DIRICHLET FORMS ON LOOP SPACE

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Abstract. We use the Malliavin calculus to define Dirichlet forms along the fibres of a map $\pi : B \to M$ from a Wiener space to a Riemannian manifold, satisfying Malliavin's regularity condition

$$\det((d\pi)(d\pi)^*)^{-1} \in L^p(B) \quad \text{for all } p < \infty.$$

In particular, we obtain Dirichlet forms on the based loop spaces of a Riemannian manifold.

For these Dirichlet forms, we calculate the Ricci curvature in the sense of Bakry and Emery.

In this article, we construct analogues of the Ornstein-Uhlenbeck diffusion on the space of based loops L_xM on a Riemannian manifold M, generalizing the results of Gaveau [6]. The proof follows very closely the construction by Kusuoka [9] of the Ornstein-Uhlenbeck diffusion on a Wiener space B:

(1) we embed L_xM in a Wiener space B by the Ito development map, and form the cylindrical compactification $\hat{B}$ of B;

(2) if $d\mu_x$ is the Wiener measure of L_xM pushed forward to $\hat{B}$ by the inclusion

$$L_xM \hookrightarrow B \hookrightarrow \hat{B},$$

we construct a Dirichlet form on $\hat{B}$ which has $d\mu_x$ as a stationary measure;

(3) we prove, using Kusuoka's technique, that the complement of L_xM in $\hat{B}$ has capacity 0, and thus that the Dirichlet form that we have constructed may be thought of as a Dirichlet form on L_xM.

We calculate the Ricci curvature of these Dirichlet forms, in the sense of Bakry and Emery; in a notation that will be explained in Section 3, we find that the Ricci curvature, thought of as an operator on the Hilbert space H, equals

$$\mathbf{Ric} = P + \mathrm{Tr}(S^*S) + S^*(d^*N) \in I + W^\infty(B, \mathrm{HS}(H))$$

where S is the the second fundamental form S of the map π.

We have given a summary of those parts of the Malliavin calculus that we use in Section 1; this section is included to make the article more complete. In Section 4, we work out in detail a simple example, the loop space of a compact Lie group; this was inspired by Freed's thesis [4], and much of the calculation is due to him.

This work was partially funded by the NSF and the Centre for Mathematical Analysis of the Australian National University.

§1. The Malliavin Calculus

There are several good introductions to the Malliavin calculus [8], [10]. In this section, we will summarize some of the more important results of the subject.

Definition 1.1. *A Wiener space B is a Banach space with a Gaussian Radon measure $d\mu$; that is, there is a bounded non-degenerate quadratic form $C(\alpha,\alpha)$ on the topological dual B' of B, the covariance of the measure, such that*

$$\int_B e^{i(\alpha,x)}\,d\mu(x) = e^{-C(\alpha,\alpha)/2} \quad \textit{for all } \alpha \in B'.$$

The example on which this definition is modeled is the classical Wiener space $C^\alpha_*([0,1],\mathbb{R}^n)$, $(\alpha < 1/2)$, of α-Hölder continuous paths in $\mathbb{R}^n$ starting at 0. In this case, the measure $d\mu$ is the Wiener measure, with covariance

$$C(\alpha,\alpha) = \int_0^1 \int_0^1 \min(s,t)\,(\alpha(s),\alpha(t))\,ds\,dt;$$

the corresponding Hilbert space H equals the space $L^{2,1}_*([0,1],\mathbb{R}^n)$ of paths satisfying

$$|x|_H^2 = \int_0^1 |\dot{x}(t)|^2\,dt < \infty.$$

The definition of a Wiener space is due to Gross, who proved that they possess the following general property, a proof of which may be found in the paper of Kusuoka [9].

Proposition 1.2. *If $(B,d\mu)$ is a Wiener space, then there is a reflexive Banach space B_0 and a compact injection $B_0 \hookrightarrow B$ such that $\mu(B \setminus B_0) = 0$.* □

For the special case of the classical Wiener space $C^\alpha_*([0,t],\mathbb{R}^n)$, there are many explicit choices for B_0; for example, we may take the Sobolev space $L^{p,s}_*([0,t],\mathbb{R}^n)$ with $\alpha + p^{-1} < s < 1/2$ and $p < \infty$.

The completion of B' in the inner product defined by the covariance $C(\alpha,\alpha)$ is a Hilbert space, which we denote by H. Taking the adjoint of the inclusion $B' \hookrightarrow H$, we obtain a canonical inclusion of H in B.

Definition 1.3. *A cylinder function on B is a function of the form $\tau^*\varphi$, where τ is a bounded linear map from B to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and φ is an element of $C_0^\infty(\mathbb{R}^n)$.*

The space of cylinder functions is dense in $L^p(B) = L^p(B,d\mu)$ for each $p < \infty$. If G is a Hilbert space, we will denote by $L^p(B;G)$ the space of L^p-functions on B with values in G. We will often make use of the Hilbert tensor product $G \otimes_2 H$ of two Hilbert spaces H and G; this is completion of the algebraic tensor product with respect to the quadratic form

$$|v \otimes_2 w|^2 = |v|^2 \cdot |w|^2.$$

If $f = \tau^*\varphi$ is a cylinder function, then its gradient, denoted by df, is the element of $C_0^\infty(B) \otimes B'$ defined by applying the map

$$\tau^* : C_0^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \to C_0^\infty(B) \otimes B'$$

to $d\varphi \in C_0^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$. Forming the closure of this operator, we obtain an unbounded operator from $L^p(B)$ to $L^p(B;H)$, which we will also denote by d; its adjoint d^* is then a closed unbounded operator from $L^p(B;H)$ to $L^p(B)$.

The composition of the operators d and d^* acting on the cylinder functions is the Ornstein-Uhlenbeck operator d^*d, which is formally self-adjoint.

Definition 1.4.

(1) *The Sobolev space $L^p_s(B)$, where $1 < p < \infty$ and $s \geq 0$, is defined as follows: $f \in L^p(B)$ is in $L^p_s(B)$ if and only if $(d^*d)^{s/2} f \in L^p(B)$, and $L^p_s(B)$ has the norm*

$$\|f\|_{p,s} = \|(d^*d)^{s/2} f\|_p + \|f\|_p.$$

The Sobolev space $L^p_{-s}(B)$ is defined to be the dual of $L^q_s(B)$, where $p^{-1}+q^{-1} = 1$. Similarly, if G is a Hilbert space, we define the Sobolev spaces $L^p_s(B; G)$, of functions with values in G.

(2) *The space of Malliavin test functions is*

$$W^\infty(B) = \bigcap_{p,s<\infty} L^p_s(B).$$

The operator d^*d is essentially self-adjoint on the Hilbert space $L^2(B)$, and has as a core the space $W^\infty(B)$. It is important to note that W^∞-functions are not all continuous.

Meyer has proved that d is bounded from $L^p_s(B)$ to $L^p_{s-1}(B; H)$ and that d^* is bounded from $L^p_s(B; H)$ to $L^p_{s-1}(B)$, for all $s \in \mathbb{R}$ and $p < \infty$. The proof is more or less the same as that of the boundedness of the Riesz operators on a compact Lie group given in Chapter 2 of Stein's book [11] (see also [1]).

We can also define W^∞-maps from a Wiener space B to a manifold M: a measurable map $\pi : B \longrightarrow M$ is W^∞ if $\pi^* : C^\infty(M) \to W^\infty(B)$ is bounded. Equivalently, if $\rho : M \longrightarrow \mathbb{R}^n$ is an embedding of M in a Euclidean space, then $\pi \in W^\infty(B, M)$ if and only if $\rho \circ \pi \in W^\infty(B, \mathbb{R}^n)$.

If $\pi : F \longrightarrow M$ is a smooth map of compact Riemannian manifolds, then π is a fibration if and only if

$$\det((d\pi)(d\pi)^*)^{-1} \in C^\infty(F).$$

Malliavin observed that an analogue of this property for W^∞-maps is a good replacement for the notion of fibration. If B is a Wiener space and M is a Riemannian manifold of dimension n, define the tangent-map

$$\Pi = d\pi \in W^\infty(B, \mathrm{Hom}(H, \pi^*TM))$$

by means of the embedding $\rho : M \longrightarrow \mathbb{R}^n$, by requiring that $d\rho \circ \Pi \in W^\infty(B, \mathrm{Hom}(H, \mathbb{R}^n))$ is equal to $d(\rho \circ \pi)$. Since the composition $\Pi\Pi^*$ is well-defined and in $W^\infty(B, \mathrm{End}(\pi^*TM))$, we can form the determinant $\det(\Pi\Pi^*)$; Malliavin's condition is that

$$(*) \qquad \det(\Pi\Pi^*)^{-1} \in W^\infty(B).$$

Proposition 1.5.

(1) *Malliavin's condition is implied by the following condition:*

$$\det(\Pi\Pi^*)^{-1} \in L^p(B) \quad \text{for } p < \infty.$$

(2) *If π satisfies Malliavin's condition, then the operator $(\Pi\Pi^*)^{-1}$ is in $W^\infty(B, \mathrm{End}(\pi^*TM))$, and $\det(\Pi\Pi^*)^k$ is in $W^\infty(B)$ for all $k \in \mathbb{Z}$.*

proof. Using the inequality

$$\lambda_1^{-2} + \cdots + \lambda_n^{-2} \leq (\lambda_1 \ldots \lambda_n)^{-2} \cdot (\lambda_1^2 + \cdots + \lambda_n^2)^{n-1},$$

where $\{\lambda_i\}$ are the eigenvalues of $\Pi\Pi^*$, we see that

$$|(\Pi\Pi^*)^{-1}| \leq \det(\Pi\Pi^*)^{-1} \cdot |\Pi\Pi^*|^{n-1},$$

where we use the Hilbert-Schmidt norm on $\mathrm{End}(T_xM)$. In particular, condition $(*)$ implies that $(\Pi\Pi^*)^{-1} \in L^p(B, \mathrm{End}(\pi^*TM))$ for all $p < \infty$.

Since $W^\infty(B)$ is an algebra, it is clear that we have only to show that $\det(\Pi\Pi^*) \in W^\infty(B)$ in order to have $\det(\Pi\Pi^*)^k \in W^\infty(B)$ for all $k \in \mathbb{Z}$. But to show that condition $(*)$ implies that $d^k \det(\Pi\Pi^*) \in W^\infty(B, H^k)$, we argue by induction on k, using the formula

$$d \det(\Pi\Pi^*) = \det(\Pi\Pi^*) \cdot \mathrm{Tr}\left((\Pi\Pi^*)^{-1} d(\Pi\Pi^*)\right).$$

To show that

$$(\Pi\Pi^*)^{-1} = \det(\Pi\Pi^*)^{-1} \times (\text{matrix of cofactors})$$

lies in $W^\infty(B, \mathrm{End}(\pi^*TM))$, it suffices to show that $\det(\Pi\Pi^*)^{-1}$ and the cofactor matrix lie respectively in $W^\infty(B)$ and $W^\infty(B, \mathrm{End}(\pi^*TM))$. For $\det(\Pi\Pi^*)^{-1}$, we have already shown this, while for the cofactor matrix it is clear, since this matrix is a polynomial in matrix elements of $\Pi\Pi^*$, which is itself W^∞. □

One sees immediately from this result that the projector N on the normal bundle to the fibres of the map π is well defined.

Proposition 1.6. *The operator $N = \Pi^*(\Pi\Pi^*)^{-1}\Pi$ is, for a.e. $x \in B$, a projector in H of rank n, and $N \in W^\infty(B, \mathrm{HS}(H))$. (Here, $\mathrm{HS}(H)$ denotes the space of Hilbert-Schmidt operators on H.)*

Proof. The operator N is indeed a projector, since it is self-adjoint, and

$$\begin{aligned} N^2 &= \Pi^*(\Pi\Pi^*)^{-1}\Pi \times \Pi^*(\Pi\Pi^*)^{-1}\Pi \\ &= \Pi^*(\Pi\Pi^*)^{-1}\Pi = N. \end{aligned}$$

Furthermore, it is clear that the kernel of N at $x \in B$ is precisely the kernel of Π, which is the tangent space to the fibre of π at x. □

Thus the projector $P = 1 - N$ orthogonal to N is the projector onto the tangent bundle to the fibres of π.

The following basic result is due to Malliavin.

Proposition 1.7. *Integration along the fibre of π defines a bounded map*

$$\pi_* : W^\infty(B) \to C^\infty(M).$$

proof. Let $f \in W^\infty(B)$ and $\varphi \in C^\infty(M)$. Then we have

$$\begin{aligned} \int_M (\pi_* f)(\Delta^k \varphi)\, dx &= \int_B f \cdot L^k(\pi^*\varphi)\, d\mu \\ &= \int_B (L^*)^k f \cdot (\pi^*\varphi)\, d\mu \le \|f\|_{p,2k} \|\varphi\|_\infty, \end{aligned}$$

for p large enough, where L is the operator

$$L = d^*(\Pi\Pi^*)^{-1} d,$$

and thus L^* is bounded from $W^\infty(B)$ to $W^\infty(B)$. This shows that π_* is bounded from $\cup_{p<\infty} L^p_{2k}(B)$ to $L^2_{2k}(M)$, from which the result follows by Sobolev's lemma, which shows that $C^\infty(M) = \cup_k L^2_{2k}(M)$. □

Despite the fact that B is not locally compact, the following result gives a good substitute for this property.

Proposition 1.7. *There is a positive function $\Psi \in W^\infty(B)$ such that the sets $\{\Psi \leq \lambda\}$ are precompact, for all $\lambda \geq 0$.*

Proof. This result may be found in Kusuoka's paper [9], where he gives a proof based on ideas of L. Gross. However, for the special case of the classical Wiener spaces associated to Brownian motion, there is a simple, explicit, construction of Ψ which was pointed out to the author by Malliavin.

On the classical Wiener space $B = C_*^\alpha([0,T], \mathbb{R}^n)$, consider the function

$$\Psi_{2k,a}(\gamma) = \int_0^T ds \int_0^T dt \, \frac{|\gamma(s) - \gamma(t)|^{2k}}{(s-t)^{1+a}} \quad \text{where } 0 < s < 1,$$

which is finite on the subspace of B for which the Besov B_{pp}^s-norm is finite, where $a = 2ks$ [12]. This function is seen to be in $W^\infty(B)$ if $s < 1/2$ as follows. It is easy to show that

$$(d^*d)^l \Psi_{2k,s} \propto \begin{cases} \Psi_{2(k-l),a-l} & \text{if } l \leq k \\ 0 & \text{if } l > k; \end{cases}$$

this uses the fact that $C(s,t) = \min(s,t)$, the kernel of the covariance of B, satisfies

$$C(s,s) - 2C(s,t) + C(t,t) = |s-t|.$$

Since $\Psi_{2k,a}$ lies in $\bigcap_{p<\infty} L^p(B)$ when $a < k$, it follows that $\Psi_{2k,a} \in W^\infty(B)$ when $a < k$.

Finally, the sets of the form $\Psi_{2k,a} \leq \lambda$ are compact in B if $2k\alpha + 1 < a$, by Sobolev's lemma, so that by choosing k large enough, we can always find suitable s and k. $\square$

§2. The vertical Dirichlet form

Let $\pi : F \longrightarrow M$ be a fibre bundle of finite-dimensional Riemannian manifolds, and let $P \in \Gamma(\mathrm{End}(TF))$ be the projection onto the vertical tangent bundle of the fibration; as in the last section,

$$P = 1 - \Pi(\Pi\Pi^*)^{-1}\Pi \quad \text{where } \Pi = d\pi.$$

Let $d_\pi : W^\infty(B) \longrightarrow W^\infty(B; H)$ be the operation of differentiation along the fibres of π; that is, $d_\pi = P \cdot d$. There is a natural Dirichlet form on F, namely

$$\mathcal{E}_\pi(f,f) = \int_F |d_\pi f|^2 \, d\mu.$$

In this formula, $d\mu$ is the Riemannian volume element on F. This Dirichlet form has the following property: the associated diffusion preserves the fibres of the map π. In other words, we may think of $\mathcal{E}_\pi$ as a family of Dirichlet forms on the fibres of $\pi : F \longrightarrow M$.

This is the idea behind the construction in this section, except that F will be replaced by a Wiener space B, and π will be a W^∞-map satisfying Malliavin's condition $(*)$ of Section 1. The first step in this program is the construction of a convenient compactification of B.

Definition 2.1. *The cylindrical compactification $\hat{B}$ of the Wiener space B is the spectrum of the commutative C^*-algebra obtained by completing the algebra of smooth cylinder functions with respect to the sup-norm, and adjoining an identity.*

Note that the map $B \hookrightarrow \hat{B}$ induces by pushforward of $d\mu$ a Radon measure on B which we shall denote $d\hat{\mu}$.

The cylindrical compactification should not be thought of as being very interesting in itself, but rather as a technical tool, enabling us to invoke various technical results only true for compact topological spaces. For example, using it, Malliavin showed that one can decompose the Wiener measure $d\mu$ along the fibres of π; this gives a nice construction of the Brownian bridge.

Theorem 2.2. *There is a family of measures $d\mu_x$ on $\pi^{-1}(x)$ such that*

$$\int f\, d\mu = \int_M \left(\int_{\pi^{-1}(x)} f\, d\mu_x \right) dx.$$

If $f \in W^\infty(B)$, then $\int_{\pi^{-1}(x)} f\, d\mu_x = (\pi_* f)(x)$, and this property characterizes the measure $d\mu_x$.

Proof. We start by decomposing the measure $d\hat{\mu}$ on $\hat{B}$. By Proposition 1.5, if $\tau^* f$ is a cylinder function on B, then $\pi_*\tau^* f \in C^\infty(M)$. It follows that the map $\tau^* f \mapsto (\pi_*\tau^* f)(x)$ defines a positive linear map on the cylinder functions on B, and hence defines a finite positive Radon measure on $\hat{B}$. We will call this measure $d\hat{\mu}_x$. It is immediate that

$$\int_{\hat{B}} (\tau^* f)\, d\hat{\mu} = \int_M \left(\int_{\hat{B}_x} \tau^* f\, d\mu_x \right) dx,$$

where $\hat{B}_x$ denotes the support of $d\hat{\mu}_x$.

Let us now suppose that B is reflexive. Then the function $\|\cdot\|_B$ on $\hat{B}$ is Borel measurable, indeed lower semicontinuous, since it equals

$$\sup_{\alpha \in B'} |\alpha(x)|/\|\alpha\|_{B'}$$

and each $\alpha \in B'$, being a cylinder function, is measurable on $\hat{B}$. Thus each open set of B is measurable when considered as a subset of $\hat{B}$. Furthermore, by Fernique's Lemma [3], the function $\|\cdot\|_B$ is integrable with respect to the measure $d\mu$, so that $\hat{\mu}(\hat{B} \setminus B) = 0$.

It follows that there is a measurable section of the inclusion of B in $\hat{B}$, say $\rho : \hat{B} \to B$, such that $\rho_* d\hat{\mu} = d\mu$. Namely, send each point in $\hat{B} \setminus B$ to 0 and each point of B to itself. We now simply define $d\mu_x$ to be the measure $\rho_* d\hat{\mu}_x$—it is easy to see that it has all of the desired properties, in particular, that

$$\mu_x(B \setminus \pi^{-1}(x)) = 0.$$

To complete the proof, we must remove the assumption that B is reflexive; to do this, it suffices to apply Proposition 1.2. □

We now turn to the definition of the Dirichlet forms which correspond to diffusions along the fibres: if $f \in W^\infty(B)$, and $x \in M$, then let

$$\mathcal{E}_x(f,f) = \int_B |d_\pi f|^2\, d\mu_x,$$

where P is the projector onto the vertical tangent bundle defined in the last section. Observe that this Dirichlet form, when considered as a Dirichlet form on $\hat{B}$, is regular in the sense of Fukushima [5], so that it leads to a diffusion (Hunt process with a.s. continuous paths) on $\hat{B}$ that never explodes.

Proposition 2.3. *This process has as stationary measures the measures $d\hat{\mu}_x$. Thus, this process remains a.s. in the set $\hat{B}_x$.*

Proof. If $\varphi \in C^\infty(M)$, then $\mathcal{E}_x(f, \pi^*\varphi) = 0$, because

$$d_\pi(\pi^*\varphi) = P \cdot \Pi^*(d\varphi) = 0.$$

Thus, the measures $(\pi^*\varphi)d\mu$ are stationary for each $\varphi \in C^\infty(M)$, and, by approximation, so is the measure $d\hat{\mu}_x$. □

We will now show that the diffusion associated to the Dirichlet form $\mathcal{E}_x$ is in fact a diffusion on B, that is, its paths are continuous in B and it a.s. never explodes. Our proof follows closely that of Kusuoka.

Define a capacity on $\hat{B}$ by

$$\mathrm{Cap}_x(U) = \inf\{\mathcal{E}_x(f,f) \mid f \text{ a cylinder function and } f \geq 1 \text{ a.s. on } U\},$$

for U open in $\hat{B}$, and extend it to arbitrary subset of $\hat{B}$ by the formula

$$\mathrm{Cap}_x(A) = \inf\{\mathrm{Cap}_x(U) \mid A \subset U\}.$$

Theorem 2.4. *The function Ψ of Proposition 1.8 is a quasi-continuous function on $\hat{B}$ with repect to the capacity* Cap_x*. In particular,* $\mathrm{Cap}_x\{\Psi = \infty\} = 0$*, so that the diffusion corresponding to $\mathcal{E}_x$ remains a.s. in the set $\{\Psi < \infty\} \subset B$, and has continuous paths in the topology of B. This diffusion has $d\mu_x$ as a stationary measure.*

Proof. Let $\eta(t)$ be a smooth function which equals t if $t \leq 1/2$ and 1 if $t \geq 1$. Then the function $\Psi_\lambda = \eta(\lambda^{-1}\Psi(x))$ is a continuous function on $\hat{B}$ for each $\lambda < \infty$, since it equals the constant function 1 outside the compact set $\{\Psi \leq \lambda\}$, and is continuous inside this set. Using the fact that $\Psi_\lambda \to \Psi$ in $W^\infty(B)$ as $\lambda \to \infty$, it follows by the boundedness of π_* from $W^\infty(B)$ to $C^\infty(M)$ that

$$\lim_{\lambda\to\infty} \mathcal{E}_x(\Psi - \Psi_\lambda, \Psi - \Psi_\lambda) = \lim_{\lambda\to\infty} (\pi_*|d_\pi(\Psi - \Psi_\lambda)|^2)(x) = 0,$$

from which it follows by the results of Chapter 3 of [5] that Ψ is quasi-continuous.

By Tchebycheff's inequality,

$$\mathrm{Cap}_x(\Psi \geq \lambda) \leq \lambda^{-2}\mathcal{E}_x(\Psi, \Psi)$$

since Ψ is quasi-continuous on $\hat{B}$, so that $\mathrm{Cap}_x(\Psi = \infty) = 0$. This shows that the Markov process corresponding to $\mathcal{E}_x$ stays within the set on which Ψ is finite.

As for showing that the paths of the process are continuous in the topology of B, this is shown in the same way as in Kusuoka's paper by using the fact that the functions $\|\cdot + h\|_B$ are quasi-continuous with respect to the capacity Cap_x for each $h \in H$. Finally, it is clear that this process has the measure $d\mu_x$ as a stationary measure. □

§3. The Ricci curvature

In this section, we will calculate the Ricci curvature $\mathbf{Ric}(\cdot) \in I + W^\infty(B, \mathrm{HS}(H))$ of the Dirichlet form of Section 2, in the sense of Bakry and Emery [1]. Let $L = d_\pi^* d_\pi$ be the generator of the Dirichlet form $\mathcal{E}_\pi(f,g)$, defined by the formula

$$\mathcal{E}_\pi(f,g) = \int_B f \cdot Lg \, d\mu.$$

Bakry and Emery introduce a hierarchy of bilinear operators

$$\Gamma_k(\cdot,\cdot) : W^\infty(B) \times W^\infty(B) \to W^\infty(B)$$

defined recursively by

$$\begin{aligned} \Gamma_0(f,g) &= f \cdot g \\ \Gamma_{k+1}(f,g) &= \tfrac{1}{2}\{\Gamma_k(Lf,g) + \Gamma_k(f,Lg) - L\Gamma_k(f,g)\} \end{aligned}$$

It is easy to show that

$$\mathcal{E}_\pi(f,g) = \int_B \Gamma_1(f,g)\, d\mu;$$

this is equivalent to the fact that the Markov process on B associated to Dirichlet form $\mathcal{E}_\pi$ has continuous paths. Our task is to calculate $\Gamma_2(f,g)$, which Bakry and Emery have shown is an important invariant of a Dirichlet form.

Given a map $\pi : B \longrightarrow M$ from a Wiener space to a Riemannian manifold M satisfying Malliavin's condition $(*)$, we may define the second fundamental form S by the following formula:

$$S(X) = N \cdot (d_{PX}P) \cdot P, \quad \text{where } X \text{ lies in } H.$$

In fact, since $P^2 = P$, Leibniz's formula shows that

$$(d_\pi P)P = N(d_\pi P) \quad \text{and} \quad (d_\pi P)N = P(d_\pi P),$$

so $S(X)$ is given by the simpler formulas $(d_{PX}P)P$ and $N(d_{PX}P)$. As in the finite dimensional case, $S(X)Y$ is symmetric in X and Y. Indeed, if $[PX, PY]$ is the Lie bracket of the vector fields PX and PY, then

$$S(X)Y - S(Y)X = N(d_{PX}(PY) - d_{PY}(PX)) = N[PX, PY].$$

However, $N[PX, PY] = 0$, because the Lie bracket of two vector fields tangential to the map π is itself tangential.

Proposition 3.1. *The second fundamental form S lies in $W^\infty(B, H \otimes_2 \mathrm{HS}(H))$, and is given by the following formula:*

$$S = \Pi^*(\Pi\Pi^*)^{-1}(d_\pi\Pi)P.$$

Proof. Using the formula $P = I - \Pi^*(\Pi\Pi^*)^{-1}\Pi$, we see that $d_\pi P$ equals

$$(d\Pi^*)(\Pi\Pi^*)^{-1}\Pi - \Pi^*(\Pi\Pi^*)^{-1}((d_\pi\Pi)\Pi^* + \Pi^*(d_\pi\Pi^*))(\Pi\Pi^*)^{-1}\Pi + \Pi^*(\Pi\Pi^*)^{-1}(d_\pi\Pi).$$

This simplifies to $P(d_\pi\Pi^*)(\Pi\Pi^*)^{-1}\Pi + \Pi^*(\Pi\Pi^*)^{-1}(d_\pi\Pi)$; multiplying by P on the right, we obtain the stated formula.

Since $S(X)Y$ is a tensor in X, it follows from its symmetry in X and Y that it is a tensor in both variables. The fact that $S \in W^\infty(B; H \otimes_2 \mathrm{HS}(H))$ is an immediate consequence of this and the above formula. $\square$

As in finite dimensions, the Levi-Civita covariant derivative on the tangent bundle to the leaves is given by the formula

$$\nabla_\pi Y = d_\pi PY - S(X)Y,$$

with ∇^* as is its formal adjoint. We also define the Hessian operator $\nabla_\pi^2 : W^\infty(B) \longrightarrow W^\infty(B, H \otimes_2 H)$, by

$$\nabla_\pi^2 f = \nabla_\pi d_\pi f.$$

As in finite dimensions, the tensor $\nabla_\pi^2 f$ takes its values in the symmetric part of $H \otimes_2 H$.

We are now ready to state the formula for $\Gamma_2(f,g)$.

Theorem 3.2. *The quadratic form $\Gamma_2(f,g)$ equals*

$$(\nabla_\pi^2 f, \nabla_\pi^2 g) + (\mathbf{Ric}\, d_\pi f, d_\pi g),$$

where the operator **Ric** *(the Ricci curvature of the Dirichlet form) is given by the formula*

$$\mathbf{Ric} = P + \mathrm{Tr}(S^*S) + S^*(d^*N).$$

In particular, $\mathbf{Ric} \in I + W^\infty(B, \mathrm{HS}(H))$.

Proof. Consider the Riemannian manifold M with smooth measure $d\mu = e^{-\varphi}dx$, where $\varphi \in C^\infty(M)$:

$$\mathcal{E}_\varphi(f,g) = \int_M (df, dg)\, d\mu.$$

Bakry and Emery calculate the quadratic form Γ_2 of $\mathcal{E}_\varphi$ as follows. Introduce the differential operator $\tilde{L}$ on one-forms, by the formula

$$\tilde{L} = d_\varphi^* d + d d_\varphi^*,$$

where $d_\varphi^* = d^* + \iota(d\varphi)$ is the adjoint of the exterior differential with respect to the inner product on one-forms induced by the metric on M and the measure $d\mu$. The reason for introducing the operator $\tilde{L}$ is that

$$\tilde{L}d = dL.$$

Let $\nabla_\varphi^* = \nabla^* + d\varphi$ be the adjoint of the Levi-Civita derivative ∇ on one-forms.

Lemma. *If* **Ric** *and* $\nabla^2\varphi$ *be the Ricci curvature of the manifold M and the Hessian of the function φ considered as symmetric endomorphisms of the cotangent bundle of M, then*

$$\tilde{L} = \nabla_\varphi^*\nabla + \mathbf{Ric} + \nabla^2\varphi$$

Proof. By the Weitzenböck formula for $\varphi = 0$, we see that

$$\begin{aligned}\tilde{L} &= d^*d + dd^* + (d\iota(d\varphi) + \iota(\varphi)d) \\ &= \nabla^*\nabla + \mathbf{Ric} + Ł(d\varphi) \\ &= \nabla_\varphi^*\nabla + \mathbf{Ric} + (Ł(d\varphi) - d\varphi\cdot\nabla)\end{aligned}$$

The operator $Ł(d\varphi) - d\varphi\cdot\nabla$ is local:

$$[Ł(d\varphi) - d\varphi\cdot\nabla, f] = d\varphi\cdot df - d\varphi\cdot df = 0.$$

In fact, it equals $\nabla^2\varphi$, as is shown by applying it to the one-form df:

$$\begin{aligned}(Ł(d\varphi) - d\varphi\cdot\nabla)df &= d(d\varphi\cdot df) - d\varphi\cdot\nabla^2 f \\ &= (\nabla^2\varphi\cdot df + d\varphi\cdot\nabla^2 f) - d\varphi\cdot\nabla^2 f \\ &= \nabla^2\varphi\cdot df \quad \square\end{aligned}$$

Armed with this result, it is easy to calculate Γ_2 for the Dirichlet form $\mathcal{E}_\varphi$:

$$\begin{aligned}\Gamma_2(f,f) &= \Gamma_1(Lf,f) - \tfrac{1}{2}L\Gamma_1(f,f) \\ &= (dLf, df) + |\nabla^2 f|^2 - ((\nabla_\varphi^*\nabla)df, df) \\ &= (\tilde{L}df, df) + |\nabla^2 f|^2 - (\tilde{L}df, df) + ((\mathbf{Ric} + \nabla^2\varphi)df, df)\end{aligned}$$

Thus, we obtain the following basic formula:

$$\Gamma_2(f,g) = (\nabla^2 f, \nabla^2 g) + ((\mathbf{Ric} + \nabla^2\varphi)df, df).$$

We now return to the situation in which we have a fibration $\pi : B \longrightarrow M$, where B is a Wiener space, except that we assume that B is the finite dimensional space $\mathbb{R}^N$, with Gaussian measure the standard one:

$$d\mu = (2\pi)^{-N/2} e^{-|x|^2/2}.$$

Lemma. *The Hessian* $\nabla_\pi^2\varphi$ *of* $\varphi = |x|^2/2 + N\log(2\pi)/2$ *along the fibres of* π *is equal to*

$$\nabla_\pi^2\varphi = P + S^*x.$$

Proof. Since $d_\pi\varphi = Px$, we see that

$$\nabla_\pi^2\varphi = \nabla_\pi(Px) = P + (\nabla_\pi P - S)x$$

However, $\nabla_\pi P - S = \nabla_\pi P - N\cdot\nabla_\pi P = P\cdot\nabla_\pi P = (\nabla_\pi P)\cdot N = S^*$, proving the lemma. □

Let h be the mean curvature of the fibration; in terms of the orthonormal basis e_i,

$$h = \sum_i S(e_i)e_i.$$

In particular, h is a section of the normal bundle to the fibration π. There is a formula for h in terms of the projection $N = I - P$, which is easily verified in the orthonormal frame e_i:

$$-h + Nx = N(d^*N).$$

We need one final lemma, which gives a formula for the Ricci curvature of the leaves in terms of the second fundamental form.

Lemma. *The Ricci curvature of the leaves is given by the formula*

$$\mathbf{Ric} = \mathrm{Tr}(S^*S) - S^*h.$$

Proof. In terms of the orthonormal frame e_i, in which $S(e_i)e_j = S_{ij}^k e_k$, the Riemannian curvature of the leaves equals

$$R_{ijkl} = S_{ik}^m S_{jl}^m - S_{il}^m S_{jk}^m.$$

Taking the trace over j and k gives

$$\mathbf{Ric}_{ij} = S_{in}^m S_{jn}^m - S_{ij}^m S_{nn}^m. \quad \square$$

Putting these lemmas together gives

$$\mathbf{Ric} + \nabla_\pi^2\varphi = \mathrm{Tr}(S^*S) + S^*(-h + x) + P = \mathrm{Tr}(S^*S) + S^*(d^*N) + P$$

In this way, we have succeeded in rewriting the formula for Γ_2 of the Dirichlet form $\mathcal{E}_\pi$ in such a way as to make sense for a map $\pi : B \to M$ satisfying Malliavin's condition. However, it is obvious that *a priori*, the Ricci curvature of the Dirichlet form $\mathcal{E}_\pi$ in the infinite dimensional case is given by a polynomial in the derivatives of the operator P, that is, a sum of terms of the form

$$C(D_1P \otimes_2 \dots \otimes_2 D_kP),$$

where D_i is product of operators d and d^*, and C is a map from $H \otimes_2 \dots \otimes_2 H$ (m times) to $H \otimes_2 H$ obtained by performing $m/2 - 1$ contractions. The formula for the Ricci curvature, in this form, must be independent of the Wiener space B, so will be given by the same formula as when B is finite dimensional, which we have just calculated. □

Observe that this theorem generalizes the formula for $\Gamma_2(f,g)$ for the Dirichlet form $\mathcal{E}$ on a Wiener space, which is equal to

$$\Gamma_2(f,g) = (\nabla^2 f, \nabla^2 g) + (\nabla f, \nabla g).$$

In this case, the Ricci curvature is equal to the identity operator; in particular, it is positive. It has been proved by Bakry and Emery [1] that if the Ricci form is strictly positive, then the Laplacian on $L^2(B)$ is hypercontractive and satisfies the Paley-Littlewood inequalities proved by Meyer for Wiener space, namely

$$\|\nabla f\|_p \sim \|\sqrt{\Delta} f\|_p \quad \text{for } p < \infty.$$

Unfortunately, the Ricci curvature that we calculated in Theorem 3.2 appears to be positive only in this single case of a Wiener space. It is not even clear whether the Ornstein-Uhlenbeck operator on L_xM, which is the generator of the Dirichlet form that we construct, has a positive gap between its lowest two eigenvalues. Nevertheless, the calculation is instructive.

§4. Example: the loop space of a compact Lie group

In this section, we will calculate the Ricci curvature of our Dirichlet form in the simplest non-trivial case, in which the manifold M is equal to a compact Lie group G, and the map π is the so-called path-ordered exponential. To simplify the derivation of the formulas, we will assume that the group G is a matrix group; of course, this is no restriction, since every compact Lie group has a faithful representation. We also suppose chosen an invariant Riemannian metric on G, which induces a Euclidean metric on $\mathbf{g}$, the Lie algebra of G.

Let (B,H) be the classical Wiener space $(C^{\alpha}_{*}([0,1],\mathbf{g}), L^{2,1}_{*}([0,1],\mathbf{g}))$, $(\alpha < 1/2)$. If $x_t \in H$, we solve the ordinary differential equation for $\gamma_t : [0,1] \to G$ with initial condition $\gamma_0 = e$,

$$\omega(\dot{\gamma}_t) = \gamma_t^{-1}\dot{\gamma}_t = \dot{x}_t.$$

The solution of this equation is known as the path-ordered exponential, and is denoted

$$\gamma_t = \operatorname{Pexp}(\textstyle\int_0^t dx_s).$$

The path-ordered exponential identifies the space H with the space of finite-energy paths in P_*G. Let π be the map $x_t \mapsto \gamma_1 = \operatorname{Pexp}(\int_0^1 dx_s)$; the fibre of π over e can be identified with the space L_*G of based loops in G:

$$\begin{array}{ccc} L_*G & \longrightarrow & H \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ \{e\} & \longrightarrow & G \end{array}$$

The map $x_t \mapsto \gamma_t$ is extended to a family of W^{∞}-maps from the Wiener space B to G, by introducing a mollifier on B:

$$x^{\varepsilon}_t = \varepsilon^{-1} \int_0^1 \lambda(\varepsilon^{-1}(s-t))\, x_s \, ds,$$

where λ is any positive symmetric function in $C^{\infty}_0[-1,1]$ such that $\int_{[-1,1]} \lambda = 1$.

Proposition 4.1.

(1) *For each $\varepsilon > 0$, the map $\pi^{\varepsilon}(x_t) = \pi(x^{\varepsilon}_t)$ is a W^{∞}-map from B to G.*
(2) *As $\varepsilon \to 0$, the maps π^{ε} converge in $W^{\infty}(B;G)$ to a map π.*
(3) *The measure $d\mu_e$ induced on L_*G by the decomposition of $d\mu$ over the fibres of π is the Wiener measure for G considered as a Riemannian manifold.* □

Proof. See, for example, [8]. □

We will now calculate the differential $d\pi$ of the map π explicitly.

Proposition 4.2.

(1) $\Pi = (d\pi)\pi^{-1} : W^{\infty}(B;H) \to \mathbf{g}$ *is given by the following formula:*

$$\Pi(h_t) = \int_0^1 \operatorname{Ad}\gamma_t \cdot dh_t.$$

(2) *The adjoint* $\Pi^*(X) : \mathbf{g} \to W^{\infty}(B;H)$ *of* Π *equals*

$$\Pi^*_t(X) = \int_0^t \operatorname{Ad}\gamma_s^{-1} \cdot X\, ds.$$

(3) *The composition* $\Pi\Pi^* = I$*; in particular, the map π satisfies the Malliavin condition $(*)$, since* $\det(\Pi\Pi^*) = 1$*, and* $N = \Pi^*\Pi$.

Proof. We will calculate $\Pi^\varepsilon = (d\pi^\varepsilon)(\pi^\varepsilon)^{-1}$, and then take the limit $\varepsilon \longrightarrow 0$. For $\varepsilon > 0$, the map π^ε is smooth, so we can calculate Π^ε path by path.

By du Hamel's formula, $(d\pi^\varepsilon)(\pi^\varepsilon)^{-1}$ equals

$$(d\pi^\varepsilon)(\pi^\varepsilon)^{-1} = d\gamma_1^\varepsilon \cdot (\gamma_1^\varepsilon)^{-1} = \int_0^1 \operatorname{Ad}\gamma_t^\varepsilon \cdot dh_t^\varepsilon$$

from which part (1) follows, by sending $\varepsilon \to 0$.

Since the metric on $\mathbf{g}$ is invariant, it follows that the adjoint of $\operatorname{Ad} g$ is $\operatorname{Ad} g^{-1}$, for any $g \in G$. It is easy to check that if $X \in \mathbf{g}$, then

$$(X, \Pi(h_t)) = \int_0^1 (X, \operatorname{Ad}\gamma_t \cdot dh_t) = \int_0^1 (\dot{k}_t, \dot{h}_t)\, dt = (k_t, h_t)_H,$$

where $k_t = \int_0^t \operatorname{Ad}\gamma_s^{-1} \cdot X\, ds$, from which we obtain the formula for $\Pi^*(X)$. It is clear from this that $\Pi\Pi^*(X) = X$. $\square$

We can simplify the formulas for N and P by introducing the unitary operator on H defined by

$$Uh_t = \int_0^t \operatorname{Ad}\gamma_s \cdot dh_s.$$

In terms of the operator U, we obtain the following formulas for the projectors P and N:

$$\begin{cases} UNU^{-1}h_t = th_1 \\ UPU^{-1}h_t = h_t - t \cdot h_1 \end{cases}$$

Similarly, we define U^ε, also unitary, by $U^\varepsilon h_t = \int_0^t \operatorname{Ad}\gamma_s^\varepsilon \cdot dh_s$, so that the operators $N^\varepsilon = (\Pi^\varepsilon)^*\Pi^\varepsilon$ and $P^\varepsilon = I - N^\varepsilon$ are given by the formulas

$$\begin{cases} U^\varepsilon N^\varepsilon (U^\varepsilon)^{-1}h_t = th_1^\varepsilon \\ U^\varepsilon P^\varepsilon (U^\varepsilon)^{-1}h_t = h_t^\varepsilon - t \cdot h_1^\varepsilon \end{cases}$$

We now turn to the calculation of the second fundamental form of the map π.

Proposition 4.3. *The second fundamental form S equals*

$$(\Pi^*(X), S(a_t)b_t) = \left(X, \int_0^1 \int_0^1 \chi(s,t)\,(d(UP)a_s, d(UP)b_t)\right) \quad \text{for } a_t, b_t \in H \text{ and } X \in \mathbf{g},$$

where $\chi(s,t)$ is the function

$$\chi(s,t) = \begin{cases} \frac{1}{2} & s < t, \\ -\frac{1}{2} & s > t. \end{cases}$$

Proof. If we take the formula for $d\pi^\varepsilon$ and differentiate it once more, we obtain

$$\begin{aligned} d^2\pi^\varepsilon(\pi^\varepsilon)^{-1}(a_t, b_t) &= \int\int_{s<t} (\operatorname{Ad}\gamma_s^\varepsilon \cdot da_s^\varepsilon)(\operatorname{Ad}\gamma_t^\varepsilon \cdot db_t^\varepsilon) - \int\int_{s>t} (\operatorname{Ad}\gamma_t^\varepsilon \cdot da_t^\varepsilon)(\operatorname{Ad}\gamma_s^\varepsilon \cdot db_s^\varepsilon) \\ &= \int_0^1 \int_0^1 \chi(s,t)[\operatorname{Ad}\gamma_s^\varepsilon \cdot da_s^\varepsilon, \operatorname{Ad}\gamma_t^\varepsilon \cdot db_t^\varepsilon] \end{aligned}$$

If we now let $\varepsilon \to 0$, we see that

$$d^2\pi(a_t, b_t) \cdot \pi^{-1} = \int_0^1 \int_0^1 \chi(s,t)[\operatorname{Ad}\gamma_s \cdot da_s, \operatorname{Ad}\gamma_t \cdot db_t].$$

The formula for $S(a_t)b_t$ follows easily from this, since

$$(\Pi^*(X), S(a_t)b_t) = (X, d^2(Pa_t)Pb_t). \quad \square$$

Observe that the formula for S is explicitly symmetric in a_t and b_t, since both $\chi(s,t)$ and $[\cdot,\cdot]$ are antisymmetric.

In [4], Freed has calculated the Ricci curvature of the loop space of a compact Lie group G with invariant metric $(\cdot,\cdot)$, not in the sense of Bakry and Emery but in the usual Riemannian sense $\mathbf{Ric} = \operatorname{Tr}(S^*S) - S^*h$. Freed obtains his formula by discarding the term involving the mean curvature h which, while divergent, is formally zero. In the rest of this section, we will calculate the Ricci curvature of the Dirichlet form on L_*G; although our calculations are similar to his, we do not have to discard any divergent quantities, since we are considering the Ricci curvature not of the loop space but of its Dirichlet form.

In order to calculate the Ricci curvature, we need to be able to take the trace of a quadratic form on H.

Lemma 4.4. *Let A be the quadratic form on $H \otimes_2 H$ defined by the formula*

$$A(a_t, b_t) = \int_0^1 \int_0^1 (\varphi(s,t) \cdot d(U^\varepsilon P^\varepsilon)a_s, d(U^\varepsilon P^\varepsilon)b_t),$$

where $\varphi \in C^\infty([0,1] \times [0,1], \operatorname{End}(\mathbf{g}))$. Then the trace of A equals

$$\operatorname{Tr}(A) = \int_0^1 \operatorname{Tr}(\lambda_\varepsilon * \varphi * \lambda_\varepsilon)(t,t)\, dt - \int_0^1 \int_0^1 \operatorname{Tr}(\lambda_\varepsilon * \varphi * \lambda_\varepsilon)(s,t)\, ds\, dt.$$

Proof. We start by substituting the formula for $U^\varepsilon P^\varepsilon$ into the definition of $A(a_t, b_t)$:

$$A(a_t, b_t) = \int_0^1 \int_0^1 (\varphi(s,t) \cdot dU^\varepsilon a_s^\varepsilon - \varphi(s,t) \cdot U^\varepsilon a_1^\varepsilon dt, dU^\varepsilon b_t^\varepsilon - U^\varepsilon b_1^\varepsilon dt).$$

Since the operator U^ε is unitary, replacing it by the identity operator does not change the trace of $A(\cdot,\cdot)$; we are left with calculating the trace of

$$A(a_t, b_t) = \int_0^1 \int_0^1 (\varphi(s,t) \cdot da_s^\varepsilon - \varphi(s,t) \cdot a_1^\varepsilon dt, db_t^\varepsilon - b_1^\varepsilon dt).$$

The lemma now follows from the fact that the trace of the quadratic form on H

$$\int_0^1 \int_0^1 (\varphi(s,t) \cdot da_s^\varepsilon, db_t^\varepsilon)$$

equals $\int_0^1 \operatorname{Tr}\varphi(t,t)\, dt$. $\square$

We can now calculate $S^*(d^*N)$. Recall that in the finite dimensional case, we had

$$N(d^*N) = Nx_t - h,$$

where h is the mean curvature of the fibres of π. This formula also holds in infinite dimensions if we replace N by the regularized $N^\varepsilon = (\Pi^\varepsilon)^*\Pi^\varepsilon$. Thus, let S^ε be the element of $W^\infty(B; H \otimes_2 \mathrm{HS}(H))$ constructed by replacing Π by Π^ε in the formula for S; then, if $X \in \mathbf{g}$, we have, for $a_t, b_t \in H$,

$$(X, S^\varepsilon(a_t)b_t) = \left(X, \int_0^1 \int_0^1 \chi(s,t)[d(U^\varepsilon P^\varepsilon)a_s^\varepsilon, d(U^\varepsilon P^\varepsilon)b_t^\varepsilon] \right).$$

It is clear that the quadratic form $(X, S^\varepsilon(a_t)b_t)$ on H is trace-class for $\varepsilon > 0$, since it is given by a smooth kernel. By Lemma 4.4, its trace in these variables, which we write (X, h^ε), is equal to

$$\left(\int_0^1 \chi(t,t)\, dt - \int_0^1 \int_0^1 \chi(s,t)\, ds\, dt \right) \cdot \sum_i (X, [X_i, X_i]),$$

which obviously vanishes. (Here, X_i is an orthonormal basis of $\mathbf{g}$.) Thus, the mean curvature h, defined to be $\lim_{\varepsilon\to 0} h^\varepsilon$, equals zero.

We now calculate $N^\varepsilon x_t$; for $\varepsilon > 0$, it is given by the Stratanovitch stochastic integral

$$N^\varepsilon x_t = \Pi^*\Pi(x_t) = (\Pi^\varepsilon)^\varepsilon \cdot \int_0^1 \mathrm{Ad}\, \gamma_t^\varepsilon \cdot dx_t.$$

It is a basic property of Stratanovitch integrals that they are continuous in the integrand. Thus, we have proved the following result:

Proposition 4.5. *The normal projection $N(d^*N)$ of the divergence of N equals the Stratanovitch stochastic integral*

$$\Pi^* \cdot \int_0^1 \mathrm{Ad}\, \gamma_t \cdot dx_t \in W^\infty(B; H).$$

Consequently, the quadratic form $(a_t, S^*(d^*N)b_t)$ equals

$$\left(\int_0^1 [(UP)a_t, d(UP)b_t], Z(x_t) \right).$$

where $Z(x_t) = \int_0^1 \mathrm{Ad}\, \gamma_t \cdot dx_t$. $\square$

We now turn to the calculation of $\mathrm{Tr}(S^*S)$. By Proposition 3.1, we have

$$\begin{aligned}(a_t, S(b_t)^*S(c_t)d_t) &= (d^2(Pa_s, Pb_t), d^2(Pc_u, Pd_v)) \\ &= \int_0^1\int_0^1\int_0^1\int_0^1 \chi(s,t)\,\chi(u,v)([d(UP)a_s, d(UP)b_t], [d(UP)c_u, d(UP)d_v]).\end{aligned}$$

We can integrate by parts with respect to s and u, using the fact that $\partial_s \chi(s,t) = \delta(s,t)$ and that $(UP)h_t = 0$ for $s \in \{0,1\}$:

$$(a_t, S(b_t)^*S(c_t)d_t) = \int_0^1 \int_0^1 ([(UP)a_s, d(UP)b_s], [(UP)c_t, d(UP)d_t]).$$

If we now take the trace over b_t and d_t using Lemma 4.4, we obtain

$$\int_0^1 K((UP)a_t, (UP)c_t)\, dt - \int_0^1 \int_0^1 K((UP)a_s, (UP)b_t)\, ds\, dt$$

where K is the Killing form on $\mathbf{g}$. The second term vanishes, since it equals $K(\Pi \cdot Pa_t, \Pi \cdot Pb_t)$.

Thus we have succeeded in proving the following formula for the Ricci curvature.

Theorem 4.6. *The Ricci curvature of the Dirichlet form $\mathcal{E}_\pi$ on L_*G evaluated on the vector $PU^{-1}a_t$ equals*

$$\int_0^1 \left(|\dot{a}_t|^2 + K(a_t, a_t) + (Z(x_t), [a_t, \dot{a}_t])\right) dt. \quad \square$$

As pointed out earlier, this quadratic form is only positive in the case in which the group G is abelian, in which case the based loop space is a Wiener space. It is possible that an extension of Bakry's and Emery's work can be found which will cover this Dirichlet form as well, showing that it too is hypercontractive, but this must remain a speculation for the moment.

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