THE EQUIVARIANT CHERN CHARACTER FOR NON-COMPACT LIE GROUPS

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To Raoul Bott on his seventieth birthday

Let G be a topological group, with classifying bundle EG. If M is a topological space with left G-action, the equivariant cohomology of M, denoted $H^*_G(M)$, is the cohomology of a homotopy quotient $M_G = EG \times_G M$. Equivariant cohomology satisfies axioms similar to those of ordinary cohomology: the analogue of the dimension axiom asserts that $H^*_G(G/H) \cong H^*(BH)$ if H is a closed subgroup of G.

If E is an equivariant complex vector bundle over M, then $E_G = EG \times_G E$ is a vector bundle over M_G, with Chern character

$$\mathrm{Ch}(E_G) = \sum_{\ell=0}^{\infty} \mathrm{Ch}^{2\ell}_G(E_G),$$

where $\mathrm{Ch}^{2\ell}(E_G) \in H^{2\ell}(M_G) = H^{2\ell}_G(M)$. This class is called the equivariant Chern character of E, and denoted $\mathrm{Ch}_G(E)$. In this article, we present a new model for $H^*_G(M)$ when G is a Lie group acting smoothly on a differentiable manifold M, and obtain an explicit formula for the equivariant Chern class $\mathrm{Ch}_G(E)$ of an equivariant vector bundle, similar to the Chern-Weil formula for the ordinary Chern class.

In the case that the group G is discrete, our construction specializes to the equivariant Chern character constructed by Bott [4]. If M is a smooth manifold on which a discrete group G acts smoothly, the double complex $C^*(G, \mathcal{A}^*(G))$ of cochains on G with values in the G-module of differential forms $\mathcal{A}^*(M)$ has two commuting differentials: the coboundary $\bar{d}$ and the exterior differential d. Bott shows that the cohomology of the total complex of $C^*(G, \mathcal{A}^*(G))$ with the differential $\bar{d} + d$ is naturally isomorphic to the equivariant cohomology $H^*_G(M)$ of M, and gives a formula for the equivariant Chern character of an equivariant vector bundle E as an element of the complex $C^*(G, \mathcal{A}^*(M))$. We now recall his formula.

Denote by Δ^k the k-simplex $\Delta^k = \{0 \le t_1 \le \cdots \le t_k \le 1\}$, and by

$$\int_{\Delta^k} : \mathcal{A}^*(\Delta^k \times M) \longrightarrow \mathcal{A}^{*-k}(M)$$

The author is partially supported by a fellowship of the Sloan Foundation, and a research grant of the NSF..

the operation of integration over the fibres of the projection $\Delta^k \times M \to M$, normalized in such a way that the integral of the differential form $dt_1 \wedge \ldots \wedge dt_k \wedge \alpha(t)$ equals the Riemann integral of $\alpha(t)$ over the simplex. Stokes's theorem takes the form

$$\int_{\Delta^k} d - (-1)^k d \int_{\Delta^k} + \int_{\partial\Delta^k} = 0.$$

Given connections ∇_i on a vector bundle E over M, we form a connection

$$d_{\Delta^k} + \nabla(t) = d_{\Delta^k} + t_1(\nabla_0 - \nabla_1) + \cdots + t_k(\nabla_{k-1} - \nabla_k) + \nabla_k$$

on the pull-back of E by the projection $\Delta^k \times M \to M$, with curvature

$$(d_{\Delta^k} + \nabla(t))^2 = \sum_{i=1}^k dt_i \wedge (\nabla_{i-1} - \nabla_i) + F(t),$$

where $F(t)$ is the curvature of the connection $\nabla(t)$

The Chern-Simons form of the connections $(\nabla_0, \ldots, \nabla_k)$ is the integral of the Chern character of the connection $d_{\Delta^k} + \nabla(t)$ over the fibres of the projection from $\Delta^k \times M$ to M:

$$\operatorname{cs}(\nabla_0, \ldots, \nabla_k) = \int_{\Delta_k} \operatorname{Tr}\bigl(\exp(d_{\Delta^k} + \nabla(t))^2\bigr).$$

Stokes's Theorem shows that

$$d \operatorname{cs}(\nabla_0, \ldots, \nabla_k) = \sum_{i=0}^k (-1)^i \operatorname{cs}(\nabla_0, \ldots, \widehat{\nabla}_i, \ldots, \nabla_k).$$

Suppose that M is a differentiable manifold on a Lie group G acts smoothly, and that E is an equivariant bundle with connection ∇. Define an element $\operatorname{Ch}_G(E, \nabla)$ of $C^*(G, \mathcal{A}^*(M))$ by the formula

$$\operatorname{Ch}_G(E, \nabla)(g_1, \ldots, g_k) = (-1)^k \operatorname{cs}(\nabla^{\gamma_0}, \ldots, \nabla^{\gamma_k});$$

here, $\nabla^{\gamma_i} = \gamma_i \nabla \gamma_i^{-1}$ is the conjugate of the connection ∇ by $\gamma_i = g_{i+1} \ldots g_k$. The formula for $d \operatorname{cs}(\nabla_0, \ldots, \nabla_k)$ shows that $\operatorname{Ch}_G(E, \nabla)$ is a cocycle, $(\bar{d} + d) \operatorname{Ch}_G(E, \nabla) = 0$, that is, that for each $k \geq 0$,

$$\begin{aligned}\operatorname{Ch}_G(E, \nabla)(g_2, \ldots, g_k) + \sum_{i=1}^{k-1} (-1)^i \operatorname{Ch}_G(E, \nabla)(g_1, \ldots, g_i g_{i+1}, \ldots, g_k) \\ + (-1)^k g_k \operatorname{Ch}_G(E, \nabla)(g_1, \ldots, g_{k-1}) + (-1)^k d \operatorname{Ch}_G(E, \nabla)(g_1, \ldots, g_k) = 0,\end{aligned}$$

Results of Dupont [5] show that $\operatorname{Ch}_G(E, \nabla)$ represents the Chern equivariant character of E.

Our model for equivariant cohomology is inspired by H. Cartan's model for equivariant cohomology when G is compact (see [1], Chapter 7). He considers the graded tensor product $\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M)$ of polynomials on the Lie algera $\mathbf{g}$ of G with values in the differential forms on M, where a differential form is assigned its usual degree, and a polynomial on $\mathbf{g}$ is assigned twice its degree. There are two natural differentials on this space: the first is the exterior differential $d = 1\otimes d$, while the second may be expressed in terms of a basis X_α of $\mathbf{g}$ and dual basis ω^α of $\mathbf{g}^* \subset \mathbb{C}[\mathbf{g}]$ by the formula

$$\iota = \sum_\alpha \omega^\alpha \iota(X_\alpha^M),$$

where X_α^M is the vector field on M corresponding to the infinitesimal action of $X_\alpha \in \mathbf{g}$. Both of these operators have total degree 1.

The operators d and ι do not commute: rather, we have the basic formula

$$d\iota + \iota d = \sum_\alpha \omega^\alpha \mathcal{L}(X_\alpha^M) = \mathcal{L},$$

where $\mathcal{L}(X_\alpha^M)$ denotes the Lie derivative with respect to the vector field X_i^M.

On the space of invariants $(\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M))^G$, the operator $\mathcal{L}$ vanishes, so that $d+\iota$ is a differential. Cartan shows that the cohomology of the complex $\big((\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M))^G, d+\iota\big)$ is naturally isomorphic to $H_G^*(M)$. Berline and Vergne [2] (see also [1], Chapter 7) have given a formula for the equivariant Chern character of an equivariant vector bundle in this complex.

The complex of equivariant differential forms $\mathcal{A}_G^*(M)$ which we study in this article is the space of differentiable group cochains $C^*(G, \mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M))$: this generalizes both Bott's complex $C^*(G,\mathcal{A}^*(M))$, when G is discrete, and Cartan's complex $(\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M))^G$, when G is compact. We define the differential d_G on $\mathcal{A}_G^*(M)$ in Section 2; it has the form

$$d_G = (\bar{d}+\bar{\iota}) + (d+\iota),$$

where $\bar{\iota}$ is a new operator, introduced in order that $d_G^2 = 0$. It follows from the theory of hypercohomology of differential graded coalgebras of Eilenberg and Moore [6] that the cohomology of this complex is naturally isomorphic to $H_G^*(M)$; we recall their theory in Section 1. In Section 2, we also construct an explicit quasi-isomorphism from the cobar complex $\Omega(\mathbb{C},\mathcal{A}^*(G),\mathcal{A}^*(M))$ to $C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M))$, which intertwines $\bar{d}$ with $\bar{d}+\iota$, and d with $d+\bar{\iota}$.

Let P be an equivariant principal bundle P with compact structure group H over M, and let θ be a connection form on P. In Section 3, we construct an equivariant Chern-Weil map $H^*(BH) \longrightarrow \mathcal{A}_G^*(M)$. Our construction specializes to that of Bott if G is discrete, and to that of Berline and Vergne if G is compact and ∇ is invariant under G. Our main tool in Section 3 is the complex of differential forms $\mathcal{A}^*(|M_\bullet|)$ on a simplicial manifold introduced by Dupont [5].

We would like to thank J. Block, R. Bott, H. Miller and N. O'Brian for a number of useful ideas in connection with this article.

Conventions

We will always take cohomology with coefficients in $\mathbb{C}$, and all vector bundles will be complex. All complexes will be of cohomological type (i.e. the differential raises degree), and concentrated in positive degree. We write $[a,b]$ for the supercommutator $ab-(-1)^{|a|\,|b|}ba$.

In the definition of the Chern character, we will omit the customary factors of $-2\pi i$: thus, our Chern character lies in $\sum_{k=0}^{\infty} H^{2k}(M,(2\pi i)^k\mathbb{Q})$, just as in algebraic geometry.

1. Cohomology of differential G-modules

1.1. **Coalgebras and comodules.** In this paragraph, we recall the definition of the cohomology of DGC (differential graded coaugmented) coalgebras (Eilenberg and Moore [6] and [9]) to the setting of de Rham cohomology. We will work in the category of topological vector spaces, and denote the projective tensor product $X\otimes_\pi Y$ by $X\otimes Y$; this has the property that if M and N are manifolds, $C^\infty(M)\otimes C^\infty(N)\cong C^\infty(M\times N)$. Since we only consider complexes concentrated in positive degree, there is no ambiguity in the definition of $K\otimes L$, when K and L are topological complexes. All maps between complexes are understood to be of degree 0 and preserve the differentials.

Definition 1.1.1. A topological DGC coalgebra is a topological complex Λ, with

(1) comultiplication $\delta:\Lambda\longrightarrow\Lambda\otimes\Lambda$ and counit $\varepsilon:\Lambda\longrightarrow\mathbb{C}$;
(2) coaugmentation, that is, a map of differential graded coalgebras $\eta:\mathbb{C}\longrightarrow\Lambda$.

A (continuous) differential graded (left) comodule over a topological DGC coalgebra is a topological complex with coaction $\nabla:\mathsf{A}\longrightarrow\Lambda\otimes\mathsf{A}$, where again the tensor product is understood to mean the projective tensor product. Right comodules are defined similarly. The coaugmentation on Λ makes $\mathbb{C}$ into a comodule in a canonical way. Other examples of comodules are given by the extended comodules, namely, those of the form $\Lambda\otimes V$ for some topological complex V.

Given a right comodule A and a left comodule B, we may form the cotensor product

$$\mathsf{A}\,\square_\Lambda\,\mathsf{B}=\ker\big(\mathsf{A}\otimes\mathsf{B}\xrightarrow{\nabla\otimes1-1\otimes\nabla}\mathsf{A}\otimes\Lambda\otimes\mathsf{B}\big).$$

In particular, we have the space of coinvariants (or primitive elements) $\mathbb{C}\,\square_\Lambda\,\mathsf{A}$ of a differential graded comodule A over a DGC coalgebra Λ.

The definition of the hypercohomology $\mathbb{H}(\Lambda, -) = \operatorname{Cotor}^{\Lambda}(\mathbb{C}, \mathsf{A})$ of $\mathbb{C} \,\square_{\Lambda} -$ makes use of a notion of injectivity for topological comodules based on Eilenberg and Moore's definition of injectivity over a DGC coalgebra, and on Hochschild and Mostow's definition of injectivity over a Lie group [7].

Definition 1.1.2. A continuous differential graded comodule A over a topological DGC coalgebra is injective if there is a morphism $f : \Lambda \otimes \mathsf{A} \longrightarrow \mathsf{A}$ of continuous differential comodules such that the composition $\mathsf{A} \xrightarrow{\nabla} \Lambda \otimes \mathsf{A} \xrightarrow{f} \mathsf{A}$ is the identity.

Equivalently, a comodule is injective if and only if it is a direct summand of an extended comodule.

We can now define the hypercohomology of a comodule A. Choose an injective resolution of $\mathbb{C}$,

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathsf{X}^0 \longrightarrow \mathsf{X}^1 \longrightarrow \dots,$$

that is, a split exact sequence of topological complexes such that each comodule X^i is injective. The hypercohomology $\mathbb{H}(\Lambda, \mathsf{A})$ of a left comodule A is defined as the cohomology of the double complex

$$0 \longrightarrow \mathsf{X}^0 \,\square_{\Lambda} \mathsf{A} \longrightarrow \mathsf{X}^1 \,\square_{\Lambda} \mathsf{A} \longrightarrow \dots$$

This definition is independent of the injective resolution $\mathsf{X}^\bullet$.

The functor $\operatorname{Cotor}_{\Lambda}(\mathsf{A}, \mathsf{B})$ may be calculated by the cobar resolution

$$\Omega^k(\mathsf{A}, \Lambda, \mathsf{B}) = \mathsf{A} \otimes \bar{\Lambda}^{\otimes k} \otimes \mathsf{B},$$

where $\bar{\Lambda}$ denotes the kernel of the counit $\varepsilon : \lambda \longrightarrow \mathbb{C}$. The differential of the cobar complex equals the sum of the internal differential and the operator

$$\begin{aligned}
\bar{d}(a \otimes \lambda_1 \otimes \dots \lambda_k \otimes b) &= \tilde{\nabla}_{\mathsf{A}} a \otimes \lambda_1 \otimes \dots \otimes \lambda_k \otimes b \\
&\quad + \sum_{i=1}^{k} (-1)^{|a|+|\lambda_1|+\dots+|\lambda_{i-1}|+i} a \otimes \lambda_1 \otimes \dots \otimes \tilde{\delta}\lambda_i \otimes \dots \otimes \lambda_k \otimes b \\
&\quad + (-1)^{|a|+|\lambda_1|+\dots+|\lambda_k|+k+1} a \otimes \lambda_1 \otimes \dots \otimes \lambda_k \otimes \tilde{\nabla}_{\mathsf{B}} b.
\end{aligned}$$

Here, $\tilde{\delta} : \Lambda \longrightarrow \Lambda \otimes \Lambda$ denotes the composition of δ with the endomorphism

$$\lambda_1 \otimes \lambda_2 \mapsto (-1)^{|\lambda_1|} \lambda_1 \otimes \lambda_2,$$

of $\Lambda \otimes \Lambda$, and similar definitions hold for $\tilde{\nabla}_{\mathsf{A}}$ and $\tilde{\nabla}_{\mathsf{B}}$.

In the special case that $\Lambda = C^\infty(G)$ is the coalgebra of smooth functions on a Lie group G, we denote the cobar complex $\Omega(\mathbb{C}, C^\infty(G), \mathsf{A})$ by $C^*(G, \mathsf{A})$: this is the complex of smooth G-cochains.

1.2. **Equivariant cohomology.** The DGC coalgebra which will interest us in this article is the space of differential forms $\mathcal{A}^*(G)$ on a Lie group G: the coproduct is defined by the pullback with respect to the multiplication map $m : G \times G \longrightarrow G$ of G,

$$\delta = m^* : \mathcal{A}^*(G) \longrightarrow \mathcal{A}^*(G \times G) \cong \mathcal{A}^*(G) \otimes \mathcal{A}^*(G),$$

the counit $\varepsilon : \mathcal{A}^*(G) \longrightarrow \mathbb{C}$ is evaluation at the identity $e \in G$, and the coaugmentation $\eta : \mathbb{C} \longrightarrow \mathcal{A}^*(G)$ maps $1 \in \mathbb{C}$ to the constant function $1 \in \mathcal{A}^0(G)$.

If M is a manifold with a smooth left action $\rho : G \times M \longrightarrow M$ of G, the space of differential forms $\mathcal{A}^*(M)$ on M is a left $\mathcal{A}^*(G)$-comodule, with coaction

$$\nabla = \rho^* : \mathcal{A}^*(M) \longrightarrow \mathcal{A}^*(G \times M) \cong \mathcal{A}^*(G) \otimes \mathcal{A}^*(M).$$

Similarly, if G acts on the right of M, $\mathcal{A}^*(M)$ is a right $\mathcal{A}^*(G)$-comodule.

Let G be a Lie group. A complex (A, d) is called a differential G-module if the following additional structures are provided:

(1) a smooth action of G, with infinitesimal action $\mathcal{L} : \mathbf{g} \otimes \mathsf{A} \longrightarrow \mathsf{A}$ of the Lie algebra $\mathbf{g}$ of G;

(2) a linear map $\iota : \mathbf{g} \otimes \mathsf{A} \longrightarrow \mathsf{A}$ of degree -1 such that for all $X \in \mathbf{g}$, $\iota(X)^2 = 0$ and
$$d\iota(X) + \iota(X)d = \mathcal{L}(X).$$

If A is a differential G-module, we define its horizontal and basic subspaces

$$\begin{aligned} \mathsf{A}_{\mathrm{hor}} &= \{v \in \mathsf{A} \mid \iota(X)v = 0 \text{ for all } X \in \mathbf{g}\}, \\ \mathsf{A}_{\mathrm{bas}} &= \{v \in \mathsf{A} \mid \iota(X)v = 0 \text{ and } gv = v \text{ for all } X \in \mathbf{g} \text{ and } g \in G\}. \end{aligned}$$

The basic subspace $\mathsf{A}_{\mathrm{bas}}$ is a subcomplex of A.

Proposition 1.2.1. *There is a natural bijection between differential G-modules and differential graded left comodules over $\mathcal{A}^*(G)$, under which the space of coinvariants $\mathbb{C}\,\square_{\mathcal{A}^*(G)}\,\mathsf{A}$ is identified with the basic complex A_{bas}.*

Proof. Let A be a differential graded comodule over $\mathcal{A}^*(G)$. The map of coalgebras $\mathcal{A}^*(G) \longrightarrow \mathcal{A}^0(G)$ induces a coaction of the coalgebra $\mathcal{A}^0(G)$ on A, in other words a smooth action of G on A. Further, given an element $X \in \mathbf{g}$, we obtain a linear map $\iota(X) : \mathsf{A} \longrightarrow \mathsf{A}$ of degree -1 by composing the arrows

$$\mathsf{A} \xrightarrow{\nabla} \mathcal{A}^*(G) \otimes \mathsf{A} \xrightarrow{\iota(X)\otimes 1} \mathcal{A}^*(G) \otimes \mathsf{A} \xrightarrow{\varepsilon\otimes 1} \mathsf{A}.$$

We leave it to the reader to check that this gives A the structure of a differential G-module, and that all $\mathcal{A}^*(G)$-comodules are obtained in this way. □

The following result is similar to Lemma 5.2 of Hochschild and Mostow.

Proposition 1.2.2. *If P carries a locally trivial action of G, then $\mathcal{A}^*(P)$ is an injective $\mathcal{A}^*(G)$-comodule.*

Proof. Choose a Haar measure dg on G. Since the action of G on P is locally trivial, there is a smooth function ϕ on P whose integral over the fibres equals 1, with respect to the vertical density on P induced by the Haar measure on G. Let $f : \mathcal{A}^*(G)\otimes\mathcal{A}^*(P) \to \mathcal{A}^*(P)$ be the map given by the formula

$$f(\omega) = \int_G g^*(\phi\omega_g)\, dg,$$

where ω_g is the restriction of ω to the subspace $\{g\} \times P$ of $G \times P$. We leave it to the reader to check that $f \cdot \nabla$ is the identity on $\mathcal{A}^*(P)$. ☐

Corollary 1.2.3. *If M is a manifold with smooth action of a Lie group G,*

$$\mathbb{H}(\mathcal{A}^*(G), \mathcal{A}^*(M)) \cong H^*_G(M).$$

Proof. Let EG be a classifying G-bundle (which we may choose to be a Hilbert manifold with smooth action of G) and let $i : \mathbb{C} \to \mathcal{A}^*(EG)$ be the inclusion of the constant functions. The map i splits, since evaluation at any point of EG gives a section $\mathcal{A}^*(EG) \to \mathbb{C}$. Since $\mathcal{A}^*(EG)$ is injective by Proposition 1.2.2, and i is a quasi-isomorphism by the de Rham theorem for Hilbert manifolds and the contractibility of EG, the result follows. ☐

1.3. **The Eilenberg-Moore spectral sequence.** Let us state an Eilenberg-Moore-type spectral sequence, dual to Theorème 2.1 of Moore [9]. If Λ is a DGC coalgebra, denote by $\Lambda^\#$ the underlying graded coalgebra, obtained by forgetting the differential. Similarly, if A is a Λ-comodule, denote by $\mathsf{A}^\#$ the underlying $\Lambda^\#$-comodule, again obtained by forgetting the differential.

Proposition 1.3.1. *There is a spectral sequence*

$$E_1 = \mathrm{Cotor}_{\Lambda^\#}(\mathsf{A}^\#, \mathsf{B}^\#) \Rightarrow \mathrm{Cotor}_\Lambda(\mathsf{A}, \mathsf{B}).$$

In particular, if $\mathsf{A}^\#$ an extended $\Lambda^\#$-comodule, then

$$\mathrm{Cotor}_\Lambda(\mathsf{A}, \mathsf{B}) \cong H^*(\mathsf{A} \,\square_\Lambda\, \mathsf{B}).$$

Proof. If we filter the cobar complex $\Omega^k(\mathsf{A}, \Lambda, \mathsf{B})$ by degree k, we easily see that the E_1-term is equal to $\mathrm{Cotor}_{\Lambda^\#}(\mathsf{A}^\#, \mathsf{B}^\#)$. If $\mathsf{A}^\#$ is an extended $\Lambda^\#$-comodule, then the E_1-term equals $E_1 = \mathsf{A}^\# \,\square_{\Lambda^\#}\, \mathsf{B}^\#$ as a graded vector space, with differential induced by the internal differentials of A and B. Thus, the spectral sequence degenerates at the E_2-term $E_2 = H^*(\mathsf{A} \,\square_\Lambda\, \mathsf{B})$. ☐

2. Equivariant differential forms

2.1. The complex of equivariant differential forms. Let A be a differential G-module, with differential d. An element of $C^k(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})$ may be identified with a smooth map

$$f(g_1,\dots,g_k|X) : G^k\times\mathbf{g}\to\mathsf{A}$$

which vanishes if any of the arguments g_i equals the identity $e\in G$. The degree of an element of $C^k(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})$ is the sum of three numbers: its cochain degree k, its degree as an element of A, and twice its degree as a polynomial on $\mathbf{g}$.

The operators d and ι are extended to act on $C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})$ by the formulas

$$\begin{aligned}(df)(g_1,\dots,g_k|X)&=(-1)^k df(g_1,\dots,g_k|X),\\(\iota f)(g_1,\dots,g_k|X)&=(-1)^k\iota(X)f(g_1,\dots,g_k|X).\end{aligned}$$

Both of these operators have degree 1, and $d\iota+\iota d=\mathcal{L}$, where $\mathcal{L}$ is the operator

$$(\mathcal{L}f)(g_1,\dots,g_k|X)=\frac{\partial}{\partial t}\Big|_{t=0}e^{tX}f(g_1,\dots,g_k|X).$$

The coboundary

$$\bar{d}:C^k(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})\to C^{k+1}(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})$$

is given by the formula

$$\begin{aligned}(\bar{d}f)(g_0,\dots,g_k|X)=f(g_1,\dots,g_k|X)+\sum_{i=1}^k(-1)^i f(g_0,\dots,g_{i-1}g_i,\dots,g_k|X)\\+(-1)^{k+1}g_kf(g_0,\dots,g_{k-1}|\operatorname{Ad}(g_k^{-1})X).\end{aligned}$$

Its cohomology is the differentiable group cohomology $H^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})=\operatorname{Cotor}^{\mathcal{A}^0(G)}(\mathbb{C},\mathsf{A})$.

Denote by

$$\bar{\iota}:C^k(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})\to C^{k-1}(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A})$$

the operator given by the formula

$$(\bar{\iota}f)(g_1,\dots,g_{k-1}|X)=\sum_{i=0}^{k-1}(-1)^i\frac{\partial}{\partial t}\Big|_{t=0}f(g_1,\dots,g_i,e^{tX_i},g_{i+1},\dots,g_{k-1}|X),$$

where $X_i=\operatorname{Ad}(g_{i+1}\dots g_{k-1})X$.

Lemma 2.1.1. *$\bar{\iota}^2=0$ and $\bar{d}\bar{\iota}+\bar{\iota}\bar{d}=-\mathcal{L}$*

Proof. If $f \in C^k(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A})$, it is easily seen that

$$
\begin{aligned}
(\bar\iota^2 f)(X|g_1,\dots,g_{k-2}) = \sum_{0\le i\le j\le k-2} &(-1)^{i+j} \frac{\partial^2}{\partial s\,\partial t}\Big|_{s=t=0} \\
&\Big(f(g_1,\dots,g_i,e^{sX_i},g_{i+1},\dots,g_j,e^{tX_j},g_{j+1},\dots,g_{k-2}|X) \\
&\quad - f(g_1,\dots,g_i,e^{tX_i},g_{i+1},\dots,g_j,e^{sX_j},g_{j+1},\dots,g_{k-2}|X)\Big).
\end{aligned}
$$

The two sums cancel, proving that $\bar\iota^2 = 0$.

To calculate $\bar d\bar\iota + \bar\iota\bar d$, observe that

$$
\begin{aligned}
(\bar\iota\bar d f)(X|g_1,\dots,g_k) &= \sum_{i=0}^{k} (-1)^i \frac{\partial}{\partial t}\Big|_{t=0} (\bar d f)(g_1,\dots,g_i,e^{tX_i},g_{i+1},\dots,g_k|X) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} f(g_1,\dots,g_k|X) \\
&\quad + \sum_{i=1}^{k} (-1)^i \frac{\partial}{\partial t}\Big|_{t=0} f(g_2,\dots,g_i,e^{tX_i},g_{i+1},\dots,g_k|X) \\
&\quad + \sum_{1\le j<i\le k} (-1)^{i+j} \frac{\partial}{\partial t}\Big|_{t=0} f(g_1,\dots,g_jg_{j+1},\dots,g_i,e^{tX_i},g_{i+1},\dots,g_k|X) \\
&\quad + \sum_{0\le i<j<k} (-1)^{i+j-1} \frac{\partial}{\partial t}\Big|_{t=0} f(X|g_1,\dots,g_i,e^{tX_i},g_{i+1},\dots,g_jg_{j+1},\dots,g_k) \\
&\quad + \sum_{i=1}^{k} \frac{\partial}{\partial t}\Big|_{t=0} f(g_1,\dots,g_ie^{tX_i},g_{i+1},\dots,g_k|X) \\
&\quad - \sum_{i=1}^{k} \frac{\partial}{\partial t}\Big|_{t=0} f(g_1,\dots,g_{i-1},e^{tX_{i-1}}g_i,\dots,g_k|X) \\
&\quad + \sum_{i=0}^{k-1} (-1)^{i+k+1} \frac{\partial}{\partial t}\Big|_{t=0} g_k f(g_1,\dots,g_i,e^{tX_i},g_{i-1},\dots,g_{k-1}|\operatorname{Ad}(g_k^{-1})X) \\
&\quad - \frac{\partial}{\partial t}\Big|_{t=0} e^{tX} f(g_1,\dots,g_k|X)
\end{aligned}
$$

The last term equals $-(\mathcal{L}f)(g_1,\dots,g_k|X)$, while the first term vanishes, since it is independent of t. Two of the terms cancel by the equality

$$
f(g_1,\dots,g_{i-1},g_ie^{tX_i},g_{i+1},\dots,g_k|X) = f(g_1,\dots,g_{i-1},e^{tX_{i-1}}g_i,g_{i+1},\dots,g_k|X).
$$

The remaining terms cancel against various terms of $\bar d\bar\iota f$. $\square$

Corollary 2.1.2. *On $C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A})$, the operator $d_G = (\bar d + \bar\iota) + (d + \iota)$ is a differential.*

Proof. Since d and ι are invariant under the action of G, we see that $[\bar{d}, d] = [\bar{d}, \iota] = 0$. On the other hand, $\bar{\iota}$ commutes with any endomorphism of $\mathbb{C}[\mathbf{g}] \otimes \mathsf{A}$ over $\mathbb{C}[\mathbf{g}]$. Thus, the two operators $\bar{d} + \bar{\iota}$ and $d + \iota$ commute, and

$$\big((\bar{d} + \bar{\iota}) + (d + \iota)\big)^2 = (\bar{d} + \bar{\iota})^2 + (d + \iota)^2 = -\mathcal{L} + \mathcal{L} = 0. \quad \square$$

When G is a compact group, the Weil complex

$$W(\mathbf{g}) = C^0(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G)) \cong \mathbb{C}[\mathbf{g}] \otimes \Lambda^* \mathbf{g},$$

with differential $d + \iota$, is a model for the complex of differential forms $\mathcal{A}^*(EG)$ on the classifying space of G. However, $W(\mathbf{g})$ is not an injective comodule over $\mathcal{A}^0(G)$, so it cannot be used to calculate the equivariant cohomology if G unless G is compact. We will now show that the complex $(C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G)), d_G)$ is a model for $\mathcal{A}^*(EG)$ which plays the role of the Weil complex when G is not compact.

Let A be a differential G-module. Filtering $C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A})$ by cochain degree, we obtain a spectral sequence

$$E_1 = \big(H^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A}), d + \iota\big) \Rightarrow H^*(C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A}), d_G).$$

Let us take $\mathsf{A} = \mathcal{A}^*(G)$. Since $\mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G)$ is an injective $\mathcal{A}^0(G)$-comodule, we may identify (E_1, d_1) with the Weil complex $W(\mathbf{g})$. This proves the following lemma.

Lemma 2.1.3. *There is a spectral sequence converging to $H^*(C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G)), d_G)$ with E_1-term isomorphic to the Weil complex:*

$$E_1 = H^k(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G)) \cong \begin{cases} W(\mathbf{g}), & k = 0, \\ 0, & k > 0. \end{cases}$$

Thus, the spectral sequence degenerates, with $E_\infty \cong E_2 \cong H^*(W(\mathbf{g})) \cong \mathbb{C}$.

This lemma is a key step in the proof of the main result of this section.

Theorem 2.1.4. *If A is a differential G-module, $\mathbb{H}(\mathcal{A}^*(G), \mathsf{A}) \cong H^*(C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A}), d_G)$.*

Proof. Observe that

$$C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A}) \cong C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G)) \,\square_{\mathcal{A}^*(G)}\, \mathsf{A}.$$

The coaugmentation $\eta : \mathbb{C} \to C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G))$, defined by sending $1 \in \mathbb{C}$ to the constant function $1 \in C^0(G, \mathcal{A}^0(G))$, is a quasi-isomorphism by Lemma 2.1.3. It follows that

$$\mathbb{H}(\mathcal{A}^*(G), \mathsf{A}) \cong \mathrm{Cotor}_{\mathcal{A}^*(G)}(C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G)), \mathsf{A}).$$

We now apply the spectral sequence of Proposition 1.3.1. Since $C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(G))^{\#}$ is an extended $\mathcal{A}^*(G)^{\#}$-comodule, we see that

$$\begin{aligned}\mathrm{Cotor}_{\mathcal{A}^*(G)}(C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(G)),\mathsf{A}) &\cong H^*(C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(G))\,\square_{\mathcal{A}^*(G)}\,\mathsf{A})\\ &\cong H^*(C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{A}),d_G). \quad\square\end{aligned}$$

Motivated by this theorem, we define the complex of equivariant differential forms $\mathcal{A}^*_G(M)$ to be the complex $C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M))$ with differential $d_G=(\bar d+\bar\iota)+(d+\iota)$. In this case, the spectral sequence becomes a spectral sequence of Bott [3]:

$$E_1=\big(H^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M)),d+\iota\big)\Rightarrow H^*_G(M).$$

In the special case that M is a point, the differential $d_1=d+\iota$ vanishes, and we obtain a spectral sequence $E_2=H^*(G,\mathbb{C}[\mathbf{g}])\Rightarrow H^*(BG)$.

2.2. **Equivariant differential forms and the cobar complex.** In this paragraph, we construct an explicit quasi-isomorphism from the cobar complex $\Omega(\mathbb{C},\mathcal{A}^*(G),\mathsf{M})$ to the complex $C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{M})$ of the last section. This map may be motivated by thinking of the DGC coalgebra $\mathcal{A}^*(G)$ as a twisted product of coalgebras $\Lambda^*\mathbf{g}^*$ and $C^\infty(G)=\mathcal{A}^0(G)$.

Definition 2.2.1. The map $\mathcal{J}:\Omega^*(\mathbb{C},\mathcal{A}^*(G),\mathsf{M})\to C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{M})$ is defined by the formula

$$\mathcal{J}(\omega_1\otimes\ldots\otimes\omega_k\otimes\mathsf{m})(g_1,\ldots,g_\ell|X)=\sum_{\pi\in S(\ell,k-\ell)}\prod_{i=1}^{\ell}\omega_{\pi(i)}(g_i)\prod_{j=\ell+1}^{k}\iota(X_j^M)\omega_{\pi(j)}(e)\mathsf{m}.$$

Here, $S(\ell,k-\ell)\subset S_k$ is the set of shuffles, that is, permutations π such that

$$\pi(1)<\cdots<\pi(\ell)\text{ and }\pi(\ell+1)<\cdots<\pi(k),$$

and $X_j=\mathrm{Ad}(g_i\ldots g_\ell)X$, where i is the least integer less than ℓ such that $\pi(j)<\pi(i)$.

In this definition, note that $\omega(g)$ depends only on the zero-form component of ω, while $\iota(X)\omega(e)$ depends only on the one-form component of ω at the identity $e\in G$.

Lemma 2.2.2. *$\mathcal{J}\cdot\bar d=(\bar d+\iota)\cdot\mathcal{J}$ and $\mathcal{J}\cdot d=(d+\bar\iota)\cdot\mathcal{J}$*

Proof. Four different types of terms contribute to $\mathcal{J}\cdot\bar d$:

(1) those terms where $\bar d$ acts by the comultiplication $\mathcal{A}^*(G)\to\mathcal{A}^*(G)\otimes\mathcal{A}^*(G)$ and $\mathcal{J}$ acts on at least one factor of $\mathcal{A}^*(G)\otimes\mathcal{A}^*(G)$ by taking the one-form component at the identity of G — these terms cancel by symmetry considerations;
(2) those terms where $\bar d$ acts by the comultiplication $\mathcal{A}^*(G)\to\mathcal{A}^*(G)\otimes\mathcal{A}^*(G)$ and $\mathcal{J}$ acts on $\mathcal{A}^*(G)\otimes\mathcal{A}^*(G)$ by taking the zero-form component in each factor;

(3) those terms where $\bar{d}$ acts by the coaction $\mathsf{M} \to \mathcal{A}^*(G) \otimes \mathsf{M}$ and $\mathcal{J}$ acts on the factor $\mathcal{A}^*(G)$ of $\mathcal{A}^*(G) \otimes \mathsf{M}$ by taking its zero-form component — these terms and those of the second type combine to give $\bar{d} \cdot \mathcal{J}$;
(4) those terms where $\bar{d}$ acts by the coaction $\mathsf{M} \to \mathcal{A}^*(G) \otimes \mathsf{M}$ and $\mathcal{J}$ acts on the factor $\mathcal{A}^*(G)$ of $\mathcal{A}^*(G) \otimes \mathsf{M}$ by taking its one-form component at the identity of G — these terms contribute $\iota \cdot \mathcal{J}$.

Two types of terms contribute to $\mathcal{J} \cdot d$:

(1) those terms in $\mathcal{J} \cdot d$ where d acts on $\mathsf{m} \in \mathsf{M}$ — these terms contribute $d \cdot \mathcal{J}$;
(2) those terms in $\mathcal{J} \cdot d$ where d acts on ω_i — these terms contribute $\bar{\iota} \cdot \mathcal{J}$. □

Theorem 2.2.3. *The map $\mathcal{J} : \Omega(\mathbb{C}, \mathcal{A}^*(G), \mathsf{M}) \to C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{M})$ is a quasi-isomorphism.*

Proof. It suffices to prove the theorem with $\mathsf{M} = \mathcal{A}^*(G)$, for which it is evident, since both complexes $\Omega(\mathbb{C}, \mathcal{A}^*(G), \mathcal{A}^*(G))$ and $C^*(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(G))$ are contractible. □

2.3. **The cup product.** In this paragraph, we define a product on the equivariant differential forms $\mathcal{A}^*_G(M)$; note that the construction of the equivariant Chern character of the next section makes no use of this product. Consider a DGC bialgebra Λ, that is, a DGC coalgebra with an associative product $m : \Lambda \otimes \Lambda \to \Lambda$ which is a map of DGC coalgebras. The tensor product $\mathsf{A} \otimes \mathsf{B}$ of two differential graded (left) Λ-comodules A and B over Λ is again a differential graded comodule over Λ, with coaction

$$\mathsf{A} \otimes \mathsf{B} \xrightarrow{\nabla_{\mathsf{A}} \otimes \nabla_{\mathsf{B}}} \Lambda \otimes \mathsf{A} \otimes \Lambda \otimes \mathsf{B} \xrightarrow{1 \otimes S \otimes 1} \Lambda \otimes \Lambda \otimes \mathsf{A} \otimes \mathsf{B} \xrightarrow{m \otimes 1 \otimes 1} \Lambda \otimes \mathsf{A} \otimes \mathsf{B}.$$

Here, S is the map $S : \mathsf{A} \otimes \Lambda \to \Lambda \otimes \mathsf{A}$ defined by the formula $S(a \otimes \lambda) = (-1)^{|a|\,|\lambda|} \lambda \otimes a$.

If a comodule A has an associative product $\mu : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A}$ which is a map of differential graded Λ-comodules, we say that A is a (differential graded) comodule algebra over Λ. An example of this abstract nonsense is the comodule algebra $\mathcal{A}^*(M)$ over the differential graded bialgebra $\mathcal{A}^*(G)$, where M is a manifold on which G acts smoothly.

The graded vector space $\operatorname{Cotor}^{\Lambda}(\mathsf{A}, \mathsf{B})$ of a pair of comodule algebras A and B is itself an algebra, in a natural way. This may be seen by constructing the product explicitly on the cobar resolution $\Omega(\mathsf{A}, \Lambda, \mathsf{B})$, as in Miller [8]. As a special case of this formula, we obtain the cup product

$$C^k(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A}) \otimes C^\ell(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A}) \to C^{k+\ell}(G, \mathbb{C}[\mathbf{g}] \otimes \mathsf{A}),$$

defined by the formula

$$(a \cup b)(g_1, \dots, g_{k+\ell} | X) = (-1)^{\ell(|a|-k)} \gamma a(g_1, \dots, g_k | \operatorname{Ad}(\gamma^{-1}) X)\, b(g_{k+1}, \dots, g_{k+\ell} | X),$$

where $\gamma = g_{k+1} \cdots g_{k+\ell}$.

Proposition 2.3.1. *With the cup product and the differential d_G, $\mathcal{A}^*_G(M)$ is a differential graded algebra.*

Proof. It is well-known that $\cup$ is associative. We must show that each of the operators d, ι, $\bar{d}$ and $\bar{\iota}$ are derivations with respect to this product.

For d and ι, this is clear since they are induced by invariant derivations of $\mathbb{C}[\mathbf{g}]\otimes\mathcal{A}^*(M)$. For $\bar{d}$, it is a basic property of the cup product. Thus, it only remains to deal with $\bar{\iota}$. If a is a k-cochain and b is a ℓ-cochain, we calculate that

$$\begin{aligned}
&\bar{\iota}(a\cup b)(g_1,\dots,g_{k+\ell-1}|X)\\
&\qquad = \sum_{i=1}^{k+\ell}(-1)^{i-1}\frac{\partial}{\partial t}\Big|_{t=0}(a\cup b)(g_1,\dots,g_{i-1},e^{tX_i},g_i,\dots,g_{k+\ell-1}|X)\\
&\qquad = \sum_{i=1}^{k}(-1)^{\ell(|a|-k)+i-1}\frac{\partial}{\partial t}\Big|_{t=0}\gamma a(g_1,\dots,g_{i-1},e^{tX_i},g_i,\dots,g_{k-1}|\operatorname{Ad}(\gamma^{-1})X)\\
&\qquad\qquad\qquad b(g_k,\dots,g_{k+\ell-1}|X)\\
&\qquad + \sum_{i=k+1}^{k+\ell}(-1)^{\ell(|a|-k)+i}\frac{\partial}{\partial t}\Big|_{t=0}\gamma e^{tX}a(g_1,\dots,g_k|\operatorname{Ad}(\gamma^{-1})X)\\
&\qquad\qquad\qquad b(g_{k+1},\dots,g_{i-1},e^{tX_i},g_i,\dots,g_{k+\ell-1}|X).
\end{aligned}$$

Since b is a normalized cochain, $b(g_{k+1},\dots,g_{i-1},e^{tX_i},g_i,\dots,g_{k+\ell-1}|X)$ vanishes when $t=0$. We see that in the second sum, $\gamma e^{tX}a$ may be replaced by γa, and hence that

$$\bar{\iota}(a\cup b)=\bar{\iota}a\cup b+(-1)^{|a|}a\cup\bar{\iota}b. \quad \square$$

The product on $\operatorname{Cotor}_\Lambda(\mathsf{A},\mathsf{B})$ may be calculated in the following way: if $\tilde{\mathsf{A}}$ is a comodule algebra such that $\tilde{\mathsf{A}}^\#$ is an extended comodule over $\Lambda^\#$, and if there is a quasi-isomorphism $\mathsf{A}\to\tilde{\mathsf{A}}$ which is a homomorphism of differential graded comodule algebras, then the induced map

$$\operatorname{Cotor}_\Lambda(\mathsf{A},\mathsf{B})\to\operatorname{Cotor}_\Lambda(\tilde{\mathsf{A}},\mathsf{B})\cong H^*(\tilde{\mathsf{A}}\,\square_\Lambda\,\mathsf{B})$$

is an isomorphism of algebras. Thus, we obtain the following addendum to Corollary 1.2.3.

Proposition 2.3.2. *The isomorphism $\mathbb{H}(\mathcal{A}^*(G),\mathcal{A}^*(M))\cong H^*_G(M)$ is compatible with products.*

It is easily verified that if M is an algebra comodule for the differential bialgebra $\mathcal{A}^*(G)$, the quasi-isomorphism $\mathcal{J}:\Omega(\mathbb{C},\mathcal{A}^*(G),\mathsf{M})\to C^*(G,\mathbb{C}[\mathbf{g}]\otimes\mathsf{M})$ sends the shuffle product to the cup product. Since we will have no need for this result, we leave the details to the reader.

3. The equivariant Chern character

3.1. **Differential forms on simplicial manifolds.** A simplicial manifold $M_\bullet$ is a contravariant functor $[k] \mapsto M_k$ from the simplicial category Δ to the category of differentiable manifolds. We will assume that the degeneracy maps of our simplicial manifolds are embeddings: it follows that they are good, in the sense of Segal [10], so that the geometric realization $|M_\bullet|$ correctly reflects the homotopy type of $M_\bullet$.

To a simplicial manifold $M_\bullet$ is associated the complex of simplicial differential forms: this is the total complex $\operatorname{Tot}\mathcal{A}^*(M_\bullet)$ of the double complex $C^n_k = \mathcal{A}^n(M_k)$, with differentials the coboundary

$$\bar{d} = \sum_{i=0}^{k} (-1)^i \partial_i^* : C^n_k \longrightarrow C^n_{k-1},$$

induced by the face maps $\partial_i : M_k \longrightarrow M_{k-1}$, $0 \le i \le k$, and the exterior differential $d : C^n_k \longrightarrow C^{n+1}_k$. De Rham's theorem for good simplicial manifolds shows that the cohomology of the complex $\operatorname{Tot}\mathcal{A}^*(M_\bullet)$ is naturally isomorphic to the ordinary cohomology $H^*(|M_\bullet|, \mathbb{C})$ of the geometric realization $|M_\bullet|$.

Also associated to a simplicial manifold is Dupont's complex [5] of differential forms $\mathcal{A}^*(|M_\bullet|)$: this is the subcomplex of the direct sum

$$\sum_{k=0}^{\infty} \mathcal{A}^*(\Delta^k \times M_k)$$

such that for all morphisms $f \in \Delta([k], [\ell])$ in the simplicial category,

$$(f_* \times 1)^* \omega_\ell = (1 \times f^*)^* \omega_k \in \mathcal{A}^*(\Delta^k \otimes M_\ell).$$

Here, $f_* : \Delta^k \longrightarrow \Delta^\ell$ and $f^* : M_\ell \longrightarrow M_k$ are the induced actions on the cosimplicial space $\Delta^\bullet$ and simplicial manifold $M_\bullet$. The exterior differential d preserves $\mathcal{A}^*(|M_\bullet|)$.

Stokes's Theorem shows that there is a map of complexes

$$\mathcal{I} : (\mathcal{A}^*(|M_\bullet|), d) \longrightarrow (\operatorname{Tot}\mathcal{A}^*(M_\bullet), \bar{d} + d),$$

defined on $\mathcal{A}^*(\Delta^k \times M_k)$ to be $(-1)^k$ times the integral over the fibre of the projection $\Delta^k \times M_k \longrightarrow M_k$. This map is a quasi-isomorphism by Theorem 2.3 of Dupont [5].

A simplicial principal bundle with structure group H on a simplicial manifold $M_\bullet$ is a map of simplicial manifolds $P_\bullet \longrightarrow M_\bullet$ together with a locally trivial right action of H on $P_\bullet$ such that $P_\bullet/H \cong M_\bullet$. A connection form θ on $P_\bullet$ is an invariant one-form $\theta \in \mathcal{A}^1(|P_\bullet|, \mathbf{h})^H$ with values in the Lie algebra $\mathbf{h}$ of H such that if X denotes both an element of $\mathbf{h}$ and the associated vector field on $\Delta^k \times P_k$, then $\iota(X)\theta = X$. The curvature of the connection form θ is the differential form

$$\Omega = d\theta + \tfrac{1}{2}[\theta, \theta] \in \mathcal{A}^2(|P_\bullet|, \mathbf{h}).$$

The following result is Proposition 3.7 of Dupont [5].

Theorem 3.1.1. *Let Φ be an invariant polynomial on the Lie algebra $\mathbf{h}$ of H. The differential form $\Phi(\Omega) \in \mathcal{A}^*(|P_\bullet|)$ is a closed basic form on $|P_\bullet|$, and hence descends to a closed differential form $\Phi(\Omega) \in \mathcal{A}^*(|M_\bullet|)$. The cohomology class of $\Phi(\Omega)$ represents the image of the class $\Phi \in H^*(BH)$ under the Chern-Weil map $H^*(BH) \to H^*(|M_\bullet|)$ associated to the principal bundle $|P_\bullet| \to |M_\bullet|$.*

3.2. **Application to the homotopy quotient.** Suppose that M is a manifold with a left action of the Lie group G. Let $E_\bullet G \times_G M$ be the nerve of the differentiable groupoid $G \times M$ with source and target maps $s(g, x) = x$ and $t(g, x) = gx$, and composition $(g, hx) \cdot (h, x) = (gh, x)$. Explicitly, this is the simplicial manifold with $E_k G \times_G M = G^k \times M$, with face maps

$$\partial_i(g_1, \dots, g_k, x) = \begin{cases} (g_2, \dots, g_k, x), & i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_k, x), & 0 < i < k, \\ (g_1, \dots, g_{k-1}, g_k x), & i = k, \end{cases}$$

and degeneracies

$$\sigma_i(g_1, \dots, g_k, x) = (g_1, \dots, g_i, e, g_{i+1}, \dots, g_k, x), \quad 0 \le i \le k.$$

Let $p_i : G^k \times M \to G$, $1 \le i \le k$, be the projection onto the i-th factor of G, and let $p_M : G^k \times M \to M$ be the projection onto M. The complex $\operatorname{Tot} \mathcal{A}^*(E_\bullet G \times_G M)$ may be identified with the cobar complex $\Omega(\mathbb{C}, \mathcal{A}^*(G), \mathcal{A}^*(M))$, by the map which sends a chain $\omega_1 \otimes \dots \otimes \omega_k \otimes \alpha \in \mathcal{A}^*(G)^{(k)} \otimes \mathcal{A}^*(M)$ of the cobar complex to the differential form

$$(-1)^{k|\omega_1| + (k-1)|\omega_2| + \dots + 2|\omega_{k-1}| + |\omega_k|} p_1^* \omega_1 \wedge \dots \wedge p_k^* \omega_k \wedge p_M^* \alpha \in \mathcal{A}^*(G^k \times M).$$

This sign may be absorbed into the definition of $\mathcal{J}$, which thus becomes a map of complexes

$$\mathcal{J} : \operatorname{Tot} \mathcal{A}^*(E_\bullet G \times_G M) \to \mathcal{A}^*_G(M).$$

If P is a G-equivariant principal bundle over M with structure group H, $E_\bullet G \times_G P$ is a simplicial principal bundle over $E_\bullet G \times_G M$. If $\theta \in \mathcal{A}^1(P, \mathbf{h})^H$ is a connection form on P, we may define a connection form on $E_\bullet G \times_G P$ by the formula

$$\Theta = t_1(\theta_0 - \theta_1) + \dots + t_k(\theta_{k-1} - \theta_k) + \theta_k,$$

where θ_i is the pull-back of θ to $\Delta^k \times E_kG \times_G P \cong \Delta^k \times G^k \times P$ by the map which sends $(t_1, \dots, t_k|g_1, \dots, g_k|p) \in \Delta^k \times G^k \times P$ to $g_{i+1} \dots g_k p \in P$. If the group G is discrete, this connection specializes to the connection of Bott which we discussed in the Introduction.

Combining Theorem 3.1.1 with the results of Section 2, we obtain the following generalization of Bott's result: here, we have restricted attention to the case in which the structure group H of P is $\mathrm{GL}(N)$, and the invariant polynomial is $X \mapsto \mathrm{Tr}(e^X)$.

Theorem 3.2.1. *The equivariant Chern character* $\mathrm{Ch}_G(P)$ *of the equivariant principal bundle* P *over* M *is represented in the complex* $\mathcal{A}^*_G(M)$ *by the equivariant differential form*

$$\mathrm{Ch}_G(\theta) = \sum_{k=0}^{\infty} (-1)^k \mathcal{J} \int_{\Delta^k} \mathrm{Tr}(\exp(d\Theta + \Theta \wedge \Theta)).$$

3.3. **The equivariant Chern character of an invariant connection.** We will now show that the equivariant Chern character of the last paragraph specializes to the equivariant Chern character of Berline and Vergne when the connection form θ on the principal bundle P is G-invariant: in practice, this means that we restrict attention to compact Lie groups G. Denote by Ω the curvature form $d\theta + \theta \wedge \theta$ of the connection form θ.

If $X \in \mathbf{g}$, denote by X^P the associated vector field on the principal bundle P. The moment of the connection P is the function $\mu \in \mathbf{g}^* \otimes \mathcal{A}^0(P, \mathbf{h})^H$ defined by the formula

$$\iota(X^P)\theta = \mu(X) \in \mathbf{h}.$$

Berline and Vergne define the equivariant Chern character associated to an invariant connection by the formula

$$\mathrm{Ch}_G(\theta) = \mathrm{Tr}(e^{\Omega+\mu});$$

this is a cycle in $(\mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(P))^G$, basic with respect to the action of H, which represents the equivariant Chern character of P in Cartan's complex. The proof that $\mathrm{Ch}_G(\theta)$ is closed uses the formula

$$\iota(X^P)\Omega + (d + \mathrm{ad}(\theta))\mu(X) = 0,$$

which may be viewed as a generalization of the formula $\iota(X_f)\omega = -df$ for the Hamiltonian vector field associated to a function f on a symplectic manifold M.

Let X_α be a basis of $\mathbf{g}$, with dual basis ω^α of $\mathbf{g}^*$: identify ω^α with a left-invariant one-form on G. Let $\pi_i : G^k \to G$ be the projection onto the i-th factor, and define the element ω_i of $\mathcal{A}^1(G^k) \otimes \mathcal{A}^0(M, \mathbf{h})$ by the formula

$$\omega_i = \sum_\alpha \pi_i^* \omega^\alpha \otimes \mu(X_\alpha^M).$$

We have the formula

$$\theta_i = \theta_k + \omega_{i+1} + \dots + \omega_k,$$

from which we see that

$$\Theta = t_1\omega_1 + \dots t_k\omega_k + \theta_k.$$

Denoting by $O(i)$ a term of degree $\geq i$ as a differential form on G^k, we see that

$$d\Theta + \Theta \wedge \Theta = \sum_{i=1}^{k} dt_i \wedge \omega_i + \Omega + O(2).$$

It follows that

$$\mathrm{Ch}_G(\theta) = \sum_{k=0}^{\infty} (-1)^{k(k+1)/2} \mathcal{J} \int_{0\leq s_1 \leq \dots \leq s_k \leq 1} \mathrm{Tr}\big(e^{s_1\Omega}\omega_1 \dots \omega_k e^{(1-s_k)\Omega} + O(k+2)\big)\, ds_1 \dots ds_k.$$

Here, the volume of the simplex Δ^k has been cancelled against the factor $k!$ coming from the different order in which the one-forms ω_i can occur on expanding the exponential. The sign comes from the formula

$$dt_1 \wedge \omega_1 \wedge \dots \wedge dt_k \wedge \omega_k = (-1)^{k(k-1)/2} dt_1 \wedge \dots \wedge dt_k \wedge \omega_1 \wedge \dots \wedge \omega_k,$$

combined with the sign $(-1)^k$ in the definition of $\mathcal{I}$.

The map $\mathcal{J}$ vanishes on the term $O(k+2)$. The sign $(-1)^{k(k+1)/2}$ is cancelled by that coming from the identification of $\mathrm{Tot}\,\mathcal{A}^*(E_\bullet G \times_G M)$ with $\Omega(\mathbb{C}, \mathcal{A}^*(G), \mathcal{A}^*(M))$. We are left with a sum of zero-cochains, each corresponding to the unique shuffle in $S(0,k)$:

$$\mathrm{Ch}_G(\theta) = \sum_{k=0}^{\infty} \int_{0\leq s_1 \leq \dots \leq s_k \leq 1} \mathrm{Tr}\big(e^{s_1\Omega}\mu(X)e^{(s_2-s_1)\Omega} \dots e^{(s_k-s_{k-1})\Omega}\mu(X)e^{(1-s_k)\Omega}\big)\, ds_1 \dots ds_k.$$

This equals the equivariant Chern character

$$\mathrm{Tr}(e^{\Omega+\mu(X)}) \in C^0(G, \mathbb{C}[\mathbf{g}] \otimes \mathcal{A}^*(M)) \subset \mathcal{A}^*_G(M)$$

of Berline and Vergne.

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