G u_r^2\,\pi_*(\varphi(F))\,d(\pi_*\mu) - (1 + O(T + r^2)) \\ &\ge (1 + O(T + r))\,\langle F^2 \log F\rangle_* - O(1), \end{aligned}$$

which after a little rearrangement gives the lemma. □

If we apply the logarithmic Sobolev inequality for the Wiener space B to the function $(\pi^* u_r)F \in W^\infty(B)$, which satisfies $\langle |(\pi^* u_r)F|^2\rangle = 1 + O(T + r^2)$, we obtain the inequality

$$\langle |(\pi^* u_r)F|^2 \log((\pi^* u_r)F)\rangle \le \langle |d((\pi^* u_r)F)|^2\rangle + O(T + r^2).$$

Since

$$|(\pi^* u_r)F|^2 \log((\pi^* u_r)F) = (\pi^* u_r)^2 F^2 \log F + \pi^*(u_r^2 \log u_r) F^2$$

and $\pi_*(F^2) = 1$, we see that

$$\begin{aligned}\int_{B_r} u_r^2 \, \pi_*(F^2 \log F)\, d(\pi_*\mu) &\le \int_{B_r} u_r^2 \, \pi_*(|dF|^2)\, d(\pi_*\mu) \\ &\quad + \int_{B_r} \pi_*(|d(\pi^* u_r)|^2)\, d(\pi_*\mu) - \int_{B_r} u_r^2 \log u_r \, d(\pi_*\mu) + O(T + r^2).\end{aligned}$$

To handle the second term on the right-hand side, we use (2.5), which shows that

$$\int_{B_r} \pi_*(|d(\pi^* u_r)|^2)\, d(\pi_*\mu) = O(r^{-2}),$$

while to bound the third term, we use the fact that $x^2 \log x \ge -(2e)^{-1}$. Thus, we see that

$$\int_{B_r} u_r^2 \, \pi_*(F^2 \log F)\, d(\pi_*\mu) \le \int_{B_r} u_r^2 \, \pi_*(|dF|^2)\, d(\pi_*\mu) + O(r^{-2}). \tag{4.5}$$

To complete the proof of Theorem 4.3, we will imitate the proof of Lemma 4.4 to obtain an upper bound for

$$\int_{B_r} u_r^2 \, \pi_*(|dF|^2 + \varepsilon F^2 \log F)\, d(\pi_*\mu)$$

in terms of $\langle |dF|^2 + \varepsilon F^2 \log F\rangle_*$, where ε is a small positive constant.

Lemma 4.6. *Let $F \in W^\infty(B)$ be a positive function satisfying* (4.2) *and such that $\pi_*(F^2) = 1$ on the ball B_r. Then there is a constant C such that the following inequality holds uniformly for small T and r:*

$$\int_G u_r^2 \, \pi_*(|dF|^2 + \varepsilon F^2 \log F)\, d(\pi_*\mu) \le (1 + O(T + r))\, \langle |dF|^2 + \varepsilon F^2 \log F\rangle_* + O(1).$$

Proof. If $F \in W^\infty(B)$ satisfies the ordinary differential equation (4.2) along the one-parameter semigroup $\exp(tX) \subset G$, it follows that

$$\begin{aligned} X\pi_*(|dF|^2) &= \pi_*(\tilde{X}|dF|^2) - \pi_*(\alpha(X)\,|dF|^2) \\ &= 2\pi_*(dF, [\tilde{X}, d]F) - \pi_*(F\,dF, d\alpha(X)). \end{aligned}$$

The first term of the right-hand side is bounded by means of the formula

$$[d, \tilde{X}] = T^{-1}(\Pi^*\rho(\nabla X)\Pi + X \cdot d\Pi, d),$$

where $\rho(\nabla X)$ is the section of the bundle $\operatorname{End}(TM)$ over M corresponding to $\nabla X \in \Gamma(M, TM \otimes TM)$. It is clear that the Hilbert-Schmidt norm of $T^{-1}\Pi^*\rho(\nabla X)\Pi$ is uniformly bounded for small T, and the same is true for $T^{-1}X \cdot d\Pi$ by Proposition 2.6. In this way, we obtain the inequality

$$X\pi_*(|dF|^2) \le \pi_*\big(C|dF|^2 + |F|\,|dF|\,|d\alpha(X)|\big).$$

Applying the Cauchy inequality

$$|F|\,|dF|\,|d\alpha(X)| \le T^{-1/2}|dF|^2 + T^{1/2}F^2\,|d\alpha(X)|^2,$$

we see that

$$X\pi_*(|dF|^2) \le \pi_*\big((T^{-1/2} + C)|dF|^2 + T^{1/2}F^2\,|d\alpha(X)|^2\big).$$

We can now bound the radial derivative of $\pi_*(|dF|^2 + r\varphi(F))$, in the direction $x \in \mathfrak{g}$, where $\varphi(x) = x^2 \log x + 1$. On the set B_r, it satisfies the bound, uniform in T for T small,

$$\begin{aligned} &\frac{d}{dt}\pi_*(|dF|^2 + \varepsilon\varphi(F))(tx) \\ &\qquad \le O(r)\pi_*(|dF|^2 + \varepsilon\varphi(F))(tx) + O(r)\pi_*\big(F^2\,(T|d\alpha(\hat{x})|^2 + rT^{1/2}|\alpha(\hat{x})|)\big)(tx) \\ &\qquad \le O(r)\pi_*(|dF|^2 + \varepsilon\varphi(F))(tx) + O(\varepsilon r)\pi_*\big(e^{\varepsilon^{-1}T|d\alpha(\hat{x})|^2} + e^{T^{1/2}|\alpha(\hat{x})|}\big)(tx), \end{aligned}$$

where we have applied the Hausdorff-Young inequalities

$$\begin{aligned} F^2\,(T|d\alpha(\hat{x})|^2) &\le \frac{\varepsilon}{2}\varphi(F) + \varepsilon e^{\varepsilon^{-1}T|d\alpha(\hat{x})|^2}, \quad \text{and} \\ F^2\,(T^{1/2}|\alpha(\hat{x})|) &\le \tfrac{1}{2}\varphi(F) + e^{T^{1/2}|\alpha(\hat{x})|}. \end{aligned}$$

By Gronwall's inequality,

$$\pi_*(|dF|^2 + \varepsilon\varphi(F))(x) \geq (1 + O(r))\langle |dF|^2 + \varepsilon\varphi(F)\rangle_* + O(\varepsilon r)\int_0^1 \pi_* J(tx)\,dt,$$

where

$$J = \sup_{\hat{x}\in S^{n-1}} \big(e^{\varepsilon^{-1}T|d\alpha(\hat{x})|^2} + e^{T^{1/2}|\alpha(\hat{x})|}\big)(tx).$$

The rest of the proof is the same as that of Lemma 4.4, except that we must bound $\|J\|_s$ instead of $\|H\|_s$. Let $X \in \mathfrak{g}$. Since $d(\pi^* f) = T^{1/2}\pi^*(df)$, we see that

$$\begin{aligned} d\alpha(X) &= T^{-1}(dy(T), \pi^* X) - d\pi^*(X(\log k(T))) \\ &= T^{-1}(dy(T), X) - T^{1/2}\pi^*(d(X(\log k(T)))) \end{aligned}$$

It follows that

$$\begin{aligned} T|d\alpha(X)|^2 &\leq \frac{2\|X\|_0^2}{T}|dy(T)|^2 + 2T^2|d(X(\log k(T)))|^2 \\ &\leq \frac{2\|X\|_0^2}{T}|dy(T)|^2 + C_0(G)\|X\|_0^2 + C_1(G)T\|X\|_1^2, \end{aligned}$$

where the constants $C_i(G)$ depend only on the group G. The uniform bound on $\langle e^{\varepsilon^{-1}T|d\alpha(X)|^2}\rangle$ for T small enough follows from Corollary 2.7, and we see that $\|J\|_s < \infty$ uniformly. $\square$

Let us assemble the results obtained so far in this section. Under the conditions on the function F of Lemma 4.4, we see by combining Lemma 4.4 and (4.5) that

$$\langle F^2 \log F\rangle_* \leq (1 + O(T + r))\int_{B_r} u_r^2\,\pi_*(|dF|^2)\,d(\pi_*\mu) + O(r^{-2}).$$

Combining this with Lemma 4.6, we see that

$$\langle F^2 \log F\rangle_* \leq (1 + O(T + r))\langle |dF|^2 + \varepsilon F^2 \log F\rangle_* + O(r^{-2}).$$

If we choose ε sufficiently small (so that $(1 + O(T + r))\varepsilon \leq \frac{1}{2}$), we obtain the logarithmic Sobolev inequality

$$\langle F^2 \log F\rangle_* \leq (1 + O(T + r))\langle |dF|^2\rangle_* + O(r^{-2})\langle F^2\rangle_* + \tfrac{1}{2}\langle F^2\rangle_* \log\langle F^2\rangle_*,$$

where we have now removed the condition that $\langle F^2\rangle_* = 1$. This immediately leads to Theorem 4.3.

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