CYCLIC HOMOLOGY AND THE BEILINSON-MANIN-SCHECHTMAN CENTRAL EXTENSION.

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Abstract. We construct central extensions of the Lie algebra of differential operators on a one-dimensional affine variety over a field of characteristic zero, generalizing the Virasoro extension. The construction is an application of recent calculations of the Hochschild and cyclic homology of algebras of differential operators.

If M is a smooth affine variety with function ring $\mathcal{O}_M$ over a field of characteristic zero, let $\mathcal{D}_M$ denote the algebra of differential operators over $\mathcal{O}_M$, let $\mathcal{V}_M \subset \mathcal{D}_M$ denote the Lie subalgebra of vector fields on M, and let Ω^k_M denote the space of k-forms on M. Recall that if M is one-dimensional and the ground field is $\mathbb{R}$, then the universal central extension of the Lie algebra of $\mathcal{V}_M$ has centre isomorphic to $\mathrm{H}^1(M) = \Omega^1_M/d\Omega^0_M$ (this extension is called the Virasoro extension):

$$0 \to \Omega^1_M/d\Omega^0_M \to \hat{\mathcal{V}}_M \to \mathcal{V}_M \to 0$$

In the article [1], Beilinson, Manin and Schechtman found an analogue of this extension for the Lie algebra $\mathcal{V}_M$ when M is one-dimensional over the complex numbers. It was suggested by Witten [10] that this extension of the Lie algebra of vector fields is the restriction of a central extension of the Lie algebra of all differential operators $\mathcal{D}_M$ on M with the same centre:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_M/d\Omega^0_M & \longrightarrow & \hat{\mathcal{V}}_M & \longrightarrow & \mathcal{V}_M & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_M/d\Omega^0_M & \longrightarrow & \hat{\mathcal{D}}_M & \longrightarrow & \mathcal{D}_M & \longrightarrow & 0
\end{array}$$

The purpose of this note is to show that this extension of $\mathcal{D}_M$ exists in the above generality, using some results from the theory of cyclic homology, in particular, recent calculations of the Hochschild and cyclic homology of algebras of differential operators.

Cyclic Homology.

We start by recalling the definitions of the Hochschild and cyclic homology of an algebra A. The best references for all of this are the paper of Loday and Quillen [9], and of course the original paper of Connes [4].

Definition 1. *The Hochschild complex of an algebra A over a field F of characteristic zero is the complex*

$$C_k(A) = \underbrace{A \otimes_F \cdots \otimes_F A}_{\text{k+1 times}},$$

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with the differential

$$b(a_0, a_1, \dots, a_k) = \sum_{i=0}^{k-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_k) + (-1)^k (a_k a_0, a_1, \dots, a_{k-1}).$$

Here, by $(a_0, \dots, a_k)$*, we mean the element* $a_0 \otimes \dots \otimes a_k$ *of* $C_k(A)$*. The Hochschild homology of the algebra* A *is the homology of the complex* $(C_*(A), b)$*; we will denote it by* $\mathrm{HH}_*(A)$*.*

In [6], Hochschild et al. have calculated the Hochschild cohomology of the algebra of regular functions on a smooth affine variety. This is probably the most interesting example of Hochschild homology, since it shows that there is a link between Hochschild homology and de Rham theory.

Theorem 2. *If* M *is a smooth affine variety, there is a natural isomorphism*

$$\mathrm{HH}_k(\mathcal{O}_M) \cong \Omega_M^k,$$

induced by following the map from $C_k(\mathcal{O}_M)$ *to* Ω_M^k*:*

$$a_0 \otimes \dots \otimes a_k \mapsto a_0 \, da_1 \dots da_k. \quad \square$$

Let $t : C_k(A) \to C_k(A)$ be the operator

$$t(a_0, \dots, a_k) = (-1)^k (a_k, a_0, \dots, a_{k-1}).$$

The cyclic homology of A, denoted $\mathrm{HC}_*(A)$, is obtained by taking the homology of the complex $C_*^\lambda(A)$, defined as follows:

$$C_k^\lambda(A) = C_k(A) / \operatorname{im}(1 - t).$$

It is easily verified that the boundary operator b sends the image of $1 - t$ into itself, and thus descends to a boundary $b^\lambda : C_k^\lambda(A) \to C_{k-1}^\lambda(A)$.

Fundamental to understanding cyclic homology is Connes's long exact sequence

$$\dots \xrightarrow{S} \mathrm{HC}_{k-1}(A) \xrightarrow{B} \mathrm{HH}_k(A) \to \mathrm{HC}_k(A) \xrightarrow{S} \mathrm{HC}_{k-2}(A) \xrightarrow{B} \dots$$

which is constructed in the above references.

Our interest in cyclic homology stems from the following result, which has its origins in the theory of the Schur multiplicator, or $H^2(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

Theorem 3 (Kassel and Loday [7]). *There is a functor from algebras* A *to Lie algebras* $L(A)$*, such that the Lie algebra* $L(A)$ *fits into an exact sequence of functors*

$$0 \to \mathrm{HC}_1(A) \to L(A) \to A \to \mathrm{HC}_0(A) \to 0.$$

In this sequence, we denote the Lie algebra of the algebra A *by the same symbol* A*, and the vector spaces* $\mathrm{HC}_i(A)$ *are made into Lie algebras by giving them the zero bracket.*

Proof. Let $\mathfrak{A}$ be the subspace of $(A \otimes A) \oplus A$ spanned by elements of the form

$$(a \otimes b, [a, b]).$$

If $\rho : \mathfrak{A} \to A$ is the evident map given by projecting onto the component in A, we obtain the exact sequence

$$0 \to \{\omega \in C_1(A) \mid b\omega = 0\} \to \mathfrak{A} \xrightarrow{\rho} A \to \mathrm{HC}_0(A) \to 0$$

Define a bracket on $\mathfrak{A}$ by

$$[(*, a), (*, b)] = (a \otimes b, [a, b]).$$

Clearly, this bracket is a bilinear map from $\mathfrak{A}$ into itself; we will make it into a Lie bracket by taking a quotient of $\mathfrak{A}$ in such a way that the necessary relations are fulfilled:

(1) (Antisymmetry of $[\cdot,\cdot]$)

$$(a \otimes b, [a, b]) \sim -(b \otimes a, [a, b]),$$

which is equivalent to

$$(a \otimes b + b \otimes a, 0) \sim 0;$$

(2) (Jacobi rule)

$$([a, b] \otimes c + [b, c] \otimes a + [c, a] \otimes b, 0) \sim 0.$$

On quotienting out $\mathfrak{A}$ by the first of these relations, we obtain a new space, which we may call $\mathfrak{A}^\lambda$, which fits into an exact sequence

$$0 \longrightarrow \{\omega \in C_1^\lambda(A) \mid b\omega = 0\} \longrightarrow \mathfrak{A}^\lambda \longrightarrow A \longrightarrow \mathrm{HC}_0(A) \longrightarrow 0.$$

This is because the span of the vectors $a \otimes b + b \otimes a$ in $C_1(A)$ is just the image of $1-t$ on $C_1(A)$, and $b \circ (1-t)$ vanishes on $C_1(A)$. Now observe that Jacobi's rule is implied by the relation

$$ab \otimes c - a \otimes bc + ca \otimes b \sim 0,$$

in conjunction with the relation that we imposed to ensure the antisymmetry of the Lie bracket. Of course, this relation is the same as quotienting by the image of $C_2^\lambda(A)$ under the boundary operator b. Thus, we see that the desired Lie algebra is $L(A) = \mathfrak{A}/((1-t)C_1(A) + bC_2(A))$. $\square$

From Theorem 3, it follows that if $\mathrm{HC}_0(A) = \mathrm{HH}_0(A) = A/[A, A]$ vanishes, then the Lie algebra $L(A)$ is a central extension of A by $\mathrm{HC}_1(A)$. Furthermore, Connes's exact sequence tells us that if $\mathrm{HC}_0(A) = 0$, then $\mathrm{HC}_1(A) = \mathrm{HH}_1(A)$. Thus, we obtain as an easy corollary:

Corollary 4. *The Lie algebra $L(A)$ is a central extension of the Lie algebra $[A, A]$ by the space $\mathrm{HC}_1(A)$. If in addition $A = [A, A]$, then $L(A)$ is a central extension of the Lie algebra A by the space $\mathrm{HH}_1(A)$.*

The interesting thing about this central extension is that it fits into a coherent family of central extensions of the Lie algebras $\mathfrak{sl}_n(A) = [\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$. In other words, if we embed $\mathfrak{sl}_n(A)$ in $\mathfrak{sl}_{n+m}(A)$ by sending M to

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$$

we obtain the map of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{HC}_1(A) \arrow[r] \arrow[d, equal] & \hat{\mathfrak{sl}}_n(A) \arrow[r] \arrow[d] & \mathfrak{sl}_n(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{HC}_1(A) \arrow[r] & \hat{\mathfrak{sl}}_{n+m}(A) \arrow[r] & \mathfrak{sl}_{n+m}(A) \arrow[r] & 0
\end{tikzcd}$$

This comes about because of the Morita equivalence isomorphisms

$$\mathrm{HH}_k(A) \cong \mathrm{HH}_k(\mathfrak{gl}_n(A)) \quad \text{and} \quad \mathrm{HC}_k(A) \cong \mathrm{HC}_k(\mathfrak{gl}_n(A)).$$

Kassel and Loday [7] have proved that $\mathrm{HC}_1(A)$ is the centre of the universal central extension of $\mathfrak{sl}_n(A)$ for $n \geq 2$.

Hochschild homology of the algebra of differential operators.

The other main result that we need to construct the Beilinson-Manin-Schechtman central extension is a formula for the Hochschild homology of the algebra of differential operators.

Theorem 5. *Let M be a smooth affine variety of dimension d, and let $\mathcal{D}_M$ be the algebra of differential operators on M. The Hochschild homology groups of $\mathcal{D}_M$ are as follows:*

$$\mathrm{HH}_k(\mathcal{D}_M) \cong \mathrm{H}^{2d-k}(M) = \mathrm{H}^{2d-k}(\Omega^*_M, d). \quad \square$$

It follows from this theorem that if M is one-dimensional, for example, the circle in the real case, or a punctured Riemann surface in the complex case, there is a central extension

$$0 \to \mathrm{H}^1(M) \to L(\mathcal{D}_M) \to \mathcal{D}_M \to 0.$$

Observe that if M has dimension higher than one, $\mathrm{HH}_1(\mathcal{D}_M) = 0$; that is, it is only in the one-dimensional case that the Lie algebra of $\mathcal{D}_M$ has a non-trivial central extension.

In the case in which $A = \mathcal{D}_M$, the algebra of regular functions $\mathcal{O}_M$ is a subalgebra of $\mathcal{D}_M$, we obtain a map of central extensions of the following form:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{HH}_1(\mathcal{O}_M) \arrow[r] \arrow[d] & \hat{\mathfrak{sl}}_n(\mathcal{O}_M) \arrow[r] \arrow[d] & \mathfrak{sl}_n(\mathcal{O}_M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{HH}_1(\mathcal{D}_M) \arrow[r] & \hat{\mathfrak{gl}}_n(\mathcal{D}_M) \arrow[r] & \mathfrak{gl}_n(\mathcal{D}_M) \arrow[r] & 0
\end{tikzcd}$$

If $d = 1$, this diagram becomes

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_M/d\Omega^0_M \arrow[r] \arrow[d, equal] & \hat{\mathfrak{sl}}_n(\mathcal{O}_M) \arrow[r] \arrow[d] & \mathfrak{sl}_n(\mathcal{O}_M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_M/d\Omega^0_M \arrow[r] & \hat{\mathfrak{gl}}_n(\mathcal{D}_M) \arrow[r] & \mathfrak{gl}_n(\mathcal{D}_M) \arrow[r] & 0
\end{tikzcd}$$

whereas if $d > 1$, it becomes

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_M/d\Omega^0_M \arrow[r] \arrow[d] & \hat{\mathfrak{sl}}_n(\mathcal{O}_M) \arrow[r] \arrow[d] & \mathfrak{sl}_n(\mathcal{O}_M) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \hat{\mathfrak{gl}}_n(\mathcal{D}_M) \arrow[r] & \mathfrak{gl}_n(\mathcal{D}_M) \arrow[r] & 0
\end{tikzcd}$$

There are two proofs of Theorem 5, each obtained by a different spectral sequence. The first proof, which makes use of sheaf cohomology on the manifold $M^{\mathbf{an}}$, is modeled on the proof of de Rham's theorem for $M^{\mathbf{an}}$, and only works if M is a variety over $\mathbb{R}$ or $\mathbb{C}$. First one proves that the result is true on balls, which is an analogue of Poincaré's lemma, and then extends it to a global result by a simple double-complex argument. This proof may be found in [2], Section 5 of [3], and [8].

The second proof is more algebraic, and works for a smooth affine variety over any field of characteristic zero; this is because it does not use the topology of the field $\mathbb{R}$ or $\mathbb{C}$. The proof is based on the spectral sequence associated to the obvious filtration of the algebra $\mathcal{D}_M$, for which $F^p\mathcal{D}_M$ is the space of differential operators of order at most p. This induces a filtration on the Hochschild complex of $\mathcal{D}_M$, such that

$$E^1_k \cong \mathrm{HH}_k(\operatorname{gr}\mathcal{D}_M).$$

Here, $\operatorname{gr}\mathcal{D}_M$ is the graded algebra associated to the filtration on $\mathcal{D}_M$, or in other words, the algebra $\mathcal{O}_{T^*M}$ of regular functions on T^*M. This spectral sequence is used to calculate $\mathrm{HC}_*(\mathcal{D}_M)$ in both [11] and the first part of [3].

Brylinski has calculated the differential d^1 in this spectral sequence. It turns out to be an operator that was constructed by Ehresmann and Libermann [5]. They start by defining the symplectic dual operator $\star : \Omega^k_{T^*M} \to \Omega^{2d-k}_{T^*M}$, which is defined on any symplectic manifold, and is analogous to the Hodge star operator on Riemannian manifolds; note that $\star^2 = 1$. Then the differential d^1 equals

$$\delta = \star\, d \,\star. \tag{6}$$

The nature of this formula is not really surprising, since the sub-leading order of the product of two differential operators, given by the Poisson bracket of their symbols, only depends on the symplectic structure on T^*M. It follows from Equation (6) that the second term in the spectral sequence is equal to

$$E^2_k \cong \mathrm{H}^{2d-k}(M).$$

In order to complete the calculation of $\mathrm{HH}_k(\mathcal{D}_M)$, we must show that the spectral sequence degenerates at the E^2-term. To do this, we use the following lemma [3, p. 392].

Lemma 7. *Let $\mathcal{R}$ be the radial vector field on T^*M,*

$$\mathcal{R} = p \cdot \frac{\partial}{\partial p},$$

and let α be the canonical 1-form $p\,dq = \iota(\mathcal{R})\omega$. If $\varepsilon(\alpha) : \Omega^k_{T^*M} \to \Omega^{k+1}_{T^*M}$ is the exterior product with α, then we have the identity

$$\delta\varepsilon(\alpha) + \varepsilon(\alpha)\delta = \mathcal{L}_{\mathcal{R}} + (d-k).$$

Proof. Since $\star\iota(\mathcal{R})\star = \varepsilon(\alpha)$ and $\star\mathcal{L}_{\mathcal{R}}\star = \mathcal{L}_{\mathcal{R}} + (d-k)$, the lemma follows on conjugating Cartan's formula $d\iota(\mathcal{R}) + \iota(\mathcal{R}) = \mathcal{L}_{\mathcal{R}}$ by the operator $\star$. □

By Lemma 7, it is easy to show that all of the higher differentials in the spectral sequence vanish, from which Theorem 5 follows. Let $E^1_{kl} \subset \Omega^k_{T^*M}$ denote the space of k-forms on T^*M such that

$$\mathcal{L}_{\mathcal{R}}\omega = l.$$

By a standard homotopy argument, Lemma 7 shows that $E^2_{kl} = 0$ for $l \neq k-d$. But it is quite easy to show that $d^i : E^i_{kl} \to E^i_{k+1,l-i}$, and thus that d^i vanishes for $i > 1$.

This second proof has the advantage that it calculates the cyclic homology of $\mathcal{D}_M$ at the same time, as well as extending straightforwardly to calculate the cyclic homology of the algebra of pseudodifferential symbols on a smooth real manifold. Also, although we choose not to write out the details, it should be clear that the same methods may also be used to establish the existence of the Beilinson-Manin-Schechtman central extension for the Neveu-Schwartz algebra as well.

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