CYCLIC OPERADS AND CYCLIC HOMOLOGY

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The cyclic homology of associative algebras was introduced by Connes [4] and Tsygan [22] in order to extend the classical theory of the Chern character to the non-commutative setting. Recently, there has been increased interest in more general algebraic structures than associative algebras, characterized by the presence of several algebraic operations. Such structures appear, for example, in homotopy theory [18], [3] and topological field theory [9]. In this paper, we extend the formalism of cyclic homology to this more general framework.

This extension is only possible under certain conditions which are best explained using the concept of an operad. In this approach to universal algebra, an algebraic structure is described by giving, for each $n \geq 0$, the space $\mathcal{P}(n)$ of all n-ary expressions which can be formed from the operations in the given algebraic structure, modulo the universally valid identities. Permuting the arguments of the expressions gives an action of the symmetric group $\mathbb{S}_n$ on $\mathcal{P}(n)$. The sequence $\mathcal{P} = \{\mathcal{P}(n)\}$ of these $\mathbb{S}_n$-modules, together with the natural composition structure on them, is the operad describing our class of algebras. In order to define cyclic homology for algebras over an operad $\mathcal{P}$, it is necessary that $\mathcal{P}$ is what we call a *cyclic operad*: this means that the action of $\mathbb{S}_n$ on $\mathcal{P}(n)$ extends to an action of $\mathbb{S}_{n+1}$ in a way compatible with compositions (see Section 2). Cyclic operads are a natural generalization of associative algebras with involution (see (2.2)).

For associative algebras over a field of characteristic 0, it is a result of Feigin and Tsygan [6] that cyclic homology is the non-abelian derived functor of the functor $A \mapsto A/[A, A]$, the target of the universal trace on A. The notion of a trace does not make sense for more general algebras. However, for an associative algebra A *with unit,* there is a bijection between traces T and invariant bilinear forms B on A, obtained by setting $T(x) = B(1, x)$. It turns out that the structure of a cyclic operad on $\mathcal{P}$ is precisely the data needed to speak about invariant bilinear forms on algebras over an operad $\mathcal{P}$. The cyclic homology of an algebra over a cyclic operad $\mathcal{P}$ is now defined as the non-abelian derived functor of the target of the universal invariant bilinear form.

In making this definition, we were strongly influenced by ideas of M. Kontsevich: he emphasized that invariant bilinear forms are more fundamental than traces [15]. Furthermore, he constructed an action of $\mathbb{S}_{n+1}$ on the space $\mathsf{Lie}(n)$ of n-ary operations on Lie algebras.

We construct an explicit complex $\mathrm{CA}(\mathcal{P}, A)$ calculating the cyclic homology. There is a long exact sequence which involves not only the cyclic homology $\mathrm{HA}_\bullet(\mathcal{P}, A)$ but two other functors $\mathrm{HB}_\bullet(\mathcal{P}, A)$ and $\mathrm{HC}_\bullet(\mathcal{P}, A)$:

$$\cdots \to \mathrm{HC}_{n+1}(\mathcal{P}, A) \to \mathrm{HA}_n(\mathcal{P}, A) \to \mathrm{HB}_n(\mathcal{P}, A) \to \mathrm{HC}_n(\mathcal{P}, A) \to \dots$$

In the general case, $\mathrm{HC}(\mathcal{P}, A)$ may be identified with the cyclic homology of the dual algebra over the dual operad (in the sense of [12] and [11]). For unital associative algebras, this sequence may be identified with the sequence of Connes and Tsygan

$$\cdots \to H^\lambda_{n+2}(A) \xrightarrow{S} H^\lambda_n(A) \xrightarrow{B} H_{n+1}(A, A) \xrightarrow{I} H^\lambda_{n+1}(A) \to \dots,$$

by an isomorphism $\mathrm{HA}_n(\mathsf{UAs}, A) \cong \mathrm{HC}_{n+1}(\mathsf{UAs}, A)$ which has no analogue for a general operad. For Lie algebras, we obtain the long exact sequence

$$\cdots \to H_{n+2}(\mathfrak{g}) \to \mathrm{HA}_n(\mathsf{Lie}, \mathfrak{g}) \to H_n(\mathfrak{g}, \mathfrak{g}) \to H_{n+1}(\mathfrak{g}) \to \dots,$$

where $\mathrm{HA}_\bullet(\mathsf{Lie}, \mathfrak{g})$ is the Lie cyclic homology of the Lie algebra $\mathfrak{g}$.

Using the long exact sequence for the commutative operad Com, we are able to prove an isomorphism of $\mathbb{S}_{n+1}$-modules

$$\mathsf{Lie}(n+1) \cong_{\mathbb{S}_{n+1}} \mathsf{Lie}(n) \otimes V_{n,1},$$

where $V_{n,1}$ is the n-dimensional irreducible representation of $\mathbb{S}_{n+1}$ on the hyperplane $x_0 + \cdots + x_n = 0$.

Another example of a cyclic operad is the operad whose algebras are A_∞-algebras: cyclic homology of A_∞-algebras was studied in [10].

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1. Operads and trees

Here we mostly review the background material referring the reader to May [18] and to introductory chapters of [11], [12] for more details.

(1.1). **Notation.** Throughout this paper, we work over a fixed field k of characteristic 0. We denote the symmetric group $\mathrm{Aut}\{1, \dots, n\}$ by $\mathbb{S}_n$. We shall also need the group $\mathrm{Aut}\{0, 1, \dots, n\}$ which we shall denote $\mathbb{S}_{n+1}$. To avoid confusion, we shall always explicitly write "+1" in the subscript when we deal with this latter group. For example, $\mathbb{S}_{2+1}$ will denote the group $\mathrm{Aut}\{0, 1, 2\}$.

For $n \geq 0$, we denote by $\tau_n \in \mathbb{S}_{n+1}$ the cycle $(0\ 1 \dots n)$; we will often write it τ if n is clear from the context. The element τ_n generates a cyclic group $\mathbb{Z}_{n+1} \subset \mathbb{S}_{n+1}$ which, together with $\mathbb{S}_n$, generates $\mathbb{S}_{n+1}$.

The category of *graded* k*-vector spaces* $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a symmetric monoidal category, with tensor product

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j$$

and symmetry maps $S : V \otimes W \to W \otimes V$ given by the formula

$$v \otimes w \mapsto (-1)^{|v|\,|w|} w \otimes v;$$

here, we write $|v| = i$ if $v \in V_i$.

A *chain complex* (dg-vector space) is a graded vector space $V_\bullet$ together with a differential $\delta : V_i \to V_{i-1}$, such that $\delta^2 = 0$. The category of chain complexes is again a symmetric monoidal category with tensor product and symmetry defined as for graded vector spaces and the differential given by the standard Leibniz formula (see, for example, Dold [5]). Given two chain complexes V and W, there is a chain complex $\hom(V, W)$, such that $\hom_n(V, W)$ is the space of all linear maps homogeneous of degree n from V to W [5].

The *suspension* $\Sigma V_\bullet$ of a chain complex $V_\bullet$ has components $(\Sigma V)_n = V_{n+1}$, and differential equal to minus that of $V_\bullet$. By $\Sigma^n V_\bullet$, $n \in \mathbb{Z}$, we denote the n-fold iterated suspension of $V_\bullet$.

The *determinant* $\det(V)$ of a finite dimensional vector space V is its highest exterior power $\Lambda^{\dim(V)} V$.

(1.2). **$\mathbb{S}$-modules.** An $\mathbb{S}$-*module* is a sequence of vector spaces $\mathcal{V} = \{\mathcal{V}(n) \mid n \geq 0\}$, together with an action of $\mathbb{S}_n$ on $\mathcal{V}(n)$ for each n. Denote the category of $\mathbb{S}$-modules, with the evident notion of morphisms, by $\mathbb{S}$-mod.

A *cyclic* $\mathbb{S}$-*module* is a sequence of vector spaces $\mathcal{V} = \{\mathcal{V}(n) \mid n \geq 0\}$, together with an action of $\mathbb{S}_{n+1}$ on $\mathcal{V}(n)$. The category of cyclic $\mathbb{S}$-modules will be denoted $\mathbb{S}_+$-mod. Since $\mathbb{S}_n$ is a subgroup of $\mathbb{S}_{n+1}$, any cyclic $\mathbb{S}$-module may be regarded as an $\mathbb{S}$-module.

An $\mathbb{S}$-module $\mathcal{V}$ gives rise to a functor, which we also denote by $\mathcal{V}$, from the category of finite sets and their bijections to the category of vector spaces. Explicitly, if I is a set with n elements, we define

$$\mathcal{V}(I) = \left(\bigoplus_{f:\{1,\dots,n\}\to I} \mathcal{V} \right)_{\mathbb{S}_n}, \tag{1.3}$$

where the sum is over all bijections f and the space of coinvariants is taken with respect to the total action. For example, if $V = W^{\otimes n}$ is the n-th tensor power of another vector space W, with the natural action of $\mathbb{S}_n$, then the formula (1.3) gives the space which we shall denote $W^{\otimes I}$. In a similar way one defines the tensor product $\bigotimes_{i\in I} W_i$ of a family of vector spaces labelled by a set I.

A similar construction to (1.3) extends a cyclic $\mathbb{S}$-module to a functor on the category of non-empty finite sets and bijections: if I is a set with $n+1$ elements, we define

$$\mathcal{V}(\!(I)\!) = \left(\bigoplus_{f:\{0,\dots,n\}\to I} \mathcal{V} \right)_{\mathbb{S}_{n+1}}. \tag{1.4}$$

(1.5). **Operads.** A k-linear *operad* is an $\mathbb{S}$-module $\mathcal{P}$ together with a map $\eta : \mathsf{k} \to \mathcal{A}(1)$ (called the unit map) and maps

$$\mu_{i_1\dots i_k} = \mu^{\mathcal{P}}_{i_1,\dots,i_k} : \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \dots \otimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \dots + i_k)$$

(called composition maps) given for any $k \geq 1$ and $i_1, \dots, i_k \geq 0$. These data are required to satisfy the axioms of equivariance, associativity and unit (see May [18], Hinich-Schechtman [13], and Getzler-Jones [11]).

We denote the product $\mu_{i_1\dots i_k}(p \otimes q_1 \otimes \dots q_k)$ by $p(q_1, \dots, q_k)$.

A *morphism of operads* $\Phi : \mathcal{P} \to \mathcal{Q}$ is a morphism of $\mathbb{S}$-modules preserving the unit and composition maps.

The *endomorphism operad* $\mathcal{E}_V$ of a vector space V consists of the spaces $\mathcal{E}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$ with the obvious action of $\mathbb{S}_n$, unit and composition maps.

An *algebra* over an operad $\mathcal{P}$ (or $\mathcal{P}$-algebra) is a vector space A together with a morphism of operads $\mathcal{P} \to \mathcal{E}_A$.

An important example of a $\mathcal{P}$-algebra is the *free* $\mathcal{P}$-*algebra* generated by a vector space V: this is the direct sum

$$\mathsf{F}(\mathcal{P}, \mathcal{V}) = \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n},$$

where $\mathbb{S}_n$ acts on $V^{\otimes n}$ by permuting the factors.

Classes of algebras such as associative, commutative, Lie and Poisson, can be regarded as algebras over suitable operads, as we explain in Section 2.

(1.6). **Operads in monoidal categories.** The notion of an operad makes sense in any symmetric monoidal category $(\mathcal{C}, \otimes)$ in the sense of [17]. The cases of interest in this paper are as follows:

(a) k-*linear operads:* $\mathcal{C}$ is the category of k-vector spaces, with $\otimes$ the tensor product, the case described above.

(b) *topological operads:* $\mathcal{C}$ is the category of topological spaces, with $\otimes$ the Cartesian product.

(c) *graded operads:* $\mathcal{C}$ is the category of graded vector spaces. If $\mathcal{P}$ is a topological operad, the homology spaces $H_\bullet(\mathcal{P}(n), \mathsf{k})$ form a graded operad, called the *homology operad* of $\mathcal{P}$ and denoted $H_\bullet(\mathcal{P}, \mathsf{k})$.

(d) *dg-operads:* $\mathcal{C}$ is the category of chain complexes $(V_\bullet, \delta)$ over k.

(1.7). **Trees.** A *tree* T is a contractible graph with finitely many vertices and edges. We denote the set of vertices of a tree by $\mathsf{v}(T)$. There are two sorts of edges: those terminated at both ends by a vertex are called internal, and the set of these is denoted $\mathsf{e}(T)$, while those terminated at either one end or none by a vertex are called external edges, and the set of these is denoted $\mathrm{ex}(T)$. (External edges are called leaves in [11].) For any vertex $v \in T$, the set of internal and external edges incident with v is denoted $\mathrm{ex}(v)$, and the *valence* of the vertex v is the cardinality of $\mathrm{ex}(v)$.

If T is a tree, we denote by $\det(T)$ the determinant of the vector space

$$\bigoplus_{e \in \mathsf{e}(T)} \mathrm{or}(e),$$

where $\mathrm{or}(e)$ is the orientation line of the edge e. Since the chain complex

$$0 \to \mathsf{k}^{\mathsf{v}(T)} \to \mathsf{k}^{\mathsf{e}(T)} \to 0$$

has $H^0 \cong \mathsf{k}$ and $H_1 \cong 0$, we see that $\det(T)$ may be canonically identified with the determinant of the vector space $\mathsf{k}^{\mathsf{v}(T)}$.

The tree T with no vertices (and hence no internal edges) is slightly exceptional, in that its external edge is not canonically oriented. We define $\det(T)$ in this case to be the orientation line of this external edge.

A *rooted tree* is a tree together with a choice of one element (the root) $\mathrm{out}(T) \in \mathrm{ex}(T)$. In a rooted tree, there is a canonical orientation of the edges, in such a way that the unique path from any edge to the root is oriented positively. This orientation has the following properties:

(1) for every vertex $v \in \mathsf{v}(T)$, there is exactly one edge $\mathrm{out}(v) \in \mathrm{ex}(v)$ emerging from v;
(2) $\mathrm{out}(T)$ is the unique external edge which emerges from T.

This orientation is illustrated in Fig. 1. We denote by $\mathrm{in}(T) = \mathrm{ex}(T) \setminus \{\mathrm{out}(T)\}$ the set of incoming external edges of the rooted tree T, and for $v \in T$, by $\mathrm{in}(v) = \mathrm{ex}(v) \setminus \{\mathrm{out}(v)\}$ the set of incoming edges at a vertex v.

If I is a finite set, an unrooted (respectively rooted) I-tree is a tree T together with a bijection $I \to \mathrm{ex}(T)$ (resp. tree T together with a bijection $I \to \mathrm{in}(T)$). Clearly, a rooted I-tree is the same thing as an unrooted I_+-tree, where I_+ is the union of I and an additional base point. We will refer to rooted $\{1, \ldots, n\}$-trees as rooted n-trees, and to unrooted $\{0, \ldots, n\}$-trees as unrooted $n+1$-trees, following the same convention on subscripts as in ((1.1)).

A tree T is *binary* if all of its vertices have valence three, that is, $|\,\mathrm{ex}(v)| = 3$ for all vertices $v \in \mathsf{v}(T)$.

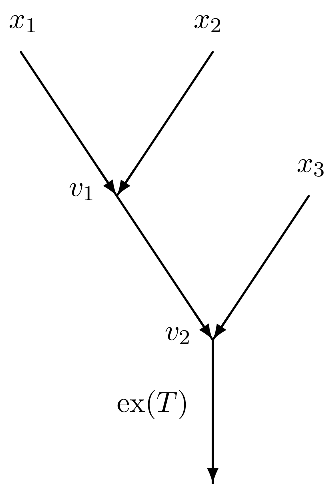

Fig. 1

(1.8). **Triples associated to trees.** Let $\mathcal{V}$ be an $\mathbb{S}$-module, and let T be a rooted tree. Define

$$\mathcal{V}(T) = \bigotimes_{v \in \mathsf{v}(T)} \mathcal{V}(\mathrm{in}(v)),$$

where $\mathcal{V}(\mathrm{in}(v))$ is defined by (1.3). We now define a functor $\mathbb{T}$ from $\mathbb{S}$-mod to itself, by summing over isomorphism classes of rooted trees:

$$\mathbb{T}\mathcal{V}(n) = \bigoplus_{\text{rooted } n\text{-trees T}} \mathcal{V}(T). \tag{1.9}$$

Similarly, let $\mathcal{W}$ be a cyclic $\mathbb{S}$-module, and let T be an unrooted tree. We define $\mathcal{W}(\!(T)\!)$ to be the tensor product over the vertices $v \in \mathsf{v}(T)$ of $\mathcal{W}(\!(\mathrm{ex}(v))\!)$. We now define a functor $\mathbb{T}_+$ from $\mathbb{S}_+$-mod to itself, by summing over isomorphism classes of unrooted trees:

$$\mathbb{T}_+\mathcal{W}(n) = \bigoplus_{\text{unrooted } n+1\text{-trees T}} \mathcal{W}(\!(T)\!). \tag{1.10}$$

Under the forgetful functor $\mathbb{S}_+$-mod $\to \mathbb{S}$-mod, the functor $\mathbb{T}_+$ reduces to the functor $\mathbb{T}$.

Grafting of trees defines, for any $\mathbb{S}$-module $\mathcal{V}$, a natural map $\mathbb{T}\mathbb{T}\mathcal{V} \to \mathbb{T}\mathcal{V}$. We also have embeddings $\mathcal{V} \to \mathbb{T}\mathcal{V}$, which identify $\mathcal{V}(n)$ with the summand in (1.9) corresponding to the unique rooted n-tree with one vertex and $n+1$ external edges.

The following result is proved in Section 1.4 of [11].

(1.11)**. Theorem.** *The maps $\mathbb{T}\mathbb{T}\mathcal{V} \to \mathbb{T}\mathcal{V}$ and $\mathcal{V} \to \mathbb{T}\mathcal{V}$ make the functor $\mathbb{T} : \mathbb{S}\text{-mod} \to \mathbb{S}\text{-mod}$ into a triple (or monad). An algebra over the triple $\mathbb{T}$ in the category of $\mathbb{S}$-modules is the same as an operad.*

This theorem amounts to the statement that if $\mathcal{P}$ is an operad and T is a rooted n-tree, there is induced a map $\mu_T : \mathcal{P}(T) \to \mathcal{P}(n)$, and these maps determine the operad structure of $\mathcal{P}$.

(1.12). **Free operads.** If $\mathcal{V}$ is an $\mathbb{S}$-module, the $\mathbb{S}$-module $\mathbb{T}\mathcal{V}$ has a natural structure of an operad, by Theorem (1.11): it is called the *free operad* generated by $\mathcal{V}$. It is convenient to represent elements of $\mathbb{T}\mathcal{V}(n)$ in a more algebraic fashion as follows. An element $p \in \mathcal{V}(n) \subset \mathbb{T}\mathcal{V}(n)$ is thought of as a formal "operation" $p(x_1, ..., x_n)$ on n indeterminates. If T is a rooted n-tree and for each vertex $v \in T$ we have assigned a vector $p_v \in \mathcal{V}(\mathrm{in}(v))$, then the element $\bigotimes p_v \in \mathcal{V}(T) \subset \mathbb{T}\mathcal{V}(n)$

is represented as an operation obtained by iterated application of the p_v, taking the tree T as a scheme of calculation. As an example, with the 3-tree T of Fig. 1 and $p_{v_1}, p_{v_2} \in E(2)$, the product $p_{v_1} \otimes p_{v_2} \in \mathbb{T}E(3)$ is represented by the expression $p_{v_2}(p_{v_1}(x_1, x_2), x_3)$.

2. Cyclic operads

In the same way as we defined in (1.8) a structure of a triple on the functor $\mathbb{T}$, we may define a structure of a triple on $\mathbb{T}_+$, acting on the category of cyclic $\mathbb{S}$-modules: grafting of unrooted trees defines, for any cyclic $\mathbb{S}$-module $\mathcal{V}$, a natural map $\mathbb{T}_+\mathbb{T}_+\mathcal{V} \to \mathbb{T}_+\mathcal{V}$, and we also have embeddings $\mathcal{V} \to \mathbb{T}_+\mathcal{V}$.

(2.1)**. Definition.** A k-linear cyclic operad is an algebra over the triple $\mathbb{T}_+$ in the category of cyclic $\mathbb{S}$-modules.

Similarly, we may define a cyclic operad in an arbitrary symmetric monoidal category $(\mathcal{C}, \otimes)$.

A cyclic operad $\mathcal{A}$ may be considered to be an ordinary operad by forgetting part of its structure. A cyclic operad structure on an operad $\mathcal{P}$ is determined by an extension of the $\mathbb{S}$-module structure of $\mathcal{P}$ to a cyclic $\mathbb{S}$-module structure, such that certain compatibility conditions are satisfied. For simplicity, we restrict attention to ungraded operads. As in (1.1), τ_n is the cycle $(0\,1 \dots n)$ in $\mathbb{S}_{n+1}$.

(2.2)**. Theorem.** *A cyclic operad $\mathcal{P}$ is an operad $\mathcal{P}$ together with an extension of the $\mathbb{S}$-module structure of $\mathcal{P}$ to a cyclic $\mathbb{S}$-module structure, such that:*

(1) $\tau_1(1) = 1$*, where* $1 \in \mathcal{P}(1)$ *is the unit of the operad* $\mathcal{P}$*;*
(2) *If* $p \in \mathcal{P}(m)$ *and* $q \in \mathcal{P}(n)$*, then*

$$\tau_{m+n-1}(p(1, \dots, 1, q)) = (\tau_n q)(\tau_m(p), 1, \dots, 1).$$

(If $\mathcal{P}$ *is a graded operad, there is an additional factor of* $(-1)^{|p|\,|q|}$*.)*

If $\mathcal{P}$ is an operad, the space $\mathcal{P}(1)$ is an algebra, with product $\mu_1 : \mathcal{P}(1) \otimes \mathcal{P}(1) \to \mathcal{P}(1)$. The above theorem implies that if $\mathcal{P}$ is a cyclic operad, $\mathcal{P}(1)$ is an algebra with involution $\tau = \tau_1$; that is, $\tau^2 = 1$ and $\tau(pq) = \tau(q)\tau(p)$ for all $p, q \in \mathcal{P}(1)$.

We give the proof of Theorem (2.2) later in this section, but first we present some examples of cyclic operads.

(2.3)**. Examples.** (a) *Endomorphism operads:* If V is a finite-dimensional k-vector space with a non-degenerate inner product $B(x, y)$, the endomorphism operad $\mathcal{E}_V$ of V is a cyclic operad. The cyclic $\mathbb{S}$-module structure of $\mathcal{E}_V$ is defined as follows: using B, we identify $\mathcal{E}_V(n) = \mathrm{Hom}(V^{\otimes(n)}, V)$ with $\mathrm{Hom}(V^{\otimes(n+1)}, \mathsf{k})$, and give this space the obvious action of $\mathbb{S}_{n+1}$.

(b) *The associative operad:* Associative k-algebras (possibly without unit) are algebras over an operad Ass defined as follows: $\mathsf{Ass}(n)$, $n \geq 1$, is the subspace of the space of non-commutative polynomials $\mathsf{k}\langle x_1, \dots, x_n\rangle$ spanned by monomials of the form $x_{\sigma(1)} \dots x_{\sigma(n)}$, where $\sigma \in \mathbb{S}_n$, in other words, monomials linear in each letter x_i, while $\mathsf{Ass}(0) = 0$. Thus, $\mathsf{Ass}(n)$ has dimension $n!$ and is isomorphic to the regular representation of $\mathbb{S}_n$. There is also an operad UAs which describes the category of unital associative algebras: UAs differs from Ass only in that $\mathsf{UAs}(0) = \mathsf{k}$, the space of non-commutative polynomials in zero variables.

(2.4)**. Proposition.** *The operads* Ass *and* UAs *are cyclic.*

Proof. We give an explicit formula for the action of $\mathbb{S}_{n+1}$ on $\mathsf{UAs}(n)$. The action of the subgroup $\mathbb{S}_n$ being known, it suffices to define the action of the cycle $\tau = (0\ 1 \dots\ n)$, which is

$$\tau(x_{\sigma(1)} \dots x_{\sigma(n)}) = x_{\sigma(\sigma^{-1}(n)+1)+1} \cdots x_{\sigma(n)+1} x_1 x_{\sigma(1)+1} \cdots x_{\sigma(\sigma^{-1}(n)-1)+1}. \tag{2.5}$$

Since $x_1 \dots x_n$ is invariant under τ, we see that

$$\mathsf{UAs}(n) \cong \operatorname{Ind}_{\mathbb{Z}_{n+1}}^{\mathbb{S}_{n+1}} \mathbb{1}. \tag{2.6}$$

It is not hard to see that the conditions of Theorem (2.2) are verified, and the proposition follows. $\square$

(c) *The A_∞-operad:* From the operad UAs we may construct an operad $\mathcal{M}$ in the category of sets, whose n-th space $\mathcal{M}(n)$ may be identified with the symmetric group $\mathbb{S}_n$ itself: the product on $\mathcal{M}$ is defined by identifying $\mathcal{M}(n)$ with the subset

$$\{x_{\sigma(1)} \dots x_{\sigma(n)} \mid \sigma \in \mathbb{S}_n\} \subset \mathsf{UAs}(n).$$

It is clear that $\mathcal{M}$ is a cyclic operad in the category of sets, since the composition maps and $\mathbb{S}_{n+1}$-actions preserve the basis vectors. An $\mathcal{M}$-algebra in the category of sets is a monoid with unit.

Stasheff [21] introduced an operad A_∞ in the category of regular cellular complexes with the homotopy type of $\mathcal{M}$; an algebra over A_∞ in the category of topological spaces is an H-space associative up to an explicit collection of higher homotopies.

The topological operad A_∞ is defined as follows: let K_n be the nth *Stasheff polytope*, whose vertices correspond to complete parenthesizations of the product $x_1 \dots x_n$ (Stasheff [21]). Such parenthesizations are in bijective correspondence with triangulations of an $(n+1)$-gon P_{n+1} into triangles whose vertices are among vertices of P_{n+1}, while faces of K_n correspond to decomposition of P_{n+1} into polygons (see, for example, Kapranov [14]). Taking P_{n+1} to be a regular polygon, we obtain an action of $\mathbb{Z}_{n+1}$ on P_{n+1} by rotation through multiples of the angle $2\pi/(n+1)$, inducing an action on K_n. We now define $A_\infty(n)$ to be the induced $\mathbb{S}_{n+1}$-space $\operatorname{Ind}_{\mathbb{Z}_{n+1}}^{\mathbb{S}_{n+1}} K_n$, analogous to (2.6). The operad structure on A_∞ is induced by the embeddings

$$K_n \times K_{a_1} \times \cdots \times K_{a_n} \hookrightarrow K_{a_1+\ldots+a_n} \tag{2.7}$$

described in [21].

(2.8)**. Proposition.** *The topological operad A_∞ and the dg-operad $C_\bullet(A_\infty, \mathsf{k})$ are cyclic.*

Proof. The cyclicity of A_∞ follows from the fact that the embeddings (2.7) satisfy the conditions of Theorem (2.2), as may be explicitly checked. Since the action of $\mathbb{S}_{n+1}$ on $A_\infty(n)$ and all of the compositions of the operad A_∞ are cellular maps, it follows immediately that $C_\bullet(A_\infty, \mathsf{k})$ is a cyclic dg-operad. $\square$

Another proof that $C_\bullet(A_\infty, \mathsf{k})$ is cyclic will be given in (5.4), using the cobar-construction from dg-cooperads to dg-operads.

(d) *Moduli spaces of stable curves:* Let $\overline{\mathcal{M}}_{g,n+1}$ be the Grothendieck-Knudsen moduli space (or rather, stack) of stable $(n+1)$-pointed curves of genus g (for genus $g = 0$, this is described in [12], and the higher genus case is similar). By definition, a point of $\overline{\mathcal{M}}_{g,n+1}$ is a system $(C, x_0, \dots, x_n)$, where C is a projective curve of arithmetic genus g with at most nodal singularities, and x_i are distinct smooth points; it is required that C has no infinitesimal automorphisms preserving the points x_i. The cyclic $\mathbb{S}$-space $\overline{\mathcal{M}}$ has

$$\overline{\mathcal{M}}(n) = \begin{cases} \coprod_{g\geq 0} \overline{\mathcal{M}}_{g,n+1}, & n > 1, \\ \{*\} \cup \coprod_{g>0} \overline{\mathcal{M}}_{g,2}, & n = 1, \\ \coprod_{g>0} \overline{\mathcal{M}}_{g,1}, & n = 0, \end{cases}$$

with $\mathbb{S}_{n+1}$ acting by permuting the points $\{x_i\}$. Then $\overline{\mathcal{M}}$ is a cyclic operad, with unit the point $* \in \overline{\mathcal{M}}(1)$, and composition induced by gluing curves together along marked points (see [12], 1.4.3). We may also define a cyclic suboperad $\overline{\mathcal{M}}_0$ of $\overline{\mathcal{M}}$ by restricting attention to curves of genus 0.

(e) *Moduli space of curves with holes:* We will only define the genus 0 version of this cyclic operad, although there is an analogue with unrestricted genus.

Let $\widehat{\mathcal{M}}_0(n)$ be the moduli space of complex analytic embeddings

$$\phi : \coprod_{i=0}^{n} \bar{D} \to S^2,$$

where $\bar{D}$ is the closed unit disk in $\mathbb{C}$; we let $\widehat{\mathcal{M}}_0(1)$ be a single point and $\widehat{\mathcal{M}}_0(0)$ be the empty set. Then the spaces $\widehat{\mathcal{M}}_0(n)$ assemble to form a cyclic $\mathbb{S}$-space. Although $\widehat{\mathcal{M}}_0(n)$ is infinite-dimensional, it has the homotopy type of a finite-dimensional CW complex: in fact, for $n > 1$, we have a fibration in the homotopy category

$$\widehat{\mathcal{M}}_0(n) \longrightarrow \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\},$$

with homotopy fiber the torus $(S^1)^{n+1}$.

The cyclic $\mathbb{S}$-module $\widehat{\mathcal{M}}_0$ is a cyclic operad: the composition is given by gluing along the boundaries of embedded disks (see [9]).

(f) *The Batalin-Vilkovisky operad:* This is the operad $\mathsf{BV} = H_\bullet(\widehat{\mathcal{M}}_0, \mathsf{k})$ in the category of graded vector spaces over k, defined by taking the homology of the topological operad $\widehat{\mathcal{M}}_0$. Since $\widehat{\mathcal{M}}_0$ is cyclic, it is automatic that BV is as well.

More explicitly, a BV-algebra (Batalin-Vilkovisky algebra) is a graded commutative algebra A together with a linear operator $\Delta : A \to A$ of degree 1 such that $\Delta^2 = 0$ and

$$\begin{aligned} \Delta(abc) &= \Delta(ab)c + (-1)^{|a|} a\Delta(bc) + (-1)^{(|a|-1)|b|} b\Delta(ac) \\ &\quad - (\Delta a)bc - (-1)^{|a|} a(\Delta b)c - (-1)^{|a|+|b|} ab(\Delta c). \end{aligned}$$

The graded vector space $\mathsf{BV}(1)$ may be identified with the homology of the circle $H_\bullet(S^1, \mathsf{k})$: the cyclic group $\mathbb{S}_{1+1}$ acts by the antipode map on $\widehat{\mathcal{M}}_0(1) \simeq S^1$, and hence acts trivially on $\mathsf{BV}(1)$. See [9] for more information on Batalin-Vilkovisky algebras and some applications to mathematical physics.

Proof of Theorem (2.2). Let $\mathcal{P}$ be an operad, and suppose that we are given an extension of the $\mathbb{S}$-module structure of $\mathcal{P}$ to a cyclic $\mathbb{S}$-module structure. In order for $\mathcal{P}$ to be a cyclic operad, we must define, for each unrooted $(n+1)$-tree T, a morphism of $\mathbb{S}_{n+1}$-modules $\mu_T : \mathcal{P}(T) \to \mathcal{P}(n)$ such that the maps

$$\mu_n : \mathbb{T}(\mathcal{P})(n) = \bigoplus_{\text{unrooted } (n+1)\text{-trees } T} \mathcal{P}(T) \xrightarrow{\oplus \mu_T} \mathcal{P}(n)$$

make $\mathcal{P}$ into a $\mathbb{T}$-algebra in the category $\mathbb{S}_+$-mod. Any unrooted $(n+1)$-tree may be considered as a rooted n-tree. For any vertex $v \in \mathsf{v}(T)$, we have $\mathcal{P}(\!(\mathrm{ex}(v))\!) = \mathcal{P}(\mathrm{in}(v))$, where on the left-hand side, we use the cyclic $\mathbb{S}$-module structure, while on the right-hand side we use the $\mathbb{S}$-module structure. This implies that the structure of an operad on $\mathcal{P}$ determines the maps μ_T. Moreover, $\mathcal{P}$ is a cyclic operad if and only if each μ_n thus constructed is equivariant with respect to $\mathbb{S}_{n+1}$.

In terms of the individual μ_T, equivariance means that for any $\sigma \in \mathbb{S}_{n+1}$ and $x \in \mathcal{P}(T) \cong \mathcal{P}(\sigma(T))$, we have

$$\mu_{\sigma(T)}(x) = \sigma(\mu_T(x)), \tag{2.9}$$

where $\sigma(T)$ is the unrooted $(n+1)$-tree obtained from T by renumbering the external edges according to σ. Condition (2) of the theorem is just (2.9) for the unrooted $(m+l)$-tree T depicting the composition $p(1,\dots,1,q)$. Thus it is certainly necessary: we will now prove that it is sufficient.

By construction, (2.9) already holds for any $\sigma \in \mathbb{S}_n$. We have the factorization $\mathbb{S}_{n+1} = \mathbb{S}_n \cdot \mathbb{Z}_{n+1}$. By means of the action of $\mathbb{S}_n$, we may transform T to a planar tree, that is, an unrooted tree embeddable in the plane in such a way that the labels $\{0,\dots,n\}$ on the external edges occur in clockwise order. Thus, it suffices to prove (2.9) for the case where T is planar and $\sigma = \tau$ is the cycle generating $\mathbb{Z}_{n+1}$, and from now on we make this assumption. Let $\Gamma(T)$ be the unique path between the external edges labelled 0 and n, and let $\gamma(T)$ be the number of vertices in $\Gamma(T)$. We will prove invariance of μ_T by induction on $\gamma(T)$.

The cases $\gamma(T) = 0$ and $\gamma(T) = 1$ are hypotheses (1) and (2) of the theorem. If $\gamma(T) \geq 2$, let e be the the internal edge of $\Gamma(T)$ adjacent to the external edge labelled by 0. Cut e in two, and denote by T' and T'' the components of the resulting graph containing the external edges labelled 0 and n respectively. If

$$x = \bigotimes_{v \in T} x_v, \quad x_v \in \mathcal{P}(\mathrm{ex}(v)),$$

is a decomposable element of $\mathcal{P}(T)$, let $x' = \bigotimes_{v \in \mathsf{v}(T')} x_v \in \mathcal{P}(T')$ and $x'' = \bigotimes_{v \in \mathsf{v}(T'')} x_v \in \mathcal{P}(T'')$. Let S be the tree with two vertices, obtained by contracting all of the vertices of T' into one vertex, and all of the vertices of T'' into another. Then we can write

$$\tau(\mu_T(x)) = \tau(\mu_S(\mu_{T'}(x') \otimes \mu_{T''}(x''))) = \mu_{\tau(S)}(\mu_{T'}(x') \otimes \mu_{T''}(x''))) = \mu_{\tau(T)}(x).$$

Here, the first equality follows from the associativity of composition in $\mathcal{P}$, the second from hypothesis (2) of the theorem, and the third from the inductive assumption that (2.9) holds for T' and T'', and associativity. This proves Theorem (2.2). □

(2.10). **Anticyclic operads and suspension.** The *suspension* of a dg-operad $\mathcal{P}$ is the dg-operad $\Lambda\mathcal{P}$ whose n-th space is

$$\Lambda\mathcal{P}(n) = \Sigma^{1-n}\operatorname{sgn}_n \otimes \mathcal{P}(n),$$

Here, sgn_n is the sign representation of $\mathbb{S}_n$. The compositions in $\Lambda\mathcal{P}$ are defined as follows: if $q_0 \in \mathcal{P}(n)$ and $q_j \in \mathcal{P}(i_j)$, $1 \leq j \leq n$, and we denote by ε_n the standard basis vector of sgn_n, then

$$\mu^{\Lambda\mathcal{P}}_{i_1\dots i_n}\big((\varepsilon_n \otimes q_0) \otimes (\varepsilon_{i_1} \otimes q_1) \otimes \dots \otimes (\varepsilon_{i_n} \otimes q_n)\big) = (-1)^\sigma \varepsilon_{i_1+\dots+i_n} \otimes \mu^{\mathcal{P}}_{i_1\dots i_n}(q_0 \otimes q_1 \otimes \dots \otimes q_n),$$

where $\sigma = \sum_{j=0}^n (i_{j+1} + \dots + i_n + n - j)|q_j|$. If A is a dg-algebra over a dg-operad $\mathcal{P}$, then ΣA is naturally an algebra over $\Lambda\mathcal{P}$. We may summarize the above discussion by saying that there is a canonical isomorphism of triples $\Lambda^{-1}\mathbb{T}\Lambda \cong \mathbb{T}$.

There is an evident extension of the *suspension* functor Λ to the category of cyclic $\mathbb{S}$-modules:

$$\Lambda\mathcal{P}(n) = \Sigma^{1-n}\operatorname{sgn}_{n+1} \otimes \mathcal{P}(n).$$

However, the triple $\mathbb{T}_- = \Lambda^{-1}\mathbb{T}_+\Lambda$ is *not* isomorphic to the triple $\mathbb{T}_+$, since

$$\mathbb{T}_-\mathcal{W}(n) \cong \bigoplus_{\text{unrooted } n+1\text{-trees } T} \det(T) \otimes \bigotimes_{v\in \mathsf{v}(T)} \mathcal{W}(\!(\mathrm{ex}(v))\!).$$

An *anticyclic* operad is an algebra over the triple $\mathbb{T}_-$. The analogue of Theorem (2.2) for anticyclic operads is obtained by multiplying the right-hand side of conditions (1) and (2) by -1.

3. Cyclic quadratic operads

(3.1). **Quadratic operads and quadratic duality.** Let E be a graded vector space with action of $\mathbb{S}_2$. We regard it as an $\mathbb{S}$-module with $E(n) = E$ for $n = 2$ and $E(n) = 0$ otherwise. Let $\mathbb{T}E$ be the free graded operad generated by E and let $R \subset \mathbb{T}E(3)$ be an $\mathbb{S}_3$-invariant subspace. The *quadratic operad* $\mathcal{P}(E, R)$ is formed by factoring $\mathbb{T}E$ by the ideal generated by the relations R.

This definition differs from the one in [12] in that we allow vector spaces to be graded, but restrict ourselves to the case of operads with $\mathcal{P}(1) = \mathsf{k}$. The case where the grading on E and R is trivial, so that $\mathcal{P}(E, R)$ is a k-linear operad, is the most important for us.

If $\mathcal{P}$ is a quadratic operad, the graded vector spaces E and R may be recovered from $\mathcal{P}$ by

$$E = \mathcal{P}(2), \quad R = \ker(\mathbb{T}E(3) \to \mathcal{P}(3)).$$

Note that $\mathbb{T}E(3) \cong \mathrm{Ind}_{\mathbb{S}_2}^{\mathbb{S}_3} E \otimes E$ is isomorphic to the sum of three copies of $E \otimes E$, corresponding to the three binary trees with 3 leaves.

If $\mathcal{P} = \mathcal{P}(E, R)$ is a quadratic operad, its *quadratic dual* is the operad $\mathcal{P}^! = \mathcal{P}(E^* \otimes \mathrm{sgn}, R^\perp)$, where sgn is the sign representation of $\mathbb{S}_2$ and $R^\perp$ is the orthogonal complement of $R \subset \mathbb{T}E(3)$ in $\mathbb{T}(E^* \otimes \mathrm{sgn})(3)$.

If $\mathcal{P}$ is a quadratic operad (regarded as a dg-operad with trivial differential) then the suspension $\Lambda\mathcal{P}$ is also quadratic.

(3.2). **Cyclic quadratic operads.** Let $\mathcal{P} = \mathcal{P}(E, R)$ be a quadratic operad. The action of $\mathbb{S}_2$ on E extends naturally to an action of $\mathbb{S}_{2+1}$ via the sign representation:

$$\mathrm{sgn} : \mathbb{S}_{2+1} \to \{\pm 1\} = \mathbb{S}_2. \tag{3.3}$$

This makes the $\mathbb{S}$-module E into a cyclic $\mathbb{S}$-module, so that the free operad $\mathbb{T}E$ is a cyclic operad. We say that $\mathcal{P}$ is a *cyclic quadratic* operad if R is an $\mathbb{S}_{3+1}$-invariant subspace of $\mathbb{T}E(3)$. It follows immediately that $\mathcal{P}$ is a cyclic operad. The reason for this definition will be clearer when we discuss invariant bilinear forms in Section 4.

Observe that $\mathcal{P}^!$ is cyclic quadratic if and only if $\mathcal{P}$ is.

(3.4). **Cyclic quadratic operads with one generator.** We now turn to the construction of examples of cyclic quadratic operads. We use the parametrization of representations of $\mathbb{S}_n$ by Young diagrams [8]: if $\lambda_1 \geq \dots \lambda_m \geq 1$ and $\lambda_1 + \cdots + \lambda_m = n$, we denote by $V_{\lambda_1,\dots,\lambda_m}$ the corresponding irreducible representation of $\mathbb{S}_n$. For example, $V_n = 1\!\!1$ is the trivial 1-dimensional representation, $V_{1^n} = \mathrm{sgn}$ is the sign representation, and $V_{n-1,1}$ is the $(n-1)$-dimensional representation on the hyperplane

$$\{(x_1, \dots, x_n) \mid x_1 + \cdots + x_n = 0\} \tag{3.5}$$

We shall make use the character table of $\mathbb{S}_4$ (or, in our notation, $\mathbb{S}_{3+1}$); for the convenience of the reader, we display it here.

	I	$(0\ 1)$	$(0\ 1\ 2)$	(0123)	$(01)(23)$
$\mathbb{1}$	1	1	1	1	1
sgn	1	-1	1	-1	1
$V_{2,2}$	2	0	-1	0	2
$V_{3,1}$	3	1	0	-1	-1
$V_{2,1,1}$	3	-1	0	1	-1

Note that the first three representations in this table can be obtained from the representations $\mathbb{1}$, sgn, $V_{2,1}$ of $\mathbb{S}_3$ via the natural surjection $\mathbb{S}_{3+1} \to \mathbb{S}_3$.

(3.6)**. Proposition.** *Every ungraded quadratic operad* $\mathcal{P} = \mathcal{P}(E, R)$ *with* $\dim(E) = 1$ *is cyclic quadratic.*

Proof. The representation E of $\mathbb{S}_2$ is either the trivial representation or the sign representation: the operad $\mathcal{P}$ thus has either a commutative or an anticommutative product. We consider these two cases separately.

We start with $E = \mathbb{1}$, the trivial representation. We use the shorthand notation for elements of $\mathbb{T}E(3)$ explained in (1.12). Thus, we represent a basis vector of E by a formal product (x_1, x_2); the fact that $\mathbb{S}_2$ acts trivially means that $(x_1, x_2) = (x_2, x_1)$; elements of $\mathbb{T}E(3)$ are then represented by expressions in x_1, x_2, x_3 obtained by iterating this product. We see that the space $\mathbb{T}E(3)$ is a three-dimensional representation of $\mathbb{S}_{3+1}$, spanned by formal expressions $((x_1, x_2), x_3)$, $((x_2, x_3), x_1)$ and $((x_3, x_1), x_2)$. We claim that as a representation of $\mathbb{S}_{3+1}$,

$$\mathbb{T}E(3) \cong \mathbb{1} \oplus V_{2,2}. \tag{3.7}$$

To see this, observe that the trace of a transposition (for example, $(1\ 2)$) equals 1, the trace of a cycle of length 3 (for example $(1\ 2\ 3)$) vanishes, and the trace of the cycle $\tau = (0\ 1\ 2\ 3)$ equals 1. We leave it to the reader to verify that

$$\tau((x_1, x_2), x_3) = ((x_2, x_3), x_1), \quad \tau((x_2, x_3), x_1) = ((x_1, x_2), x_3),$$
$$\tau((x_3, x_1), x_2) = ((x_3, x_1), x_2),$$

from which this last fact follows.

On restriction to the subgroup $\mathbb{S}_3$, the isomorphism (3.7) becomes

$$\mathbb{T}E(3) \cong \mathbb{1} \oplus V_{2,1}.$$

Thus, the $\mathbb{S}_{3+1}$-invariant and $\mathbb{S}_3$-invariant subspaces of $\mathbb{T}E(3)$ are the same, so that any space of relations R automatically defines a cyclic quadratic operad.

The case $E = \mathrm{sgn}$, the sign representation, is similar. Here, we simply give the answer:

$$\mathbb{T}E(3) \cong_{\mathbb{S}_{3+1}} \mathrm{sgn} \oplus V_{2,2} \cong_{\mathbb{S}_3} \mathrm{sgn} \oplus V_{2,1}, \tag{3.8}$$

where the first isomorphism is that of $\mathbb{S}_{3+1}$-modules and the second one is an isomorphism of $\mathbb{S}_3$-modules. $\square$

(3.9). **List of cyclic quadratic operads with one generator.** (a) *The commutative operad* Com has one commutative product (x_1, x_2), and relations $R = V_{2,2}$ in (3.7), implementing the associativity relations

$$((x_1, x_2), x_3) = ((x_2, x_3), x_1) \text{ and } ((x_2, x_3), x_1) = ((x_3, x_1), x_2).$$

We have $\mathsf{Com}(n) = \mathbb{1}$ for all $n \geq 1$, and all of the composition maps are isomorphisms of the trivial representation. Com-algebras are nonunital commutative (associative) algebras in the usual sense.

By contrast, unital commutative algebras are described by the operad UCom, which has $\mathsf{UCom}(n) = \mathbb{1}$ for all $n \geq 0$, together with the evident structure maps; however, this operad is not quadratic.

(b) *The Lie operad* Lie has one anticommutative product $[x_1, x_2]$ and relations $R = \mathrm{sgn}$ in (3.8), implementing the Jacobi rule

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.$$

We have $\mathsf{Lie} \cong \mathsf{Com}^!$. The $\mathbb{S}_n$-module $\mathsf{Lie}(n)$ may be realized as the subspace in the free Lie algebra on generators $x_1, \dots, x_n$ spanned by Lie monomials which contain each x_i exactly once; it has dimension $(n-1)!$. Since Lie is a cyclic operad, we see that $\mathsf{Lie}(n)$ has a natural $\mathbb{S}_{n+1}$-action, as was first observed by Kontsevich [15].

(c) *The mock-Lie operad* $\widetilde{\mathsf{Lie}}$, with one commutative product and relations $R = \mathbb{1}$ in (3.7): this implements the relation

$$((x_1, x_2), x_3) + ((x_2, x_3), x_1) + ((x_3, x_1), x_2) = 0.$$

(d) *The mock commutative operad* $\widetilde{\mathsf{Com}}$ has one anticommutative product and relations $R = V_{2,2}$ in (3.8), implementing the relations

$$[[x_1, x_2], x_3] = -[x_1, [x_2, x_3]] \text{ and } [[x_2, x_3], x_1] = -[x_2, [x_3, x_1]].$$

These relations easily imply that $\widetilde{\mathsf{Com}}(3) \cong \mathrm{sgn}$, while $\widetilde{\mathsf{Com}}(n) = 0$ for $n > 3$. We have $\widetilde{\mathsf{Lie}} \cong \widetilde{\mathsf{Com}}^!$. The operads $\widetilde{\mathsf{Lie}}$ and $\widetilde{\mathsf{Com}}$ are quite pathological[1].

(d) The trivial and free operads are obtained by imposing as relations respectively $R = \mathbb{T}E(3)$ and $R = 0$.

(3.10). **The associative and Poisson operads.** The operad Ass describing nonunital associative algebras (2.3) (b) is quadratic, since it is generated by the binary operations $x_1 x_2$ and $x_2 x_1$, while the space R of relations is the image of the relation $x_1(x_2 x_3) - (x_1 x_2) x_3 = 0$ under the action of $\mathbb{S}_3$. The operad UAs describing unital associative algebras is not quadratic. We will omit the adjective nonunital in referring to nonunital associative algebras.

A *Poisson algebra* is a commutative algebra P with a Lie bracket $\{-,-\}$ such that each operator $\{x, -\}$ is a derivation of the product. The operad Pois associated to Poisson algebras is defined in a similar way to the associative operad: $\mathsf{Pois}(n)$, $n \geq 1$, is a subspace in the free Poisson algebra generated by letters $\{x_1, \dots, x_n\}$ spanned by expressions which contain each letter just once, while $\mathsf{Pois}(0) = 0$. It is easily seen that Pois is a quadratic operad.

(3.11)**. Proposition.** *The operads* Ass *and* Pois *are cyclic quadratic.*

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[1]For instance, these operads are not Koszul in the sense of [12]. This is because the exponential generating function $g(t) = \sum_{n=1}^{\infty} \dim \widetilde{\mathsf{Com}}(n) t^n/n!$ is equal to $g(t) = t + t^2/2 + t^3/6$; if $\widetilde{\mathsf{Com}}$ was Koszul, the generating function of $\widetilde{\mathsf{Lie}}$ would coincide with the solution $h(t)$ of the functional equation $h(-g(-t)) = t$ (by Theorem 3.3.2 of [12]). However

$$h(t) = t + t^2/2 + t^3/3 + 5t^4/24 + t^5/12 - 7t^6/144 - 13t^7/72 + \dots$$

has some negative coefficients, giving a contradiction.

Proof. In (2.5), we defined an explicit action of $\mathbb{S}_{n+1}$ on $\mathsf{Ass}(n)$. It is straightforward to show that the natural map $\mathbb{T}(\mathsf{Ass}(2))(3) \to \mathsf{Ass}(3)$ is $\mathbb{S}_{3+1}$-equivariant: thus, its kernel R, the space of relations in Ass, is $\mathbb{S}_{3+1}$-invariant, and Ass is cyclic quadratic.

We now turn to Pois. Consider the basis of $\mathsf{Ass}(2)$ given by the two operations

$$(x_1, x_2) = x_1x_2 + x_2x_1 \quad , \quad [x_1, x_2] = x_1x_2 - x_2x_1.$$

Any elements in $\mathsf{Ass}(n)$ may be expressed as a sum of monomials in $x_1, \dots, x_n$ using the two operations $(-,-)$ and $[-,-]$; we give $\mathsf{Ass}(n)$ an increasing filtration F by setting $F_r\mathsf{Ass}(n)$ to be spanned by words involving the product $(-,-)$ not more than r times (Getzler-Jones [11], Section 5.3). Thus, $F_0\mathsf{Ass}(n) \cong \mathsf{Lie}(n)$, while $F_{n-1}\mathsf{Ass}(n) \cong \mathsf{Ass}(n)$. An alternative description of F may be obtained by regarding $\mathsf{k}\langle x_1, \dots, x_n\rangle$ as the universal enveloping algebra of the free Lie algebra generated by the x_i, and taking the intersection of $\mathsf{Ass}(n) \subset \mathsf{k}\langle x_1, \dots, x_n\rangle$ with the standard filtration of the enveloping algebra.

The filtration F is clearly compatible with the operad structure of Ass: $F_r\mathsf{Ass}(n)$ is invariant under the action of $\mathbb{S}_n$, and

$$\mu_{i_1\dots i_k}(F_r\mathsf{Ass}(k) \otimes F_{r_1}\mathsf{Ass}(i_1) \otimes \dots \otimes F_{r_k}\mathsf{Ass}(i_k)) \subset F_{r+r_1+\dots r_k}\mathsf{Ass}(i_1 + \dots i_k).$$

Using the characterization of F in terms of the filtration on the universal enveloping algebra, we see that the operad formed by the spaces[2] $\mathrm{gr}^F\mathsf{Ass}(n)$ is naturally isomorphic to the Poisson operad Pois.

It suffices to show that this filtration is preserved under the action of $\mathbb{S}_{n+1}$. Let $g \in \mathsf{Ass}(n)$ be a monomial in $x_1, \dots, x_n$ obtained by iterated application of the product $(-,-)$ and $[-,-]$. As in (1.12), we represent g as a binary rooted n-tree T in which each vertex is labelled by one of the product $(-,-)$ or $[-,-]$. If $\sigma \in \mathbb{S}_{n+1}$ is a permutation, the element σg is represented, up to sign, by the same tree with external edges renumbered by σ, and as its new root the external edge whose label is now 0; in particular, the assignment of products to the vertices remains the same. In particular, the number of occurrences of the product $(-,-)$ in σg is the same as in g. $\square$

(3.12). **Cyclic quadratic operads generated by the regular representation.** We will now classify all ungraded cyclic quadratic operads with $E = \mathbb{1} \oplus \mathrm{sgn}$ the regular representation of $\mathbb{S}_2$. The associative operad Ass and the Poisson operad Pois will emerge naturally from this classification.

(3.13). **Proposition.** *If $E = \mathbb{1} \oplus \mathrm{sgn}$, then we have an isomorphism of $\mathbb{S}_{3+1}$-modules*

$$\mathbb{T}E(3) \cong \mathbb{1} \oplus \mathrm{sgn} \oplus 2V_{2,2} \oplus V_{3,1} \oplus V_{2,1,1}.$$

Proof. We denote the commutative product in E by (a, b), and the anticommutative product by $[a, b]$. The torus $\mathrm{Aut}_{\mathbb{S}_2}(E) = \mathsf{k}^\times \oplus \mathsf{k}^\times$ acts on $\mathbb{T}E$, by rescaling the two products: we say that a monomial has weight k if there are k occurrences of the anticommutative bracket $[-,-]$ in it.

The space $\mathbb{T}E(3)$ has dimension 12, and is spanned by monomials

weight 0: $((1,2),3)$, $((2,3),1)$ and $((3,1),2)$;
weight 1: $([1,2],3)$, $([2,3],1)$, $([3,1],2)$, $[(1,2),3]$, $[(2,3),1]$ and $[(3,1),2]$;
weight 2: $[[1,2],3]$, $[[2,3],1]$ and $[[3,1],2]$.

We have already seen that the monomials of weight 0 and 2 contribute the irreducible representations $\mathbb{1} \oplus V_{2,2}$ and $\mathrm{sgn} \oplus V_{2,2}$ respectively. On the monomials of weight 1, the permutations $(1\ 2)$ and $(1\ 2\ 3)$ both have trace 0, permitting us to conclude (by looking at the character table) that it

[2] taken with trivial grading, not the grading induced by F

is isomorphic to a direct sum of either $V_{3,1}$ or $\mathbb{1}\oplus V_{2,2}$, and either $V_{2,1,1}$ or $\mathrm{sgn}\oplus V_{2,2}$. To distinguish between these different possibilities, we calculate the action of the involution $\sigma=(0\ 1)(2\ 3)$ and find it to be:

$$([1,2],3)\longleftrightarrow -[(1,2),3] \quad , \quad ([3,1],2)\longleftrightarrow [(3,1),2],$$
$$([2,3],1)\to -([2,3],1) \quad , \quad [(2,3),1]\to -[(2,3),1].$$

It follows that σ has trace -2 on the monomials of weight 1, and hence that they are isomorphic to $V_{3,1}\oplus V_{2,1,1}$ as a representation of $\mathbb{S}_{3+1}$. □

(3.14)**. Theorem.** *There are exactly* 80 *cyclic quadratic operads with* $\mathcal{P}(2)=\mathbb{1}\oplus\mathrm{sgn}$*:*

(1) 32 *operads corresponding to subsets of* $\{\mathbb{1},\mathrm{sgn},V_{2,2},V_{3,1},V_{2,1,1}\}$.
(2) 48 *operads, each corresponding to a choice of a subset of* $\{\mathbb{1},\mathrm{sgn},V_{3,1},V_{2,1,1}\}$ *and a choice of one of the three symbols* $0,1,\infty$.

Proof. By the previous proposition, the $\mathbb{S}_{3+1}$-invariant subspaces of $\mathbb{T}E(3)$ may be classified as follows:

(1) 32 subspaces corresponding to subsets of the set of irreducible representations
$$\{\mathbb{1},\mathrm{sgn},V_{2,2},V_{3,1},V_{2,1,1}\}$$
of $\mathbb{S}_{3+1}$.
(2) 16 families of subspaces corresponding to subsets of $\{\mathbb{1},\mathrm{sgn},V_{3,1},V_{2,1,1}\}$; each family is parametrized by a projective line P^1, which specifies a proper $\mathbb{S}_{3+1}$-invariant subspace of $V_{2,2}\oplus V_{2,2}$.

The torus $\mathrm{Aut}_{\mathbb{S}_2}(E)=\mathsf{k}^\times\oplus\mathsf{k}^\times$ fixes any subspace of the first type, and acts in a non-trivial way on P^1. This action has three orbits, two points denoted by symbols 0 and ∞, and one isomorphic to $\mathsf{k}^\times$, denoted by the symbol 1. □

Thus, the symbol 0 means that we are imposing the associativity relations on the commutative product $(-,-)$, the symbol ∞ means that we are imposing the mock associativity relations on the anticommutative product, while the symbol 1 means that we impose some mixture of the two.

Let us show how the associative operad Ass and Poisson operad Pois fit into this classification. As an $\mathbb{S}_{3+1}$-module,
$$\mathsf{Ass}(3)\cong\mathrm{Ind}_{\mathbb{Z}_{3+1}}^{\mathbb{S}_{3+1}}\mathbb{1}\cong\mathbb{1}\oplus V_{2,2}\oplus V_{2,1,1}.$$
Thus, the associative operad corresponds in our classification to the subset $\{\mathbb{1},V_{2,1,1}\}$ of $\{\mathbb{1},\mathrm{sgn},V_{3,1},V_{2,1,1}\}$. Furthermore, the orbit of the torus $\mathsf{k}^\times\times\mathsf{k}^\times$ is the one labelled by $1\in\{0,1,\infty\}$.

The Poisson operad lies in the same family as the associative operad, since $\mathsf{Pois}(3)\cong\mathsf{Ass}(3)$ as a representation of $\mathbb{S}_{3+1}$. However, now the set of relations forming a representation $V_{2,2}$ correspond to the orbit labelled by $0\in\{0,1,\infty\}$.

The third operad lying in this family, labelled by $\infty\in\{0,1,\infty\}$, is rather pathological. It is generated by two operations: a commutative but non-associative product $x\circ y$ and a non-commutative operation $[x,y]$ such that $[[x,y],z]=0$. These operations satisfy the Leibniz rule
$$[x,y\circ z]=[x,y]\circ z+y\circ[x,z].$$

(3.15). **The Leibniz operad is not cyclic quadratic.** A Leibniz algebra (Loday [16]) is an algebra with a product xy which satisfies the identity

$$x_1(x_2x_3) - (x_1x_2)x_3 + (x_1x_3)x_2 = 0. \tag{3.16}$$

Note that an anticommutative Leibniz algebra is the same thing as a Lie algebra. There is a quadratic operad Leib, with $\mathsf{Leib}(2) \cong \mathbb{1} \oplus \mathrm{sgn}$, describing Leibniz algebras. However, this operad is not cyclic, since the cycle $\tau \in \mathbb{S}_{3+1}$ may be checked to send the relation (3.16) to

$$(x_1x_2)x_3 - x_1(x_2x_3) + (x_3x_1)x_2 = 0,$$

which does not hold in a general Leibniz algebra.

(3.17). **The braid operad is not cyclic quadratic.** The *braid operad* Br is the suboperad of the Batalin-Vilkovisky operad generated by the operation x_1x_2, of degree 0, and the operation

$$\{x_1, x_2\} = \Delta(x_1x_2) - (\Delta x_1)x_2 - (-1)^{|x_1|}x_1(\Delta x_2),$$

of degree 1; both of these operations are invariant under the action of $\mathbb{S}_2$ on $\mathsf{Br}(2)$ (Getzler [9], Getzler-Jones [11]). In fact, Br is a quadratic operad, with relations

$$\begin{aligned}
&(x_1x_2)x_3 = x_1(x_2x_1),\\
&\{x_1, x_2x_3\} = \{x_1, x_2\}x_3 + (-1)^{(|x_1|-1)|x_2|}x_2\{x_1, x_3\},\\
&\{\{x_1, x_2\}, x_3\} + (-1)^{|x_1|(|x_2|+|x_3|)}\{\{x_2, x_3\}, x_1\} + (-1)^{|x_3|(|x_1|+|x_2|)}\{\{x_3, x_1\}, x_2\} = 0.
\end{aligned}$$

The braid operad is similar to the Poisson operad, except that the bracket $\{-,-\}$ has degree 1. The reason for calling it the braid operad is that if V is a graded vector space, the free braid algebra generated by V may be expressed in the form

$$\mathsf{F}(\mathsf{Br}, V) = \bigoplus_{n=1}^{\infty} H_\bullet(\mathbb{B}_n, V^{\otimes n}),$$

where $\mathbb{B}_n$ is the braid group on n strands, acting on $V^{\otimes n}$ through the homomorphism $\mathbb{B}_n \to \mathbb{S}_n$. This isomorphism is related to the fact that the homology of a double loop space $H_\bullet(\Omega^2 M)$ is a braid algebra. More precisely, as shown in Cohen [3], Br is isomorphic to the homology operad $H_\bullet(\mathcal{C}_2, \mathsf{k})$ of the little square operad $\mathcal{C}_2$, for which double loop spaces are algebras.

(3.18). **Proposition.** *The braid operad is not cyclic quadratic.*

Proof. We show that the Poisson identity is not preserved by the action of $\mathbb{S}_{3+1}$ on $\mathbb{T}E(3)$, where $E = \mathsf{Br}(2) \cong \mathbb{1} \oplus (\Sigma\mathbb{1})$. The space $\mathbb{T}E(3)_1$ is spanned by the monomials $\{x_1, x_2\}x_3$, $\{x_2, x_3\}x_1$, $\{x_3, x_1\}x_2$, $\{x_1x_2, x_3\}$, $\{x_2x_3, x_1\}$ and $\{x_3x_1, x_2\}$. We calculate that $\tau = (0\ 1\ 2\ 3)$ acts by

$$\begin{aligned}
&\{x_1, x_2\}x_3 \to (-1)^{|x_1|}x_1\{x_2, x_3\} \to \{x_1x_2, x_3\} \to \{x_1, x_2x_3\} \to \{x_1, x_2\}x_3,\\
&\{x_3, x_1\}x_2 \longleftrightarrow \{x_3x_1, x_2\}.
\end{aligned}$$

It follows that the Poisson relation

$$\{x_1, x_2x_3\} - \{x_1, x_2\}x_3 - (-1)^{(|x_1|-1)|x_2|}x_2\{x_1, x_3\}$$

is mapped to

$$\begin{aligned}
\{x_1, x_2\}x_3 &- (-1)^{|x_1|}x_1\{x_2, x_3\} - (-1)^{|x_1|\,|x_2|}\{x_2, x_1x_3\}\\
&= (-1)^{|x_1|\,|x_2|}\big(-\{x_2, x_1x_3\} + \{x_2, x_1\}x_3 - (-1)^{(|x_2|-1)|x_1|}x_1\{x_2, x_3\}\big),
\end{aligned}$$

which is not one of the defining relations of the braid operad Br. $\square$

4. Invariant bilinear forms

(4.1)**. Definition.** Let $\mathcal{P}$ be a cyclic (or anticyclic) dg-operad and let A be a $\mathcal{P}$-algebra. A bilinear form $B : A \otimes A \to V$ with values in a graded vector space V is *invariant* (with respect to the cyclic structure on $\mathcal{P}$) if for all $n \geq 0$, the map $B_n : \mathcal{P}(n) \otimes A^{\otimes(n+1)} \to V$ defined by the formula

$$B_n(p \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n) = (-1)^{|x_0|\,|p|} B(x_0, p(x_1, \ldots, x_n))$$

is invariant under the action of $\mathbb{S}_{n+1}$ on $\mathcal{P}(n) \otimes A^{\otimes(n+1)}$.

(4.2)**. Remarks.** (a) The special case of ungraded operads and algebras, in which the signs disappear from the formulas, arises frequently in applications.

(b) Since the map B_n of the above definition is automatically invariant under $\mathbb{S}_n$, it suffices to check that it is invariant under the action of the cycle $\tau \in \mathbb{S}_{n+1}$.

(c) Taking $p = 1 \in \mathcal{P}(1)$, we learn that an invariant bilinear form is (anti)symmetric if $\mathcal{P}$ is (anti)cyclic, that is,

$$B(x, y) = \pm(-1)^{|x|\,|y|} B(y, x),$$

with $+$ in the cyclic case and $-$ in the anticyclic case. This reflects the fact that symmetric bilinear forms B on a graded vector space V correspond to antisymmetric bilinear forms $\tilde{B}$ on its suspension ΣV, by means of the formula

$$\tilde{B}(\Sigma v, \Sigma w) = (-1)^{|v|} B(v, w).$$

(4.3)**. Proposition.** *Let $\mathcal{P}$ be a cyclic quadratic operad. A bilinear form B on a $\mathcal{P}$-algebra is invariant if and only if it is symmetric and for each $p \in \mathcal{P}(2)$,*

$$B(p(x_0, x_1), x_2) = (-1)^{|p|\,|x_0|} B(x_0, p(x_1, x_2)). \tag{4.4}$$

Proof. We will simplify the discussion by supposing that $\mathcal{P}$ and A are concentrated in degree 0. Invariance of $B_2 : \mathcal{P}(2) \otimes A^{\otimes 3} \to V$ under the cycle $\tau = (0\ 1\ 2)$ implies that

$$B(x_0, (\tau p)(x_1, x_2)) = B(x_2, p(x_0, x_1)).$$

By the definition (3.3) of the $\mathbb{S}_{2+1}$ action on $\mathcal{P}(2)$, we know that $\tau p = p$. Thus, (4.4) is necessary and sufficient for invariance of B_2 under τ. Invariance of $B_n : \mathcal{P}(n) \otimes A^{\otimes(n+1)} \to V$ for $n > 2$ follows from invariance of B_2 since $\mathcal{P}(2)$ generates $\mathcal{P}(n)$ for $n > 2$. $\square$

(4.5)**. Examples.** (a) *Associative algebras:* Let $\mathcal{P} = \mathsf{Ass}$ be the associative operad. A bilinear form on an associative algebra A is invariant if and only if it is symmetric and $B(x_0, x_1x_2) = B(x_0x_1, x_2)$: this implies that $B(x_0, x_1 \ldots x_n)$ is invariant under the action of τ on $A^{\otimes(n+1)}$. This explains the formula (2.5) for the action of the cycle $\tau \in \mathbb{S}_{n+1}$ on the space $\mathsf{Ass}(n)$.

If A has a unit, there is an identification between the invariant bilinear forms on A with values in V and traces, that is, linear maps $T : A \to V$ such that $T(xy) = (-1)^{|x|\,|y|}T(yx)$. The correspondence associates to an invariant bilinear form B the linear map $T(x) = B(1, x)$; the bilinear form may be recovered by setting $B(x, y) = T(xy)$.

(b) *Commutative algebras:* Let $\mathcal{P} = \mathsf{Com}$ be the commutative operad. If A is a commutative algebra, an invariant bilinear form on A is a symmetric map $B : A \otimes A \to V$ such that $B(x_0x_1, x_2) = B(x_0, x_1x_2)$. In this case the expression $B(x_0, x_1 \ldots x_n)$ is totally symmetric.

(c) *Lie algebras:* Let $\mathcal{P} = \mathsf{Lie}$ be the Lie operad. If $\mathfrak{g}$ is a Lie algebra, an invariant bilinear form on $\mathfrak{g}$ is a symmetric map $B : \mathfrak{g} \otimes \mathfrak{g} \to V$ such that $B([x, y], z) = B(x, [y, z])$. If $\mathfrak{g}$ is finite dimensional, an example is given by the Killing form.

(d) *Poisson algebras:* If P is a Poisson algebra, an invariant bilinear form on P is a symmetric map $B : P \otimes P \to V$ such that $B(xy, z) = B(x, yz)$ and $B(\{x, y\}, z) = B(x, \{y, z\})$. To give a geometric example, let $\mathsf{k} = \mathbb{R}$ and (M, ω) be a connected $2n$-dimensional symplectic manifold. Let $P = C_c^\infty(M)$ be the Poisson algebra of smooth real functions on M of compact support. We equip this algebra with the standard C^∞-topology.

(4.6)**. Proposition.** *The space of continuous invariant bilinear forms on $C_c^\infty(M)$ with values in $\mathbb{R}$ is spanned by the Liouville bilinear form*

$$B(f, g) = \int_M f g\, \omega^n.$$

Proof. We first check that the Liouville bilinear form is invariant: it is obvious that it is symmetric, and that $B(fg, h) = B(f, gh)$, while

$$\begin{aligned} B(f, \{g, h\}) &= \int_M f\{g, h\}\omega^n = \int_M f H_g(h)\omega^n = -\int_M H_g(f)h\omega^n \\ &= -B(\{g, f\}, h) = B(\{f, g\}, h). \end{aligned}$$

Here, we denote by H_g the Hamiltonian vector field associated to the function g, and we have used Liouville's theorem, which says that the Lie derivative $\mathcal{L}_{H_g}(\omega^n) = n\omega^{n-1}\mathcal{L}_{H_g}(\omega)$ vanishes.

It remains to check that all continuous invariant bilinear forms are proportional to the Liouville form. A continuous bilinear form on P is the same thing as a current on $M \times M$, while the condition $B(fg, h) = B(f, gh)$ says that this current is the direct image $\Delta_* \gamma$ of a current γ on M by the inclusion of the diagonal $\Delta : M \to M \times M$. The condition $B(f, \{g, h\}) = B(\{f, g\}, h)$ implies that the Lie derivative of γ with respect to any Hamiltonian vector field vanishes. Any such current is proportional to ω^n, as follows by calculation in Darboux coordinates. □

(e) *Batalin-Vilkovisky algebras:* Let $\mathcal{P} = \mathsf{BV}$ be the Batalin-Vilkovisky operad ((2.3)) (f). If A is a BV-algebra, an invariant bilinear form on A is a symmetric map $B : A \otimes A \to V$ such that $B(xy, z) = B(x, yz)$ and $B(\Delta x, y) = (-1)^{|x|} B(x, \Delta y)$. One important source of Batalin-Vilkovisky algebras is the space of functions on a *Batalin-Vilkovisky supermanifold* (M, ω, ν): this is an odd symplectic supermanifold (M, ω) together with a nowhere-vanishing section ν of the Berezinian bundle such that the operator $\Delta f = \mathrm{div}_\nu H_f$ satisfies the Batalin-Vilkovisky equation $\Delta^2 = 0$ (see [9]). By an analysis not too different from that which we gave above for the Poisson algebra of a symplectic manifold, it may be shown that any continuous invariant bilinear form on the Batalin-Vilkovisky algebra $C_c^\infty(M)$ associated to a Batalin-Vilkovisky manifold with values in $\mathbb{R}$ is proportional to $B(f, g) = \int_M f g\, \nu$.

(4.7). **The universal invariant bilinear form.** Let $\mathcal{P}$ be a cyclic graded operad and A be a graded $\mathcal{P}$-algebra. We define the (graded) vector space $\lambda(\mathcal{P}, A)$ to be the quotient of $A \otimes A$ by the subspace spanned by the images of all the maps

$$\mathcal{P}(n) \otimes A^{\otimes(n+1)} \xrightarrow{1-\sigma} \mathcal{P}(n) \otimes A^{\otimes(n+1)} \xrightarrow{B_n} A \otimes A, \quad n \geq 1, \sigma \in \mathbb{S}_{n+1}$$

where B_n is as in (4.1) and σ acts diagonally. There is a tautological bilinear form $A \otimes A \to \lambda(\mathcal{P}, A)$, which is clearly universal among all the invariant bilinear forms on A.

For the cases $\mathcal{P} = \mathsf{Ass}$, Com, and Lie, the space $\lambda(\mathcal{P}, A)$ was introduced by Kontsevich [15], who denoted it $\Omega^0(A)$.

Thus $\lambda(\mathcal{P}, -)$ is a functor from the category of $\mathcal{P}$-algebras to the category of graded vector spaces. If $\mathcal{P}$ is a dg-operad and A is a dg-algebra over $\mathcal{P}$ then $\lambda(\mathcal{P}, A)$ acquires a natural differential, becoming a chain complex. Of course, if $\mathcal{P}$ and A are ungraded, then so is $\lambda(\mathcal{P}, A)$.

(4.8)**. Example.** If $\mathcal{P} = \mathsf{Ass}$ and A is a unital associative algebra, then $\lambda(\mathsf{Ass}, A) = A/[A, A]$ is the quotient of A by its commutant. If $\mathcal{P} = \mathsf{Com}$ and A is a unital commutative algebra, then $\lambda(\mathsf{Com}, A) = A$. If $\mathcal{P} = \mathsf{Pois}$ and P is a unital Poisson algebra (this means that 1 is a unit for the product and $\{1, x\} = 0$ for all $x \in P$), then $\lambda(\mathsf{Pois}, P) = P/\{P, P\}$ is the quotient of P by the span of all Poisson brackets.

In the next proposition, we calculate the universal invariant bilinear form of a free algebra $\mathsf{F}(\mathcal{P}, V)$.

(4.9)**. Proposition.** *Let $\mathcal{P}$ be a dg-operad, and let A be a dg-algebra over $\mathcal{P}$. There is a natural isomorphism of chain complexes*

$$\lambda(\mathcal{P}, \mathsf{F}(\mathcal{P}, V)) \cong \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{\otimes(n+1)}.$$

Proof. If $p \in \mathcal{P}(n)$ and $q \in \mathcal{P}(m)$, and $x_i, y_i \in V$, the image of the element

$$p(x_1, \dots, x_n) \otimes q(y_1, \dots, y_n) \in \lambda(\mathsf{F}(\mathcal{P}, V))$$

is equal, up to a sign, to the image of

$$y_1 \otimes (\tau p)(y_2, \dots, y_m, q(x_1, \dots, x_n)).$$

We see that $\lambda(\mathcal{P}, \mathsf{F}(\mathcal{P}, V))$ is a quotient of

$$V \otimes \mathsf{F}(\mathcal{P}, V) \cong \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes(n+1)}.$$

Using once more the coinvariance under the action of the cyclic group $\mathbb{Z}_{n+1}$, we get a surjection

$$\bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{\otimes(n+1)} \to \lambda(\mathcal{P}, \mathsf{F}(\mathcal{P}, V)).$$

It is easily seen that this is in fact an isomorphism. $\square$

For the cases $\mathcal{P} \in \{\mathsf{Ass}, \mathsf{Com}, \mathsf{Lie}\}$, this result was proved by Kontsevich [15].

5. Cyclic homology

(5.1). **Non-abelian derived functors.** All operads and algebras over them in the remainder of this paper will lie in the symmetric monoidal category of positively graded chain complexes ($V_n = 0$ for $n < 0$), unless otherwise mentioned; of course, ungraded vector spaces form a subcategory of this category. The reason for this is that we wish to use the methods of Quillen's homotopical algebra [19], even if we are primarily interested in ungraded operads and algebras.

(5.2)**. Definition.** Let $\mathcal{P}$ be a cyclic operad. The cyclic homology functors $\mathrm{HA}_i(\mathcal{P}, A)$ are the non-abelian (left) derived functors of the functor $\lambda(\mathcal{P}, A)$.

This definition is motivated by the result of Feigin and Tsygan [6] who show that cyclic homology of a unital associative algebras (over a field of characteristic zero) are the derived functors of the functor $A \mapsto A/[A, A]$.

Let us briefly explain the construction of non-abelian derived functors. If V is a chain complex, denote by $V^{\#}$ the graded vector space obtained from V by forgetting the differential. The same notation will be used for dg-operads and dg-algebras. We say that a map $A \to F$ of $\mathcal{P}$-algebras (or chain complexes) is a *weak equivalence* if it induces an isomorphism on homology.

If A is an $\mathcal{P}$-algebra, an *almost free resolution* of A is a weak equivalence of $\mathcal{P}$-algebras $\tilde{A} \to A$ such that the $\mathcal{P}^{\#}$-algebra $\tilde{A}^{\#}$ is free. In Chapter 2 of [11], it is shown that every $\mathcal{P}$-algebra A has a natural almost free resolution.

Let F be a functor from $\mathcal{P}$-algebras to chain complexes. The non-abelian derived functors $\mathsf{L}_i F$ are defined as follows: if A is an $\mathcal{P}$-algebra, $\mathsf{L}_i F(A) = H_i(F(\tilde{A}))$, where $\tilde{A} \to A$ is an almost free resolution of A.

For this definition to be sensible, we must of course check that the result is independent of the choice of almost free resolution. A sufficient condition for this is that whenever $A \to F$ is a weak equivalence of almost free $\mathcal{P}$-algebras, the induced map $F(A) \to F(B)$ is a weak equivalence of chain complexes. For the functor $\lambda(\mathcal{P}, -)$ this condition will be implied by the following theorem.

(5.3)**. Theorem.** *There is a functor* $\mathrm{CA}(\mathcal{P}, A)$ *from* $\mathcal{P}$*-algebras to chain complexes, together with a natural transformation* $\mathrm{CA}(\mathcal{P}, A) \to \lambda(\mathcal{P}, A)$*, having the following two properties:*

1. *If* $A \to F$ *is a weak equivalence of* $\mathcal{P}$*-algebras, the induced map* $\mathrm{CA}(\mathcal{P}, A) \to \mathrm{CA}(\mathcal{P}, B)$ *is a weak equivalence of chain complexes.*
2. *If* A *is an almost free algebra, the induced map* $\mathrm{CA}(\mathcal{P}, A) \to \lambda(\mathcal{P}, A)$ *is a weak equivalence.*

The proof of Theorem (5.3) occupies the remainder of this section.

A standard way of constructing non-abelian derived functors is the method of simplicial resolutions Quillen [19]. Indeed, if the field k over which we were working had positive characteristic, this would be the only method available. (This is because the category of algebras is not in general a closed model category unless the characteristic is 0, whereas the category of simplicial algebras is always a simplicial category.) Another setting in which it might be of interest to investigate simplicial resolutions is that of cyclic homology for algebras over cyclic operads in categories of spectra — this would allow one to relate the theory of topological Hochschild and cyclic homology for "rings up to homotopy" developed by Bökstedt, Hsiang and Madsen [2] to our work.

(5.4). **Cooperads.** A (cyclic) *cooperad* in a symmetric monoidal category $\mathcal{C}$ is a (cyclic) operad in the opposite category $\mathcal{C}^{\mathrm{op}}$ (see Getzler-Jones [11]): thus, a cooperad $\mathcal{Z}$ has cocomposition maps

$$\nu_{i_1 \dots i_k} : \mathcal{Z}(i_1 + \dots + i_k) \to \mathcal{Z}(k) \otimes \mathcal{Z}(i_1) \otimes \dots \mathcal{Z}(i_k).$$

If $\mathcal{Z}$ is a k-linear cooperad (with grading concentrated in degree 0), its dual $\mathbb{S}$-module $\mathcal{Z}^* = \{\mathcal{Z}(n)^*\}$ is an operad; the converse is true provided $\mathcal{Z}(n)$ is finite-dimensional for all $n \geq 0$. The definition of cyclic and anticyclic cooperads is similar.

An augmented dg-operad is a dg-operad $\mathcal{P}$ together with a map $\varepsilon : \mathcal{P}(1) \to k$. Denote the kernel of the augmentation ε by $\bar{\mathcal{P}}$.

An important example of a dg-cooperad is the *bar cooperad* $\mathcal{BP}$ of an augmented dg-operad $\mathcal{P}$, whose underlying graded cooperad is the cofree cooperad (in the sense dual to that of (1.12)) generated by the $\mathbb{S}$-module $\Sigma\bar{\mathcal{P}} = \{\Sigma\bar{\mathcal{P}}(n)\}$:

$$\mathcal{BP}(n) = \bigoplus_{\text{rooted } n\text{-trees } T} (\Sigma\bar{\mathcal{P}})(T).$$

The differential δ in $\mathcal{BP}(n)$ is a sum $\delta_1 + \delta_2$, where δ_1 is induced by the differential of $\mathcal{P}$ and δ_2 is defined as follows (see Chapter 2 of [11] or Chapter 3 of [12]).

If T is a rooted tree and $e \in \mathsf{e}(T)$ is an internal edge of T, let T/e be the tree obtained from T by removing e and merging its two ends to a single vertex. The composition map $\mu_{T,e} : \mathcal{P}(T) \to \mathcal{P}(T/e)$ induces a map $\partial_{T,e} : (\Sigma\bar{\mathcal{P}})(T) \to (\Sigma\bar{\mathcal{P}})(T/e)$, and δ_2 is the sum over all rooted trees T and internal edges $e \in \mathsf{e}(T)$ of the operators $\partial_{T,e}$.

The tensor product of the suspensions of complexes $\{V(i) \mid i \in I\}$ is related to the tensor product of the complexes themselves by the canonical isomorphism

$$\bigotimes_{i\in I}(\Sigma V(i)) \cong \det(\mathsf{k}^I) \otimes \Sigma^{|I|}\left(\bigotimes_{i\in I} V(i)\right).$$

Therefore we can write

(5.5) $$\mathcal{BP}(n) = \bigoplus_{\text{rooted } n\text{-trees } T} \det(T) \otimes \Sigma^{|\mathsf{v}(T)|}\bar{\mathcal{P}}(T),$$

where $\det(T)$ is as in (1.7). This shows that if $\mathcal{P}$ is a cyclic (resp. anticyclic) dg-operad, then $\mathcal{BP}$ is an anticyclic (resp. cyclic) dg-cooperad.

There is a functor $\mathcal{B}^*$ (called the *cobar-construction*) from dg-cooperads to dg-operads, adjoint to $\mathcal{B}$ [11]. Similarly to $\mathcal{B}$, it sends cyclic cooperads to anticyclic operads and vice versa. It is proved in [11] (and, in dual language, in [12]) that $\mathcal{B}$ and $\mathcal{B}^*$ are inverse to each other, up to weak equivalence. The dg-operad $C_\bullet(A_\infty, \mathsf{k})$ of ((2.3)) (c) is isomorphic to $\mathcal{B}^*(\Lambda\mathsf{Ass})^*$, and we obtain another proof that it is cyclic; furthermore, the two cyclic structures agree.

(5.6). **Twisting cochains.** If $\mathcal{Q}$ and $\mathcal{P}$ are operads, their tensor product $\mathcal{Q} \otimes \mathcal{P} = \{\mathcal{Q}(n) \otimes \mathcal{P}(n)\}$ is again an operad in an obvious way. Similarly, if $\mathcal{Z}$ is a dg-cooperad and $\mathcal{P}$ is a dg-operad, the $\mathbb{S}$-module $\hom(\mathcal{Z}, \mathcal{P}) = \{\hom(\mathcal{Z}(n), \mathcal{P}(n))\}$ is a dg-operad; however, this operad is not necessarily concentrated in positive degrees, even if $\mathcal{Z}$ and $\mathcal{P}$ are.

A *twisting cochain* from a dg-cooperad $\mathcal{Z}$ to a dg-operad $\mathcal{P}$ is a sequence of maps $\Phi_n \in \hom_{\mathbb{S}_n}(\mathcal{Z}(n), \mathcal{P}(n))$ of degree -1 such that

$$\delta\Phi_n + \sum_{i+j+k=n} \Phi_{i+k+1}(\underbrace{1, \dots, 1}_{i \text{ times}}, \Phi_j, \underbrace{1, \dots, 1}_{k \text{ times}}) = 0,$$

where the composition is with respect to the structure of an operad on $\hom(\mathcal{Z}, \mathcal{P})$. (See Section 2.3 of [11].)

If $\mathcal{Z}$ is an anticyclic cooperad and $\mathcal{P}$ is a cyclic operad, a *cyclic twisting cochain* Φ from $\mathcal{Z}$ to $\mathcal{P}$ is a twisting cochain from $\mathcal{Z}$ to $\mathcal{P}$ such that each Φ_n is equivariant with respect to $\mathbb{S}_{n+1}$.

(5.7)**. Examples.** (a) *The universal twisting cochain:* Let $\mathcal{P}$ be an augmented dg-operad and $\mathcal{BP}$ its bar cooperad. There is a natural twisting cochain $\Phi^{\mathcal{P}}$ from $\mathcal{BP}$ to $\mathcal{P}$ defined as follows: it vanishes on summands of $\mathcal{BP}$ associated to trees with more than one vertex, while on trees with one vertex, it is the inclusion $\Sigma\bar{\mathcal{P}}(n) \hookrightarrow \mathcal{P}(n)$. As explained in [11], $\Phi^{\mathcal{P}}$ is universal, in a natural sense, among twisting cochains to $\mathcal{P}$.

If the dg-operad $\mathcal{P}$ is cyclic, then $\Phi^{\mathcal{P}}$ is a cyclic twisting cochain, as may be easily verified.

(b) *The quadratic twisting cochain:* Let $\mathcal{P} = \mathcal{P}(E, R)$ be a quadratic operad, such that E is finite dimensional, and let $\mathcal{P}^!$ be its quadratic dual. Then $\mathcal{P}^\perp = \Lambda^{-1}(\mathcal{P}^!)^*$ is a cooperad, and there is a twisting cochain $\Psi^{\mathcal{P}}$ from $\mathcal{P}^\perp$ to $\mathcal{P}$, such that $\Psi^{\mathcal{P}}_n = 0$ for $n \neq 2$, and

$$\Psi^{\mathcal{P}}_2 : \mathcal{P}^\perp(2) \cong \Sigma E \to E = \mathcal{P}(2)$$

is the identity. If $\mathcal{P}$ is cyclic quadratic, then $\mathcal{P}^\perp$ is an anticyclic cooperad, and $\Psi^{\mathcal{P}}$ is a cyclic twisting cochain.

(c) *The unital-associative twisting cochain:* There is a modification of the quadratic twisting cochain of (b) for the operad UAs describing unital associative algebras (3.10). Let $\mathsf{UCoas} = \Lambda^{-1}\,\mathsf{UAs}^*$; the only difference between the cooperads UCoas and $\mathsf{Ass}^\perp$ is that $\mathsf{UCoas}(0) = \mathsf{k}$ whereas $\mathsf{Ass}^\perp(0) = 0$. We define the twisting cochain Ψ^{UAs} from UCoas to UAs by setting $\Psi^{\mathsf{UAs}}(2)$ be the identity map as in (b) and the other components of Ψ^{UAs} to be zero.

(5.8). **Three complexes.** Let $\mathcal{P}$ be a cyclic dg-operad, A be a $\mathcal{P}$-algebra, $\mathcal{Z}$ be a cyclic dg-cooperad, and Φ be a cyclic twisting cochain from $\mathcal{Z}$ to $\mathcal{P}$. We now define a natural short exact sequence of chain complexes

$$0 \to \mathrm{CA}(\Phi, A) \to \mathrm{CB}(\Phi, A) \to \mathrm{CC}(\Phi, A) \to 0. \tag{5.9}$$

The underlying graded vector spaces of these complexes are as follows:

$$\begin{aligned}
\mathrm{CA}(\Phi, A) &= \bigoplus_{n=1}^{\infty} \mathcal{Z}(n) \otimes_{\mathbb{S}_{n+1}} (V_{n,1} \otimes A^{\otimes(n+1)}), \\
\mathrm{CB}(\Phi, A) &= \bigoplus_{n=0}^{\infty} \mathcal{Z}(n) \otimes_{\mathbb{S}^{n+1}} (\mathsf{k}^{n+1} \otimes A^{\otimes(n+1)}), \\
\mathrm{CC}(\Phi, A) &= \bigoplus_{n=0}^{\infty} \mathcal{Z}(n) \otimes_{\mathbb{S}^{n+1}} A^{\otimes(n+1)},
\end{aligned}$$

where k^{n+1} is the standard permutation representation of $\mathbb{S}_{n+1}$, and $V_{n,1} \subset \mathsf{k}^{n+1}$ is the irreducible representation defined in (3.5). The short exact sequence (5.9) is induced by the short exact sequence of $\mathbb{S}_{n+1}$-modules

$$0 \to V_{n,1} \to \mathsf{k}^{n+1} \to \mathbb{1} \to 0.$$

In the special case of the universal twisting cochain $\Phi^{\mathcal{P}}$ from $\mathcal{BP}$ to $\mathcal{P}$, we denote the corresponding complexes by $\mathrm{CA}(\mathcal{P}, A) = \mathrm{CA}(\Phi^{\mathcal{P}}, A)$, etc.

The differential δ on $\mathrm{CB}(\Phi, A)$ is the sum $\delta_{\mathcal{Z}} + \delta_A + \delta_\Phi$, where $\delta_{\mathcal{Z}}$ and δ_A are induced by the differentials of $\mathcal{Z}$ and A, and δ_Φ is defined as follows.

If $L \coprod M$ is a partition of $\{0, \dots, n\}$ into two disjoint subsets, let $L^+ = L \coprod \{M\}$ and let $M^+ = M \coprod \{L\}$. Using the cocomposition of the anticyclic cooperad $\mathcal{Z}$, we obtain a map

$$\mathcal{Z}(n) \to \mathcal{Z}(\!(L^+)\!) \otimes \mathcal{Z}(\!(M^+)\!),$$

which induces a map

$$\mathcal{Z}(n) \otimes A^{\otimes(n+1)} \otimes \mathsf{k}^{n+1} \to (\mathcal{Z}(\!(L^+)\!) \otimes A^{\otimes L}) \otimes (\mathcal{Z}(\!(M^+)\!) \otimes A^{\otimes M}) \otimes \mathsf{k}^{n+1}.$$

The twisting cochain $\Phi : \mathcal{Z}(\!(L^+)\!) \to \mathcal{P}(\!(L^+)\!)$ is a map of degree -1 which, when composed with the structure map

$$\mathcal{P}(\!(L^+)\!) \otimes A^{\otimes L} \cong \mathcal{P}(L) \otimes A^{\otimes L} \to A$$

of the $\mathcal{P}$-algebra A induces a map

$$\mathcal{Z}(n) \otimes A^{\otimes(n+1)} \otimes \mathsf{k}^{n+1} \to A \otimes \mathcal{Z}(M) \otimes A^{\otimes M} \otimes \mathsf{k}^{n+1}.$$

There is a map $\mathsf{k}^{n+1} \to \mathsf{k}^{M^+}$ defined by the composition

$$\mathsf{k}^{n+1} \cong \mathsf{k}^L \oplus \mathsf{k}^M \xrightarrow{\mathrm{sum} \oplus \mathrm{id}} \mathsf{k} \oplus \mathsf{k}^M \cong \mathsf{k}^{M^+}, \tag{5.10}$$

where $\mathrm{sum} : \mathsf{k}^L \to \mathsf{k}$ is the summation map. Composing these maps, we obtain a map

$$\mathcal{Z}(n) \otimes A^{\otimes(n+1)} \otimes \mathsf{k}^{n+1} \xrightarrow{\delta_M} \mathcal{Z}(M) \otimes A^{\otimes M^+} \otimes \mathsf{k}^{M^+}.$$

Summing over all subsets $M \subset \{0, \dots, n\}$ of cardinality m, we obtain a map

$$\mathcal{Z}(n) \otimes A^{\otimes(n+1)} \otimes \mathsf{k}^{n+1} \xrightarrow{\delta_m} \bigoplus_{|M|=m} \mathcal{Z}(M) \otimes A^{\otimes M^+} \otimes \mathsf{k}^{M^+}.$$

The group $\mathbb{S}_{n+1}$ acts on both sides, and the map δ_m is equivariant — here, we use the fact that the twisting cochain is cyclic. Thus, it descends to a map from $\mathcal{Z}(n) \otimes_{\mathbb{S}_{n+1}} (A^{\otimes(n+1)} \otimes \mathsf{k}^{n+1})$ to the space of $\mathbb{S}_{n+1}$-coinvariants of the right-hand side, which we compose with the projection

$$\left(\bigoplus_{|M|=m} \mathcal{Z}(M) \otimes A^{\otimes M^+} \otimes \mathsf{k}^{M^+} \right)_{\mathbb{S}_{n+1}} \to \mathcal{Z}(m) \otimes_{\mathbb{S}_{m+1}} (A^{\otimes(m+1)} \otimes \mathsf{k}^{m+1}).$$

The sum of these maps over all n and $m \le n$ gives an endomorphism δ_Φ of degree -1 of $\mathrm{CB}(\Phi, A)$.

(5.11). **Lemma.** (1) *The endomorphism $\delta_{\mathcal{Z}} + \delta_A + \delta_\Phi$ of $\mathrm{CB}(\Phi, A)$ is a differential.*

(2) *The differential $\delta_{\mathcal{Z}} + \delta_A + \delta_\Phi$ of $\mathrm{CA}(\Phi, A)$ preserves $\mathrm{CA}(\Phi, A)$, and projects to a differential on $\mathrm{CC}(\Phi, A)$.*

Proof. Part (1) is a straightforward consequence of the fact that Φ is a twisting cochain, and is left to the reader.

Part (2) is a consequence of the fact that the map of (5.10) takes the subspace $V_{n,1}$ onto $V_{|L|,1}$. □

For any cyclic twisting cochain Φ from an anticyclic cooperad $\mathcal{Z}$ to a cyclic operad $\mathcal{P}$, there is a natural transformation

$$\mathrm{CA}(\Phi, A) \to \lambda(\mathcal{P}, A),$$

which sends $\mathcal{Z}(n) \otimes_{\mathbb{S}_{n+1}} (V_{n,1} \otimes A^{\otimes(n+1)})$ to zero for $n \ne 1$, and for $n = 1$, is the composition

$$\mathcal{Z}(1) \otimes_{\mathbb{S}_2} (V_{1,1} \otimes A \otimes A) \xrightarrow{\varepsilon} (A \otimes A)_{\mathbb{S}_2} \to \lambda(\mathcal{P}, A),$$

induced by the counit $\varepsilon : \mathcal{Z}(1) \to \mathrm{sgn} \cong V_{1,1}$ of the anticyclic cooperad $\mathcal{Z}$. It is obvious that this natural transformation intertwines the differential $\delta_{\mathcal{Z}} + \delta_A + \delta_\Phi$ which we have constructed with the natural differential on $\lambda(\mathcal{P}, A)$.

Proof. Proof of Theorem (5.3) Part (1) follows by a simple spectral sequence argument; convergence is ensured by the assumption that all complexes in question are situated in non-negative degrees.

To prove part (2), we consider the spectral sequence relating the homology of the complex $\mathrm{CA}(\mathcal{P}, A)$ with that of the complex $\mathrm{CA}(\mathcal{P}^\#, A^\#)$ with differential δ_Φ. Comparing this with a similar spectral sequence for the homology of $\lambda(\mathcal{P}, A)$, we find that all we need to prove is the following lemma.

(5.12). **Lemma.** *Let $\mathcal{P}$ be a graded operad (with trivial differential, situated in degrees ≥ 0), and let V be a graded vector space (also situated in degrees ≥ 0). Let $A = \mathsf{F}(\mathcal{P}, V)$ be the free graded $\mathcal{P}$-algebra generated by V. Then the map*

$$\mathrm{CA}(\mathcal{P}, A) \to \lambda(\mathcal{P}, A) \cong \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} A^{\otimes(n+1)}$$

is a weak equivalence.

Proof. We have

$$\mathrm{CA}(\mathcal{P}, A) = \bigoplus_{n=1}^{\infty} \Big(V_{n,1} \otimes \mathcal{B}\mathcal{P}(n) \otimes A^{\otimes(n+1)} \Big)_{\mathbb{S}_{n+1}}. \tag{5.13}$$

For a finite set I, denote by k_0^I the vector space of functions $I \to \mathsf{k}$ whose sum is zero; thus, $\mathsf{k}_0^n \cong V_{n-1,1}$. By (5.5), and bearing in mind that rooted n-trees are the same as unrooted $(n+1)$-trees, we may rewrite the n-th summand of (5.13) as

$$\bigoplus_{\substack{\text{unlabeled, unrooted} \\ (n+1)\text{-trees } T}} \Big(\mathsf{k}_0^{\mathrm{ex}(T)} \otimes (\Sigma\bar{\mathcal{P}})(T) \otimes A^{\otimes \mathrm{ex}(T)} \Big)_{\mathrm{Aut}(T)}.$$

Since the free algebra A has the form

$$A = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n},$$

we can express $\mathrm{CA}(\mathcal{P}, A)$ directly in terms of trees. First, we need some terminology.

A vertex v of a tree T is called *extremal,* if the number of internal edges incident to v is one or zero. By an *extended tree* (T, S), we mean a tree T together with a subset S of its set of extremal vertices; we denote by $T \backslash S$ the tree obtained from T by removing all vertices in S together with all of the external edges attached to them. For an extended tree (T, S), let

$$\tilde{\mathcal{P}}(T, S) = \bigotimes_{v \in \mathsf{v}(T \backslash S)} (\Sigma\bar{\mathcal{P}})(\!(\mathrm{ex}(v))\!) \otimes \bigotimes_{v \in S} \mathcal{P}(\!(\mathrm{ex}(v))\!).$$

Using this notation, we may write

$$\mathrm{CA}(\mathcal{P}, A) = \bigoplus_{(T,S)} \Big(\mathsf{k}_0^{\mathrm{ex}(T \backslash S)} \otimes \tilde{\mathcal{P}}(T, S) \otimes V^{\otimes \mathrm{ex}(T)} \Big)_{\mathrm{Aut}(T,S)}, \tag{5.14}$$

where the summation is over isomorphism classes of extended trees.

Introduce an increasing filtration G on $\mathrm{CA}(\mathcal{P}, A)$ by setting

$$G_l \mathrm{CA}(\mathcal{P}, A) = \bigoplus_{\{(T,S) \mid |\mathsf{v}(T)| \leq l\}} \Big(\mathsf{k}_0^{\mathrm{ex}(T \backslash S)} \otimes \tilde{\mathcal{P}}(T, S) \otimes V^{\otimes \mathrm{ex}(T)} \Big)_{\mathrm{Aut}(T,S)}.$$

This filtration is preserved by the differential $\delta = \delta_{\mathcal{B}\mathcal{P}} + \delta_{\Phi}$ of $\mathrm{CA}(\mathcal{P}, A)$. The associated graded complex $\mathrm{gr}^G \mathrm{CA}(\mathcal{P}, A)$ may be identified, as a graded vector space, with $\mathrm{CA}(\mathcal{P}, A)$, while its differential $\mathrm{gr}(\delta)$ is the sum of those pieces of δ which preserve the number of vertices in an extended tree (T, S). Thus $\delta_{\mathcal{B}\mathcal{P}}$ does not contribute at all to $\mathrm{gr}(\delta)$, while δ_{Φ} contributes a sum, over extremal vertices of T not in S, of the operation which adjoins v to S and in (5.14) attaches the sum of the elements of k labelling edges in $\mathrm{ex}(v)$ to the external edge of the tree $T \setminus (S \cup \{v\})$ corresponding to v.

On the summand of $\mathrm{gr}^G \mathrm{CA}(\mathcal{P}, A)$ corresponding to the extended tree (T, S), define an operator H of degree $+1$ as a sum over vertices $v \in S$ of the operation which removes v from S and divides the element of k attached to the external edge of $T \setminus S$ corresponding to v equally among the edges in $\mathrm{ex}(v)$. It is easily seen that $I - (\mathrm{gr}(\delta)H + H\,\mathrm{gr}(\delta))$ is the projection onto summands in (5.14) corresponding to extended trees (T, S) such that T has one vertex v and $S = \{v\}$. The spectral sequence associated to the filtration on $G_l\mathrm{CA}(\Phi, A)$ collapses, and the homology of $\mathrm{CA}(\Phi, A)$ equals the homology of $\mathrm{gr}^G \mathrm{CA}(\Phi, A)$, which is

$$\bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{\otimes(n+1)}.$$

By Proposition (4.9), this is the same as $\lambda(\mathcal{P}, A)$, completing the proof of Theorem (5.3). $\square$

6. Examples

(6.1). **Associative algebras.** We first consider the theory of associative algebras with unit. Let $\mathcal{P} = \mathsf{UAs}$ be the unital associative operad (see (3.10)), with $\mathsf{UAs}(n) \cong \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\mathbb{S}_{n+1}} 1\!\!1$ for $n \geq 0$. Let UCoas be the cooperad introduced in (5.7) (c) and $\Psi = \Psi^{\mathsf{UAs}}$ be the unital-associative twisting cochain. Thus, $\mathsf{UCoas}(n) \cong \Sigma^{n-1} \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\mathbb{S}_{n+1}} \mathrm{sgn}$ for $n \geq 0$.

Let A be a unital associative algebra, which for simplicity we will suppose to have trivial grading. We abbreviate the complex $\mathrm{CA}(\Psi, A)$ by $\mathrm{CA}(A)$, and similarly for $\mathrm{CB}(\Psi, A)$ and $\mathrm{CC}(\Psi, A)$. We will now identify these complexes, relating them to the usual Connes-Tsygan complexes for cyclic homology of associative algebras, whose definition we now recall (see Loday [16]).

The *Hochschild complex* $C_\bullet(A, A)$ of an associative algebra A has $C_n(A, A) = A^{\otimes(n+1)}$ with differential b given by the formula

$$b(x_0 \otimes \ldots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i x_0 \otimes \ldots \otimes x_i x_{i+1} \ldots \otimes x_n + (-1)^n x_n x_0 \otimes \ldots \otimes x_{n-1}.$$

The homology of the complex $C_\bullet(A, A)$ may be identified with $\mathrm{Tor}_\bullet^{A\otimes A^{\mathrm{op}}}(A, A)$. We also have the complex $C_\bullet(A, b')$, with $C_n(A, b') = A^{\otimes(n+1)}$ but with the modified differential

$$b'(x_0 \otimes \ldots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i x_0 \otimes \ldots \otimes x_i x_{i+1} \ldots \otimes x_n.$$

The complex $C_\bullet(A, b')$ is contractible if A is unital.

Let $t = (-1)^n \tau \in \mathsf{k}[\mathbb{S}_{n+1}]$, and let $N = \sum_{i=0}^n t^i$ The *cyclic double complex* of A, denoted $C_{\bullet\bullet}(A)$, is the double complex

$$0 \leftarrow C_\bullet(A, A) \xleftarrow{1-t} C_\bullet(A, b') \xleftarrow{N} C_\bullet(A, A) \xleftarrow{1-t} C_\bullet(A, b') \xleftarrow{N} \ldots$$

The homology of the total complex of $C_{\bullet\bullet}(A)$ is the classical *cyclic homology* of the algebra A, denoted $H_\bullet^\lambda(A)$.

We will also need the cyclic complex $C_\bullet^\lambda(A)$, which is the cokernel of the map of complexes $C_\bullet(A, A) \xleftarrow{1-t} C_\bullet(A, b')$. Since $C_\bullet(A, b')$ is acyclic, $C_\bullet^\lambda(A)$ is weakly equivalent to the total complex of the double complex $C_{\bullet\bullet}(A)$.

We now identify the complex $\mathrm{CB}(A)$. Note that

$$\mathsf{UCoas}(n) \otimes \mathsf{k}^{n+1} \cong \Sigma^{n-1} \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\mathbb{S}_{n+1}} 1\!\!1 \otimes \mathsf{k}^{n+1} \cong \Sigma^{n-1}\mathsf{k}[\mathbb{S}_{n+1}]$$

is the $(n-1)$-fold suspension of the regular representation of $\mathbb{S}_{n+1}$. Thus

$$\mathrm{CB}(A) \cong \bigoplus_{n=0}^{\infty} \Sigma^{n-1} A^{\otimes(n+1)}.$$

If one goes through the definition of the differential δ_Ψ on this complex, one finds that it may be identified with the differential b of the Hochshild complex $C_\bullet(A, A)$ of A. Thus, we see that $\mathrm{CB}(A) \cong \Sigma^{-1} C_\bullet(A, A)$ is the desuspension of the Hochschild complex of A.

The complex $\mathrm{CC}(A)$ is identified with the cokernel of the operator $1-t$ acting on $\mathrm{CB}(A)$:

$$\mathrm{CC}(A) \cong \bigoplus_{n=1}^{\infty} A^{\otimes(n+1)} / \operatorname{Im}(1-t) = \Sigma^{-1} C_\bullet(A, A) / \operatorname{Im}(1-t).$$

It follows that $\mathrm{CC}(A) \cong \Sigma^{-1} C^\lambda_\bullet(A)$, and that $\mathrm{HC}_n(A) \cong H^\lambda_{n+1}(A)$ is the cyclic homology of A.

Finally, we identify $\mathrm{CA}(A)$. Note that $V_{n,1}$, regarded as a module over the group algebra $\mathsf{k}[\mathbb{S}_{n+1}]$, is generated by one element u, invariant under $\mathbb{S}_n \subset \mathbb{S}_{n+1}$, and with the relation

$$(1 + \tau + \dots + \tau^{n-1} + \tau^n) u = 0.$$

This implies that

$$\mathrm{CA}(A) \cong \bigoplus_{n=1}^{\infty} \Sigma^{n-1} A^{\otimes(n+1)} / \operatorname{Im}(N).$$

Unraveling the definition of the differential δ_Ψ, we see that we may identify it with b'; thus, $\Sigma\mathrm{CA}(A)$ is the cokernel of the map of complexes $C_\bullet(A, A) \xrightarrow{N} C_\bullet(A, b')$, or equivalently, of the map of complexes $C^\lambda_\bullet(A) \xrightarrow{N} C_\bullet(A, b')$. It turns out that this complex again calculates the cyclic homology.

(6.2)**. Proposition.** *If A is unital, the complex $\mathrm{CA}(A)$ is weakly equivalent to the total complex of the double complex $C_{\bullet\bullet}(A)$.*

Proof. By unitality of A, the complex $C_\bullet(A, b')$ is contractible. The result follows from the short exact sequence of complexes

$$0 \to C^\lambda_\bullet(A) \xrightarrow{N} C_\bullet(A, b') \to \Sigma\mathrm{CA}_\bullet(A) \to 0. \tag{6.3}$$

□

To summarize our results, we see that if A is a unital associative algebra, $\mathrm{HA}_\bullet(A)$ and $\Sigma\mathrm{HC}_\bullet(A)$ are both naturally isomorphic to the cyclic homology $H^\lambda_\bullet(A)$ of A, and the long exact sequence of homology associated to our sequence of complexes (5.9), is nothing other than the Connes-Tsygan *periodicity exact sequence:*

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{HC}_{n+1}(A) \arrow[r] \arrow[d, equal] & \mathrm{HA}_n(A) \arrow[r] \arrow[d, equal] & \mathrm{HB}_n(A) \arrow[r] \arrow[d, equal] & \mathrm{HC}_n(A) \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & H^\lambda_{n+2}(A) \arrow[r, "S"] & H^\lambda_n(A) \arrow[r, "B"] & H_{n+1}(A, A) \arrow[r, "I"] & H^\lambda_{n+1}(A) \arrow[r] & \dots
\end{tikzcd}$$

In particular, the map $S : H^\lambda_{n+2}(A) \to H^\lambda_n(A)$ (the additive analog of the inverse of the Bott map in K-theory) is identified with the boundary map $\mathrm{HC}_{n+1}(A) \to \mathrm{HA}_n(A)$ of our long exact sequence, after we take into account the "accidental" isomorphism $\mathrm{HC}_n(A) \cong \mathrm{HA}_{n+1}(A)$.

The relation between HA and HC for more general cyclic operads will be studied in (6.11). We will see that these theories are in a certain sense dual.

Let us now briefly consider the operad Ass; there is a surjective map of operads $\mathsf{UAs} \to \mathsf{Ass}$ which is an isomorphism $\mathsf{UAs}(n) \cong \mathsf{Ass}(n)$ except for $n = 0$, where $\mathsf{Ass}(0) = 0$. The operad Ass is cyclic quadratic, the cooperad $\mathsf{Ass}^{\perp}$ (see (5.7) (b)) embeds in UCoas, and $\mathsf{UCoas}(n) \cong \mathsf{Coass}(n)$ except for $n = 0$, where $\mathsf{Coass}(0) = 0$. The quadratic twisting cochain $\Psi = \Psi^{\mathsf{Ass}}$ from the cooperad $\mathsf{Ass}^{\perp}$ to the operad Ass is induced by the cyclic twisting cochain from UCoas to UAs by means of these maps. If A is an associative algebra (not necessarily with unit), the three complexes are closely related to those which we described above:

(1) $\mathrm{CA}(\Psi, A)$ is identical to $\mathrm{CA}(A)$;
(2) $\mathrm{CB}_n(\Psi, A)$ and $\mathrm{CC}_n(\Psi, A)$ are identical to $\mathrm{CB}_n(A)$ and $\mathrm{CC}_n(A)$, respectively, if $n \geq 0$, but the bottom of the complex, the space A in degree -1, is discarded in each case.

Thus, the long exact sequence again agrees with that of Connes and Tsygan, except at the last few terms.

In a similar way, we can put the theory of cyclic homology of A_∞-algebras developed in [10] into the framework of the present paper by using the cyclic structure of the operad A_∞ described in (2.3) (c).

(6.4). **Koszul operads.** Let $\mathcal{P} = \mathcal{P}(E, R)$ be a cyclic quadratic operad with quadratic dual $\mathcal{P}^!$. There is a natural projection of operads $\Lambda^{-1}(\mathcal{B}\mathcal{P})^* \to \mathcal{P}^!$ defined as follows (see [12], 4.1.1):

$$\Lambda^{-1}(\mathcal{B}\mathcal{P})^*(n) \cong (\mathcal{B}\Lambda\mathcal{P})^*(n) = \bigoplus_{\text{rooted } n\text{-trees } T} (\Sigma\Lambda\bar{\mathcal{P}})(T)^*$$
$$\longrightarrow \bigoplus_{\text{binary } n\text{-trees } T} (\Sigma\Lambda\bar{\mathcal{P}})(T)^* = \mathbb{T}(E^*)(n) \longrightarrow \mathcal{P}^!(n).$$

The operad $\mathcal{P}$ is called *Koszul,* if all of these maps are weak equivalences. The operads Ass, Com and Lie are Koszul [12], as is Pois [11].

Let $\mathcal{P}^{\perp}$ be the cooperad $\Lambda^{-1}(\mathcal{P}^!)^*$ (5.7) (b). The dual of the above projection of operads is an embedding of cooperads $j : \mathcal{P}^{\perp} \hookrightarrow \mathcal{B}\mathcal{P}$.

(6.5)**. Proposition.** *Let $\mathcal{P}$ be a Koszul cyclic quadratic operad, and let $\Psi = \Psi^{\mathcal{P}}$ be the quadratic twisting cochain from $\mathcal{P}^{\perp}$ to $\mathcal{P}$. Then the functor $H_i(\mathrm{CA}(\Psi, A))$ is naturally equivalent to the cyclic homology $\mathrm{HA}_i(\mathcal{P}, A)$.*

Proof. The morphism of cooperads $j : \mathcal{P}^{\perp} \to \mathcal{B}\mathcal{P}$ being a weak equivalence, we easily infer that the induced morphism of complexes $\mathrm{CA}(\Psi^{\mathcal{P}}, A) \to \mathrm{CA}(\Phi^{\mathcal{P}}, A)$, where $\Phi^{\mathcal{P}}$ is the universal twisting cochain, is as well. By Theorem (5.3), $\mathrm{CA}(\Phi^{\mathcal{P}}, A)$ calculates the cyclic homology of A, and the corollary follows. □

In the remainder of this section, we illustrate this corollary in the cases of commutative and Lie algebras; we take up Poisson and Batalin-Vilkovisky algebras elsewhere.

(6.6). **Commutative algebras.** The dual cooperad $\mathsf{Com}^{\perp}$ of the commutative operad Com is isomorphic to $\Lambda^{-1}\mathsf{Lie}^*$. We will now identify the complexes $\mathrm{CA}(\Psi, A)$ and $\mathrm{CC}(\Psi, A)$, where $\Psi = \Psi^{\mathsf{Com}}$ is the canonical cyclic twisting cochain from $\mathsf{Com}^{\perp}$ to Com.

Recall the definition of the *Harrison complex* $\mathrm{CHarr}_\bullet(A, M)$ of a commutative algebra A with coefficients in a module M [1], [11]: as a graded vector space,

$$\mathrm{CHarr}_\bullet(A, M) \cong \bigoplus_{n=1}^{\infty} \mathsf{Lie}(n)^* \otimes_{\mathbb{S}_n} (\Sigma A)^{\otimes n} \otimes M,$$

while its differential is induced from that of the Hochschild complex $C_\bullet(A, M)$, of which it is naturally a subcomplex. Denote the cohomology of the Harrison complex by $\mathrm{Harr}_\bullet(A, M)$. We will be especially concerned with the cases where $M = A$, and where $M = \mathsf{k}$ with trivial action of A. In particular, we will need the theorem of Barr [1] and Quillen [20], that $H_\bullet(A, \mathsf{k})$ are the left non-abelian derived functors of the functor $A \mapsto A/A^2$ from commutative algebras to vector spaces (see also [11]).

The complex $\mathrm{CC}_\bullet(\Phi, A)$ is isomorphic to $\Sigma\,\mathrm{CHarr}_\bullet(A, A)$; the identification of the differential proceeds as in the case of associative algebras. In particular, if A is concentrated in degree 0, we see that $\mathrm{CC}_n(\Phi, A) \cong \mathrm{CHarr}_{n+1}(A, A)$.

We now identify the complex $\mathrm{CA}(\Phi, A)$. If A is an almost free commutative algebra, the natural map from $\lambda(\mathsf{Com}, A) \to A^2$ induced by sending $x \otimes y$ to xy is an isomorphism. It follows that if A is almost free, there is a short exact sequence

$$0 \to \lambda(\mathsf{Com}, A) \to A \to A/A^2 \to 0.$$

The homology long exact sequence of this short exact sequence may be identified with

$$\cdots \to \mathrm{Harr}_{n+1}(A, \mathsf{k}) \to \mathrm{HA}_n(\mathsf{Com}, A) \to H_n(A) \to \mathrm{Harr}_n(A, \mathsf{k}) \to \dots, \tag{6.7}$$

since the homology of $\lambda(\mathsf{Com}, A)$ is by definition the cyclic homology of A, while the homology of A/A^2 is, by the theorem of Barr and Quillen, isomorphic to $\mathrm{Harr}_\bullet(A, \mathsf{k})$. Since the long exact sequence (6.7) is natural, it actually holds for all A. So we get the following proposition.

(6.8)**. Proposition.** *If A is an ungraded commutative algebra and $n > 0$,*

$$\mathrm{HA}_n(\mathsf{Com}, A) \cong \mathrm{Harr}_{n+1}(A, \mathsf{k}).$$

This proposition sheds some light on the structure of the $\mathbb{S}_{n+1}$-module $\mathsf{Lie}(n)$.

(6.9)**. Corollary.** *There is a natural isomorphism of $\mathbb{S}_{n+1}$-modules*

$$\mathsf{Lie}(n) \otimes V_{n,1} \cong \mathsf{Lie}(n+1).$$

Proof. This follows by applying the isomorphism $\mathrm{HA}_n(\mathsf{Com}, A) \cong \mathrm{Harr}_{n+1}(A, \mathsf{k})$ to the commutative algebra spanned by vectors $\{x_0, \dots, x_n\}$, with vanishing product. Then the differentials of the complexes $\mathrm{CA}_\bullet(\Phi, A)$ and $\mathrm{CHarr}_\bullet(A, \mathsf{k})$ vanish. Both complexes decompose into a direct sum of multi-homogeneous components according to the degree in each generator x_i. The summand of multi-degree $(1, \dots, 1)$ in $\mathrm{HA}_\bullet(\mathsf{Com}, A)$ is isomorphic to $\mathsf{Com}^\perp(n) \cong \Sigma^{n-1}\mathsf{Lie}(n)^* \otimes \mathrm{sgn} \otimes V_{n,1}$, while the summand of multi-degree $(1, \dots, 1)$ in $\mathrm{CHarr}_\bullet(A, \mathsf{k})$ is isomorphic to $\Sigma^n\mathsf{Lie}(n+1)^* \otimes \mathrm{sgn}$, and the result follows. □

We do not identify the complex $\mathrm{CB}(\Phi, A)$ explicitly, but simply state the long exact sequence for the three homology theories:

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{HC}_{n+1}(\mathsf{Com}, A) \arrow[r] \arrow[d, equal] & \mathrm{HA}_n(\mathsf{Com}, A) \arrow[r] \arrow[d, equal] & \mathrm{HB}_n(\mathsf{Com}, A) \arrow[r] & {} \\
& \mathrm{Harr}_n(A, A) \arrow[r, "S"] & \mathrm{Harr}_{n+1}(A, \mathsf{k}) & &
\end{tikzcd}$$

Note that the map S, analogous to the periodicity map of cyclic homology, now maps between two distinct homology theories.

(6.10). **Lie algebras.** The dual cooperad $\mathsf{Lie}^\perp$ of the Lie operad Lie is isomorphic to $\Lambda^{-1}\mathsf{Com}^*$. Let us identify the complexes $\mathrm{CA}(\Psi,\mathfrak{g})$ and $\mathrm{CC}(\Psi,\mathfrak{g})$, where Ψ is the canonical cyclic twisting cochain from $\mathsf{Lie}^\perp$ to Lie. Explicitly, $\Psi_2 : \mathsf{Lie}^\perp \cong \Sigma\mathsf{Com}(2)^* \otimes \mathrm{sgn} \to \mathsf{Lie}(2)$ is the natural isomorphism, while $\Psi_n = 0$ for $n \neq 2$.

Since $\mathsf{Lie}^\perp(n) \cong \Sigma^{n-1}\,\mathrm{sgn}$, we see that

$$\mathrm{CB}_n(\Psi,\mathfrak{g}) \cong \mathfrak{g} \otimes \Sigma^n\Lambda^n\mathfrak{g} \cong C_n(\mathfrak{g},\mathfrak{g})$$

is the Chevalley-Eilenberg chain complex of $\mathfrak{g}$ with coefficients in the adjoint module, while $\mathrm{CC}_n(\Psi,\mathfrak{g}) \cong C_{n+1}(\mathfrak{g},\mathsf{k})$ is the truncated Chevalley-Eilenberg complex of $\mathfrak{g}$ with trivial coefficients. As a consequence, $\mathrm{CA}_n(\Psi,\mathfrak{g})$ is the kernel of the map of complexes $C_n(\mathfrak{g},\mathfrak{g}) \to C_{n+1}(\mathfrak{g},\mathsf{k})$ given by the formula

$$g_0 \otimes (g_1 \wedge \ldots \wedge g_n) \mapsto g_0 \wedge g_1 \wedge \ldots \wedge g_n.$$

(6.11). **Duality between** HA **and** HC**.** By Theorem (5.3), the complex $\mathrm{CA}(\mathcal{P},A)$ calculates the cyclic homology of an algebra A over a cyclic operad $\mathcal{P}$. We will now give a similar interpretation of the homology of $\mathrm{CC}(\mathcal{P},A)$.

Let $\mathcal{Z}$ be a dg-cooperad. A *dg-coalgebra* over $\mathcal{Z}$ is a chain complex C with coaction maps $C \to \mathcal{Z}(n) \otimes C^{\otimes n}$ satisfying conditions dual to those for an algebra over an operad ([11], Section 1.7). If V is a chain complex then the *cofree $\mathcal{Z}$-coalgebra* generated by V is defined as

$$\mathsf{G}(\mathcal{Z},V) = \bigoplus_{n=0}^{\infty}(\mathcal{Z}(n) \otimes V^{\otimes n})^{\mathbb{S}_n}.$$

If Φ is a twisting cochain from a cooperad $\mathcal{Z}$ to an operad $\mathcal{P}$, there is a functor $\mathbb{B}(\Phi,A)$ from $\mathcal{P}$-algebras to $\mathcal{Z}$-coalgebras called the *bar-construction.* As a coalgebra, $\mathbb{B}(\Phi,A) = \mathsf{G}(\mathcal{Z},A)$ is the cofree $\mathcal{Z}$-coalgebra generated by A. The differential in $\mathbb{B}(\mathcal{Z},A)$ is the sum of the differential of $\mathsf{G}(\Phi,A)$ and a differential involving the twisting cochain ([11], Section 2.3).

In the special case that Φ is the universal twisting cochain from $\mathcal{BP}$ to $\mathcal{P}$, we denote the bar construction by $\mathbb{B}(\mathcal{P},A)$. It is proved in [11] that the functor $A \mapsto \mathbb{B}(\mathcal{P},A)$ induced an equivalence of categories

$$\mathrm{Ho}\{\mathcal{P}\text{-algebras}\} \equiv \mathrm{Ho}\{\mathcal{BP}\text{-coalgebras}\}$$

where Ho is the homotopy category, obtained by inverting weak equivalences.

The cyclic cohomology of a dg-coalgebra C over a cyclic dg-cooperad $\mathcal{Z}$ has a definition dual to that of the cyclic homology of an algebra over a cyclic dg-operad. We define the chain complex $\lambda(\mathcal{Z},C)$ to be the kernel of the map

$$C \otimes C \to \mathcal{Z}(n) \otimes C^{\otimes(n+1)} \to \mathcal{Z}(n) \otimes_{\mathbb{S}_n^+} C^{\otimes(n+1)},$$

where the first map is induced by the coaction $C \to \mathcal{Z}(n) \otimes C^{\otimes n}$ of C, and the second is projection to coinvariants; the differential is induced by the differential on $C \otimes C$. For a cofree coalgebra $\mathsf{G}(\mathcal{Z},V)$, have

$$\lambda(\mathcal{Z},\mathsf{G}(\mathcal{Z},V)) \cong \bigoplus_{n=0}^{\infty}\mathcal{Z}(n) \otimes_{\mathbb{S}_{n+1}} V^{\otimes(n+1)}, \tag{6.12}$$

in a way analogous to Proposition (4.9). The cyclic cohomology groups of a coalgebra C are the right non-abelian derived functors $\mathsf{R}_i\lambda(\mathcal{Z},C))$; they may be calculated by applying the functor $\lambda(\mathcal{Z},-)$ to an *almost cofree resolution* $C \to \tilde{C}$ of the coalgebra C; that is, a weak equivalence of $\mathcal{Z}$-coalgebras with $\tilde{C}^{\#}$ a cofree $\mathcal{Z}^{\#}$-coalgebra.

Since the bar coalgebra $\mathbb{B}(\mathcal{P}, A)$ is almost cofree, the following proposition follows immediately from (6.12).

(6.13)**. Proposition.** *Let $\mathcal{P}$ be an augmented cyclic dg-operad, and let A be a $\mathcal{P}$-algebra. Then the homology of the complex* $\mathrm{HC}(\mathcal{P}, A)$ *coincides with the cyclic cohomology of the coalgebra $\mathbb{B}(\mathcal{P}, A)$ over the cooperad $\mathbb{B}\mathcal{P}$.*

Note that the result of Feigin and Tsygan relating the cyclic homology of a Koszul associative algebra with the cyclic homology of its dual ([7], Theorem 2.4.1 (b)) is a special case of this proposition, corresponding to $\mathcal{P} = \mathsf{Ass}$.

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