THE HOMOLOGY OF ALGEBRAS OF PSEUDO-DIFFERENTIAL SYMBOLS AND THE NON-COMMUTATIVE RESIDUE.

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Introduction

In this paper, we will calculate the Hochschild and cyclic homology groups of the algebra of pseudo-differential symbols $\Psi^\infty(M)/\Psi^{-\infty}(M)$ on a smooth manifold M; our main result is that

$$\mathrm{HH}_*(\Psi^\infty(M)/\Psi^{-\infty}(M)) = \mathrm{H}^{2n-*}(S^*M \times S^1; \mathbb{C}).$$

We will perform the calculation by two completely different methods, each of which has some advantages. In the first, we filter the algebra in question as in [2], and use a homogeneity argument to show that the resulting spectral sequence degenerates. From this point of view, the homology is seen to be a "semi-classical" invariant, since it is calculated at first order in Planck's constant, as the homology of the differential forms on T^*M with respect to the operator $\delta : \Omega^*(T^*M) \to \Omega^{*-1}(T^*M)$ (defined by Koszul [14] and Brylinski [2]; see §1).

The other method is a sheaf-theoretic calculation, modeled on Weil's proof of de Rham's Theorem, which uses a form of Poincaré lemma for Hochschild homology of symbols. In the case of differential operators, this Poincaré lemma states that the Hochschild homology of $\mathcal{D}(\mathbb{R}^n)$, the algebra of differential operators on $\mathbb{R}^n$, satisfies $\mathrm{HH}_*(\mathcal{D}(\mathbb{R}^n)) = \mathrm{H}^{2n-*}(\mathbb{R}^n, \mathbb{C})$. For pseudo-differential operators, the Poincaré lemma is formulated for the sheaf $\mathcal{E}_\mathbb{R}$ of micro-differential operators on $S^*M \times S^1$, introduced in [16]; the copy of S^1 comes from the fact that the sheaf $\mathcal{E}_\mathbb{R}$ lives on the cotangent bundle of a complexification of M.

In the course of the paper, we also perform a number of other such calculations of Hochschild homology, for algebras of differential operators, operators with compact support, and formal deformations of $C^\infty(X)$ for X a conic symplectic manifold.

It follows directly from our calculation of $\mathrm{HH}_*(\Psi^\infty(M)/\Psi^{-\infty}(M))$ that when M is connected and of dimension greater than one, the zeroth Hochschild homology group is one-dimensional. This gives an explanation of the well-known fact that there is, up to a constant, a unique continuous trace $R : \Psi^\infty/\Psi^{-\infty} \to \mathbb{C}$, known as the residue. This trace has been studied by Guillemin and Wodzicki ([9],[19,20]), in relation to the residues of zeta-functions of elliptic pseudo-differential operators. The advantage of our approach is that we obtain the existence and uniqueness of the residue by a direct differential-geometric argument. It is then easy to find the formula for R, as we do in §4.

We now give a summary of the different sections. In §1, we recall the spectral sequence of [2] for so-called quantum algebras (that is, the algebra of functions on a symplectic manifold with a star product). In §2, we restrict our attention to the conic symplectic case, calculating the commutator of δ with the operation of exterior product with $\iota(X)\omega$, where X is a conformal Hamiltonian vector field. This identity is then used, with X equal to the radial vector field, to calculate, in a geometric way, the Hochschild homology of quantum algebras based on conic symplectic manifolds. In §3, we use the same ideas to calculate the cyclic homology of these algebras. In §4, we apply these methods to calculate the Hochschild and cyclic homology of the algebra of pseudo-differential symbols, and we derive the main properties of the residue; we also give the link between our calculation and the higher residues of Wodzicki [20]. In §5, we present the sheaf-theoretic approach to the same calculations, which makes use of Poincaré-type lemmas.

We would like to thank Alain Connes, Victor Guillemin and Mariusz Wodzicki for several helpful discussions.

§1. The homology of quantum algebras

In this section, we will give the definition of a quantum algebra, which is an algebraic model for quantum mechanics. (This theory has been explored in great detail by a number of authors, who call it the theory of star products; for a nice review of this theory, see [5].)

definition. *A Poisson bracket on a commutative algebra $\mathcal{A}$ over $\mathbb{C}$ is a bilinear map*

$$(a, b) \mapsto \{a, b\},$$

which satisfies the following conditions:

(1) *$\mathcal{A}$ is a Lie algebra with respect to the Poisson bracket,*
(2) *$\{f, gh\} = g\{f, h\} + h\{f, g\}$ for all f, g and $h \in \mathcal{A}$.*

The most typical example of a Poisson algebra is the algebra of smooth functions on a symplectic manifold (M,ω), with Poisson bracket defined by

$$\{f,g\} = \omega^{-1}(df,dg),$$

where ω^{-1} is the nondegenerate form on TM corresponding to the symplectic form ω on T^*M. We interpret the elements of $\mathcal{A}$ as quantum observables, which correspond to certain derivations of $\mathcal{A}$, called Hamiltonian vector fields, by the rule $H_g f = \{f,g\}$, in such a way that

$$[H_f, H_g] = -H_{\{f,g\}}.$$

definition. *Let $\mathcal{A}$ be Poisson algebra. Then a quantum algebra is an associative product on the algebra $\mathcal{A}\{\hbar\}$ of the form*

$$a \star b = ab + \hbar\{a,b\}/2i + \sum_{i=2}^{\infty} \hbar^i \varphi_i(a,b),$$

where φ_i are bilinear maps from $\mathcal{A}\times\mathcal{A}$ to $\mathcal{A}$ satisfying

$$\varphi_i(a,b) = (-1)^i \varphi_i(b,a) \text{ and } \varphi_i(1,a) = 0.$$

One should think of a quantum algebra as a non-commutative deformation of the original product on $\mathcal{A}$ such that

$$a \star b - b \star a = \frac{\hbar}{i}\{a,b\}.$$

We our now interested in calculating the Hochschild homology $\mathrm{HH}_*(\mathcal{A}\{\hbar\})$ of the algebra $\mathcal{A}\{\hbar\}$ over $\mathbb{C}\{\hbar\}$ with its new product $\star$. This is the functor (usually denoted by $\mathrm{H}_*^{\mathbb{C}\{\hbar\}}(\mathcal{A}\{\hbar\},\mathcal{A}\{\hbar\})$) equal to the homology of the bar complex

$$C_i^{\mathbb{C}\{\hbar\}}(\mathcal{A}\{\hbar\}) = \underbrace{\mathcal{A}\{\hbar\} \otimes_{\mathbb{C}\{\hbar\}} \mathcal{A}\{\hbar\} \otimes_{\mathbb{C}\{\hbar\}} \cdots \otimes_{\mathbb{C}\{\hbar\}} \mathcal{A}\{\hbar\}}_{i+1 \text{ times}},$$

with respect to the differential b:

$$b(a_0,a_1,\dots,a_m) = \sum_{i=0}^{m-1} (-1)^i (a_0,\dots,a_i \star a_{i-1},\dots,a_m) \\ + (-1)^m (a_m \star a_0, a_1, \dots, a_{m-1}).$$

If $\mathcal{A}$ is a commutative algebra, then the Hochschild complex of $\mathcal{A}$ is a differential graded commutative algebra, with respect to the shuffle product. Let E be the projection onto the space A_i of Hochschild cycles $(a_0,\dots,a_m)$ which are antisymmetric in all but the first entry, which is defined as follows:

$$(a_0,a_1,\dots,a_m) \overset{E}{\mapsto} (m!)^{-1} \sum_{\sigma\in\Sigma_m} (a_0, a_{\sigma(1)},\dots,a_{\sigma(m)}).$$

We also have a map from the algebra of antisymmetric Hochschild chains A_* to the algebra $\Omega^*(\mathcal{A})$ of Kähler differentials of $\mathcal{A}$ (the exterior algebra generated by $\Omega^1(\mathcal{A}) = d\mathcal{A}$ under the relations $d(fg) = fdg + gdf$), given by the formula

$$(a_0, a_1, \dots, a_m) \mapsto a_0\, da_1 \wedge \dots \wedge da_m.$$

We can assemble these maps into the following diagram of complexes:

$$\begin{array}{ccccc}
\Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle 0} \\
C_i & \xrightarrow{E} & A_i & \longrightarrow & \Omega^i(\mathcal{A}) \\
\Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle 0} \\
C_{i-1} & \xrightarrow{E} & A_{i-1} & \longrightarrow & \Omega^{i-1}(\mathcal{A}) \\
\Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle 0}
\end{array}$$

Proposition (Hochschild, Kostant and Rosenberg [10]**; Connes** [4]**).**

(1) *If $\mathcal{A}$ is a smooth, commutative algebra, then the above three complexes have the same homology, hence* $\mathrm{HH}_i(\mathcal{A}) \cong \Omega^i(\mathcal{A})$.

(2) *If $\mathcal{A}$ is the algebra of smooth functions on a manifold M (respectively, the algebra of smooth functions of compact support), we obtain* $\mathrm{HH}_*(C^\infty(M)) \cong \Omega^i(M)$, *and* $\mathrm{HH}_*(C_c^\infty(M)) \cong \Omega_c^i(M)$. *(Of course, since $\mathcal{A}$ is a topological algebra, we must be careful which definition of the tensor product we use in defining Hochschild chains in order for this to be true; this is discussed in* [4, Chapter II]*.)* □

We will consider $\mathcal{A}\{\hbar\}$ as a filtered algebra, with respect to the obvious filtration, defined as follows:

$$F_i(\mathcal{A}\{\hbar\}) = \hbar^i \mathcal{A}[[\hbar]].$$

Using this filtration, we endow the algebra $\mathcal{A}\{\hbar\}$ with the structure of a complete, locally convex algebra as follows. Write $\mathcal{A}\{\hbar\}$ in the form

$$\mathcal{A}\{\hbar\} = \operatorname*{proj\,lim}_m \left(\operatorname*{inj\,lim}_{n \geq m} F_n(\mathcal{A}\{\hbar\})/F_m(\mathcal{A}\{\hbar\}) \right).$$

Each space $F_n(\mathcal{A}\{\hbar\})/F_m(\mathcal{A}\{\hbar\})$ has a topology, as the tensor product of a topological algebra with the finite dimensional vector space spanned by $\hbar^i$, with $m \leq i \leq n$. We endow the spaces $\operatorname{inj\,lim}_n F_n(\mathcal{A}\{\hbar\})/F_m(\mathcal{A}\{\hbar\})$ with the direct limit topology, and then $\mathcal{A}\{\hbar\}$ is endowed with the projective limit topology.

We now introduce a spectral sequence first studied by [2] which reduces the calculation of $\mathrm{HH}_*(\mathcal{A}\{\hbar\})$ to that of $\mathrm{HH}_*(\mathcal{A})$. This is the spectral sequence obtained from the filtration on $\mathcal{A}\{\hbar\}$.

Theorem.

(1) *The E^1 term of this spectral sequence is isomorphic to* $\mathrm{HH}_*(\mathcal{A})\{\hbar\}$.

(2) *The differential d_1 of the E^1 term is equal to $\hbar/i$ times the operator* $\delta : \mathrm{HH}_*(\mathcal{A}) \to \mathrm{HH}_{*-1}(\mathcal{A}$ *introduced by Koszul* [14], *defined by the following formula:*

$$\delta(a_0, \ldots, a_m) = \sum_{1 \le i \le m}^{m} (-1)^{i+1}(\{a_0, a_i\}, a_1, \ldots, \widehat{a_i}, \ldots, a_m) + \sum_{1 \le i < j \le m} (-1)^{i+j+1}(a_0, \{a_i, a_j\}, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_m).$$

proof. This theorem is easy to understand if in the formula for the b operator, we expand in powers of $\hbar$ using the definition of $a \star b$:

$$b(a_0, \ldots, a_m) = b_0(a_0, \ldots, a_m) + O(\hbar).$$

From this, we see that the differential in the E^0 term of the spectral sequence may be identified with the differential b_0 which defines the Hochschild homology of $\mathcal{A}$, proving (1).

Let E denote the antisymmetrization operator on Hochschild chains. If $\alpha = (a_0, \ldots, a_m)$ is a chain which satisfies $b_0\alpha = 0$, then we can find another chain α' which differs from α by a boundary, but which satisfies $E\alpha' = \alpha'$ in addition. On this chain, the full boundary operator b is equal to $\hbar/i$ times

$$\sum_{1 \le i \le m} (-1)^{i+1} E(\{a_0, a_i\}, a_1, \ldots, \widehat{a_i}, \ldots, a_m) + \sum_{1 \le i < j \le m} (-1)^{i+j+1} E(a_0, \{a_i, a_j\}, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_m) + O(\hbar).$$

The formula for d_1 may be identified easily from this equation. □

§2. The δ operator on conic symplectic manifolds

We will now turn our attention to the Poisson algebra which arises in the study of formal pseudo-differential operators on manifolds.

Definition. *A conic symplectic manifold (X, ω) is a symplectic manifold with a free action T_t of the multiplicative group $\mathbb{R}_+$ on it, such that*

$$T_t^*\omega = t\omega.$$

Equivalently, we may define the conic structure by giving a vector field $\mathcal{R}$ such that $\mathcal{L}_{\mathcal{R}}\omega = \omega$, in which case

$$T_t = \exp t\mathcal{R}.$$

An example of a conic symplectic manifold is the cotangent bundle of a manifold M with the zero-section removed, which is usually written as $\dot{T}^*M$.

Proposition (Darboux's lemma). *Let X be a $2n$-dimensional conic symplectic manifold. Given any point $x \in X$, there is a conic neighbourhood U (that is, preserved by T_t) and a symplectomorphism $\varphi : U \to \dot{T}^*\mathbb{R}^n$.* $\square$

If X is a conic symplectic manifold, then the Poisson bracket reduces the degree of homogeneity by 1, in the sense that

$$t^{-1}\{T_t^* a, T_t^* b\} = T_t^*\{a, b\}.$$

Thus, in a defining a quantum algebra structure on the space of smooth functions on X, we are lead to assign degree of homogeneity 1 to $\hbar$, and require that in the definition of the star-product

$$a \star b = ab + \hbar\{a, b\}/2i + \sum_{i=2}^{\infty} \hbar^i \varphi^i(a, b),$$

the maps φ_i reduce the degree of homogeneity by i. We also require that the maps φ_i are local, that is, are bilinear differential operators.

Proposition (de Wilde and Lecomte [18]). *There is a unique local star-product on any conic symplectic manifold, up to conjugation by a $\mathbb{C}\{\hbar\}$-linear endomorphism of $C^\infty(X)\{\hbar\}$ of the following form:*

$$a \mapsto a + \sum_{i=2}^{\infty} \hbar^i \rho_i(a),$$

where ρ_i is a differential operator on X which reduces the degree of homogeneity by i. $\square$

We will now study in greater detail the δ operator that arises in the spectral sequence for the Hochschild homology of $(C^\infty(X)\{\hbar\}, \star)$:

$$\begin{aligned}\delta(a_0\, da_1 \wedge \ldots \wedge da_m) = &\sum_{1 \le i \le n} (-1)^{i+1}\{a_0, a_i\}\, da_1 \wedge \ldots \wedge \widehat{da_i} \wedge \ldots \wedge da_m \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j+1} a_0\, d\{a_i, a_j\} \wedge \ldots \wedge \widehat{da_i} \wedge \ldots \wedge \widehat{da_j} \wedge \ldots \wedge da_m.\end{aligned}$$

As we have seen, the E^1 term in the spectral sequence for this homology is just the space of differential forms on X, and the δ operator is an operator which reduces the degree of a differential form by 1.

If V is a symplectic vector space, then its exterior algebra has a symplectic dual operator (Brylinski [2]) mapping $\Lambda^k V$ to $\Lambda^{2n-k} V$. Before defining it, we need some more notation. If v is an element of V, let $\varepsilon(v) : \Lambda^k V \to \Lambda^{k+1} V$ denote exterior multiplication by v, and let $\varepsilon^*(v)$ be its left adjoint with respect to the symplectic product. These operators satisfy

$$[\varepsilon(v), \varepsilon^*(w)] = \omega(v, w).$$

By extension, for any element $\theta \in \Lambda^l V$ of the exterior algebra of V, we obtain operators $\varepsilon(\theta) : \Lambda^k V \to \Lambda^{k+l} V$ and $\varepsilon^*(\theta) : \Lambda^k V \to \Lambda^{k-l}$. (Similarly, if $v \in V^*$, we define the inner product $\iota(v) : \Lambda^* V \to \Lambda^{*-1} V$ and its left adjoint $\iota^*(v) : \Lambda^* V \to \Lambda^{*+1} V$.)

We can now define the dual operator:

$$* = \exp \pi i (\varepsilon(\omega) + \varepsilon^*(\omega))/2.$$

Since $[\varepsilon(\omega), \varepsilon^*(\omega)] = n - k$ on $\Lambda^k V$, the operators $\varepsilon(\omega)$ and $\varepsilon^*(\omega)$ generate a Lie algebra isomorphic to the Lie algebra $\mathfrak{sl}(2)$: the operator $\varepsilon(\omega)$ corresponds to the matrix $T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the operator $\varepsilon^*(\omega)$ to the matrix $T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and the operator $(n-k)$ to the matrix $T_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (This point of view is familiar from the theory of harmonic forms on Kähler manifolds.) It follows from the matrix identity

$$\exp \left[\pi i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

that $*^2 = 1$.

Although we will nowhere make use of the fact, it is interesting to observe the close analogy between this symplectic dual operator and the Fourier transform on an inner product space. Indeed, the operator $* : \Lambda^* V \to \Lambda^* V$ may be defined by the formula

$$(*\alpha)(x) = \int_V \exp(\omega(x, y))\, \alpha(y)\, dy,$$

where $\int$ denotes the Berezin integral on the fermionic vector space V; the analogy with the Fourier transform is clear. In addition, the ordinary Fourier transform is equal to $\exp(\pi i H/2)$ where H is the harmonic oscillator on V; the similarity of the operators H and $\varepsilon(\omega) + \varepsilon^*(\omega)$ should is evident.

Proposition.

(1) *If θ is a one-form on X, then $\varepsilon^*(\theta) = *\varepsilon(\theta)*$.*
(2) *The operator δ is equal to $*d*$.*

Proof.

(1) Let us expand the operator $\mathrm{Ad}\,\{\exp ti(\varepsilon(\omega)+\varepsilon^*(\omega))\}\,.\varepsilon(\theta)$ as a power series in t. Since $[\varepsilon(\omega)+\varepsilon^*(\omega),\varepsilon(\theta)]=\varepsilon^*(\theta)$ and $[\varepsilon(\omega)+\varepsilon^*(\omega),\varepsilon^*(\theta)]=-\varepsilon(\theta)$, we obtain

$$\begin{aligned}\mathrm{Ad}\,\{\exp ti(\varepsilon(\omega)+\varepsilon^*(\omega))\}\,.\varepsilon(\theta) &= \sum_{j=0}^{\infty}\frac{(ti)^j}{j!}\left(\mathrm{ad}(\varepsilon(\omega)+\varepsilon^*(\omega))\right)^j.\varepsilon(\theta)\\ &= \cos t.\varepsilon(\theta)+\sin t.\varepsilon^*(\theta).\end{aligned}$$

We see from this that when $t=\pi/2$, the right hand side becomes equal to $\varepsilon^*(\theta)$.

(2) Since this is a local result, we are free to assume that our symplectic manifold is a symplectic vector space V, with basis Z_i $(0\le i\le 2n)$, so that V^* has the dual basis θ_i. The vector fields Z_i commute with $*$, since ω is invariant under translation of V, so that we obtain

$$\begin{aligned}*d* &= *\left(\sum_{i=1}^{2n}\varepsilon(\theta_i)Z_i\right)*\\ &= \sum_{i=1}^{2n}\varepsilon^*(\theta_i)Z_i.\end{aligned}$$

It is straightforward to check that this expression is equal to δ. $\square$

Definition. *A conformal Hamiltonian vector field Z on a symplectic manifold is a vector field satisfying*

$$\mathcal{L}_Z\omega = c\omega \qquad \textit{where } c\in\mathbb{R}.$$

On a conic symplectic manifold, any conformal Hamiltonian vector field is locally of the form $H_f + c\mathcal{R}$ for some $c\in\mathbb{R}$, since the radial vector field $\mathcal{R}$ is conformal Hamiltonian with $c=1$.

Theorem. *If Z is a conformal Hamiltonian vector field, then on $\Omega^k(X)$,*

$$[\delta,\iota^*(Z)] = \mathcal{L}_Z + c(n-k).$$

(The bracket here denotes the supercommutator; thus, since both operators δ and $\iota^*(Z)$ are odd, it is actually the anticommutator.)

Note that $\iota^*(Z)=\varepsilon(\iota(Z)\omega)$. It follows that when Z is a Hamiltonian vector field (so that $\iota(X)\omega=-df$ for some $f\in C^\infty(X)$), this formula becomes

$$[\varepsilon(df),\delta] = \mathcal{L}_{H_f},$$

while if Z is the radial vector field $\mathcal{R}$, then $\iota(\mathcal{R})\omega$ is the canonical 1*-form α on X which satisfies $d\alpha = \omega$, and we obtain*

$$[\delta, \varepsilon(\alpha)] = \mathcal{L}_{\mathcal{R}} + (n-k).$$

Proof. Recall the identity $[d, \iota(Z)] = \mathcal{L}_Z$. Conjugating this by the duality operator $*$ gives

$$*[d, \iota(Z)]* = [\delta, \iota^*(Z)] = *\mathcal{L}_Z * .$$

We next calculate the conjugate of $\mathcal{L}_Z$ by $\exp ti(\varepsilon(\omega) + \varepsilon^*(\omega))$, when Z is a conformally Hamiltonian vector field, making use of the representation theory of $\mathfrak{sl}(2)$. Observe that in its commutation relations with the Lie algebra generated by $\varepsilon(\omega)$ and $\varepsilon^*(\omega)$, the operator $\mathcal{L}_Z$ behaves like $-cT_0/2$. Using the identity in $\mathfrak{sl}(2)$,

$$\mathrm{Ad}\left[\exp ti \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos^2 t - \sin^2 t & \sin t. \cos t \\ \sin t. \cos t & \cos^2 t - \sin^2 t \end{pmatrix}$$

the following calculation is made extremely simple to perform:

$$\begin{aligned} \mathrm{Ad}\{\exp ti(\varepsilon(\omega) + \varepsilon^*(\omega))\}.\mathcal{L}_Z &= \sum_{j=0}^{\infty} \frac{(ti)^j}{j!} (\mathrm{ad}(\varepsilon(\omega) + \varepsilon^*(\omega)))^j.\mathcal{L}_Z \\ &= \mathcal{L}_Z - \frac{ci}{2} \sin t \cos t.(\varepsilon(\omega) - \varepsilon^*(\omega)) - \frac{c}{2}(\cos^2 t - \sin^2 t - 1).(n-k) \end{aligned}$$

Setting t equal to $\pi/2$ proves the result. $\square$

This theorem enables us to calculate the Hochschild homology of the quantum algebra $C^\infty(X)\{\hbar\}$ on a conic symplectic manifold X.

Theorem. *The higher differentials d_i, $i > 1$, in the spectral sequence abutting at $\mathrm{HH}_*(C^\infty(X)\{\hbar\})$ are equal to zero. Furthermore, the spectral sequence for $\mathrm{HH}_*(\mathcal{A}\{\hbar\})$ degenerates at E^2, and is convergent.*

proof. Let us first consider the behaviour of the differentials d_i in the spectral sequence with respect to the action of $\mathbb{R}_+$ given by T_t (this action extends to each term E^i of the spectral sequence). It is easily seen that the operator $\delta : E_k^1 \to E_{k-1}^i$ satisfies $\mathrm{Ad}(T_t).\delta = t^{-1}\delta$, while the higher differentials $d_i : E_k^i \to E_{k-1}^i$ satisfy $\mathrm{Ad}(T_t).d_i = t^{-i}d_i$.

Using our results on conformal Hamiltonian vector fields, we will show that E_k^2 is the homology of the operator δ restricted to the closed subspace of E_k^1 on which T_t acts as multiplication by t^{k-n}; it follows immediately that any element of E_k^i will satisfy this identity for all $i \geq 2$, showing that d_i must vanish for $i \geq 2$.

Indeed, it was proved above that $[\delta, \varepsilon(\alpha)] = \mathcal{L}_{\mathcal{R}} + (n-k)$. This homotopy shows that the homology of δ on $\Omega^*(X)$ may be calculated by restricting it to the space of forms satisfying

$$\mathcal{L}_{\mathcal{R}}\theta + (n-k)\theta = 0,$$

or in other words, the space $\{\theta \in E^1_k \mid T_t\theta = t^{k-n}\theta\}$.

This completes the proof of the degeneration of the spectral sequence. To show the convergence, we argue as follows. From the filtration of the algebra $C^\infty(X)\{\hbar\}$, we obtain a filtration of the Hochschild complex $C_*(C^\infty(X)\{\hbar\})$, in such a way that $C_*(C^\infty(X)\{\hbar\})$ is actually isomorphic to the projective limit of the quotient complexes $C_*(C^\infty(X)\{\hbar\})/F_mC_*(C^\infty(X)\{\hbar\})$. For each such quotient complex $C_*(C^\infty(X)\{\hbar\})/F_mC_*(C^\infty(X)\{\hbar\})$, one has a spectral sequence whose E^1 term is the part of degree $\geq m$ of the Hochschild homology of the graded algebra $\mathrm{gr}(C^\infty(X)\{\hbar\})$, which is isomorphic to the "Laurent algebra", that is, the tensor product of algebras $C^\infty(X) \otimes \mathbb{C}\{\hbar\}$. By the result of Hochschild et al. quoted in Section 1, we see that E^1 is the part of degree $\geq m$ of the algebra of differential forms with coefficients in $\mathrm{gr}(C^\infty(X)\{\hbar\})$.

The differential d_1 in the term E^1 of these spectral sequences is once more given by the operator δ. We may argue as before, using the homotopy

$$[\delta, \varepsilon(\alpha)] = \mathcal{L}_{\mathcal{R}} + (n-k),$$

that the E^2_k-term of the spectral sequence of

$$C_*(C^\infty(X)\{\hbar\})/F_mC_*(C^\infty(X)\{\hbar\})$$

is independent of m for m small enough, and that $E^i_k = E^\infty_k$ if $i \geq 2$. Then, by using induction on i and the Mittag-Leffler theorem of Grothendieck ([8]; for a shorter exposition, see [12]), we may deduce that the E^i_k term of the spectral sequences of

$$C_*(C^\infty(X)\{\hbar\}) \qquad \text{and} \qquad C_*(C^\infty(X)\{\hbar\})/F_mC^*(C^\infty(X)\{\hbar\})$$

are equal for large negative m. This demonstrates the convergence of the spectral sequence for $\mathrm{HH}_*(C^\infty(X)\{\hbar\})$. $\square$

Corollary. *The Hochschild homology* $\mathrm{HH}_*(C^\infty(X)\{\hbar\})$ *of the quantum algebra* $C^\infty(X)\{\hbar\}$ *is equal to* $\mathrm{H}^{2n-*}(X)\{\hbar\}$.

proof. Consider the diagram

$$\begin{array}{ccc} \Omega^k(X) & \xrightarrow{\ *\ } & \Omega^{2n-k}(X) \\ {\scriptstyle\delta}\big\downarrow & & {\scriptstyle d}\big\downarrow \\ \Omega^{k-1}(X) & \xrightarrow{\ *\ } & \Omega^{2n-k+1}(X) \end{array}$$

Since the horizontal arrows are isomorphisms, we obtain an isomorphism between the homology $\mathrm{H}_*(\Omega^*(X),\delta)$ and the cohomology $\mathrm{H}^{2n-*}(\Omega^*(X),d)$, which is isomorphic to $\mathrm{H}^{2n-*}(X)$ by de Rham's theorem. But this is just the E^2 term of the spectral sequence for $\mathrm{HH}_*(C^\infty(X)\{\hbar\})$, which by the convergence result of the above theorem is equal to E^∞. $\square$

§3. The cyclic homology of quantum algebras.

In this section, we will follow the same steps for cyclic homology as we have already traced in the case of the Hochschild homology: we recall the definition of the cyclic homology of an algebra, obtain a spectral sequence abutting to it in the case of a quantum algebra, and, using simple differential geometric identities, prove convergence in the case of the quantum algebra of a conic symplectic manifold.

Definition. *The cyclic homology* $\mathrm{HC}_*(\mathcal{A})$ *of an algebra* $(\mathcal{A},\star)$ *is the homology of the space of polynomials in the bar complex* $C_*(\mathcal{A})[u]$, *with respect to the boundary operator* $b+u^{-1}B$, *where* B *is given by the formula*

$$B(a_0,\dots,a_m)=\sum_{i=0}^{m}(-1)^{im}(1,a_i,\dots,a_m,a_0,\dots,a_{i-1}) \\ +\sum_{i=0}^{m}(-1)^{(i+1)m}(a_i,\dots,a_m,a_0,\dots,a_{i-1},1)$$

Here, by u^{-1} *we mean the operator mapping* u^i *to* u^{i-1} *for* $i>0$, *and mapping* u^0 *to* 0. *The variable* u *is assigned degree* 2 *in the complex* $C_*(\mathcal{A})[u]$.

This definition of the cyclic homology of an algebra is explained carefully in Loday and Quillen [15].

If $\mathcal{A}\{\hbar\}$ is a quantum algebra, it will be more convenient to define its cyclic homology as being the cohomology of the complex $C_*(\mathcal{A}\{\hbar\})[u]$ with respect to the operator $b+i^{-1}\hbar^2u^{-1}B$. This is equivalent, of course, since $i^{-1}\hbar^2$ is invertible.

We now obtain a spectral sequence for $\mathrm{HC}_*(\mathcal{A}\{\hbar\})$, by considering a filtration on the complex $C_*(\mathcal{A}\{\hbar\})[u]$ analogous to that used in the calculation of $\mathrm{HH}_*(\mathcal{A}\{\hbar\})$,

$$F_i(C_*(\mathcal{A}\{\hbar\})[u])=\hbar^iC_*(\mathcal{A})[u].$$

Lemma.

(1) *The* E^1 *term of this spectral sequence is isomorphic to* $\mathrm{HH}_*(\mathcal{A})[u]$, *and its differential* d_1 *is equal to* $i^{-1}\hbar\delta$.

(2) *When* $\mathcal{A}$ *is the algebra of smooth functions on a manifold, the operator* B *may be identified with the exterior differential* d*:*

$$d(a_0\,da_1\wedge\dots\wedge da_m)=da_0\wedge\dots\wedge da_m.$$

Proof.

(1) This is clear once we expand the differential $b+i^{-1}\hbar^2u^{-1}B$ in terms homogeneous in $\hbar$:
$$b = b_0 + i^{-1}\hbar\delta + O(\hbar^2).$$

(2) It is a simple matter to identify the operator B as the exterior differential when $\mathcal{A} = C^\infty(X)$, given the explicit definition of B, since in the algebra of differential forms, the Hochschild chain $(a_0, \dots, a_m, 1)$ represents zero, while the other terms in the definition of B give precisely the exterior differential. $\square$

If $\mathcal{A}\{\hbar\}$ is the quantum algebra associated to a conic symplectic manifold X, we would like to show that the spectral sequence for the cyclic homology degenerates at the E^2 term, as did the spectral sequence for Hochschild homology. In fact, more or less the same proof does the trick.

Theorem. *The higher differentials d_i, $i > 1$, in the spectral sequence abutting at $\mathrm{HC}_*(C^\infty(X)\{\hbar\})$ are equal to zero.*

proof. As in Section 2, it follows from the homotopy identity $[\delta, \varepsilon(\alpha)] = \mathcal{L}_{\mathcal{R}} + (n-k)$ that E^2_k has a set of representatives that are homogeneous of degree $k-n$. Since d_i changes the degree of homogeneity by $-i$, for all $i \geq 1$, it follows immediately that $d_i = 0$ for $i > 1$. This shows convergence of the spectral sequence; to show that it degenerates, we argue as for the case of Hochschild homology. $\square$

Corollary. *The cyclic homology HC_* of the quantum algebra $C^\infty(X)\{\hbar\}$ is equal to*
$\mathrm{H}^{2n-*}(X)\{\hbar\}[u]$. $\square$

§4. Application to the algebra of pseudo-differential symbols

The methods that we have developed in the previous sections applies also to calculate the Hochschild and cyclic homologies of the algebra of pseudo-differential symbols $\Psi^\infty(M)/\Psi^{-\infty}(M)$. This is because the structure of the product on this algebra is very similar to that of the Poisson algebra of $\dot{T}^*M$. However, the final answer will be somewhat different; one way to explain this is to observe that the de Rham cohomology of the algebra of Laurent polynomial coefficient differential forms on the positive real axis is two dimensional, with representatives 1 and dt/t, whereas the de Rham cohomology of the algebra of smooth coefficient differential forms is one dimensional, with representative 1.

Let us recall more precisely the structure of the algebra of pseudo-differential operators on a manifold M. This is a filtered algebra $\Psi^m(M)$ of operators on

$C^\infty(M)$, such that

$$\Psi^{-\infty}(M) = \bigcap_m \Psi^m(M)$$

is the algebra of smoothing operators. If we take the quotient of $\Psi^\infty(M) = \bigcup_m \Psi^m(M)$ by $\Psi^{-\infty}(M)$, we obtain an algebra which can be quite explicitly described; it consists of formal sums of homogeneous functions on the space $\dot{T}^*M$, of the form

$$a(x,\xi) \sim \sum_{i=-\infty}^{m} a_i(x,\xi),$$

where $a_i(x,t\xi) = t^i a(x,\xi)$ for $t > 0$; the product is given by a formula of the type

$$a \star b \sim ab + \{a,b\}/2i + \sum_{i=2}^{\infty} \varphi_i(a,b),$$

where φ_i is a certain bilinear map from $\Psi^k/\Psi^{-\infty} \times \Psi^l/\Psi^{-\infty}$ to $\Psi^{k+l-i}/\Psi^{-\infty}$.

We endow the algebra $\mathcal{P} = \Psi^\infty/\Psi^{-\infty}$ with the structure of a complete, locally convex algebra as follows. Write $\mathcal{P}$ in the form

$$\mathcal{P} = \operatorname{proj\,lim}_m \left(\operatorname{inj\,lim}_{n \geq m} \Psi^n/\Psi^m \right).$$

Each space Ψ^n/Ψ^m has a topology, as the space of smooth sections of a vector bundle on the manifold S^*M. We endow the spaces $\operatorname{inj\,lim}_n \Psi^n/\Psi^m$ with the direct limit topology, and then $\mathcal{P}$ is endowed with the projective limit topology.

We also may consider the algebra $\mathcal{P}_c \subset \mathcal{P}$, consisting of symbols which are supported in a compact subset of S^*M. We have $\mathcal{P}_c = \operatorname{inj\,lim} \mathcal{P}_K$, where K runs over the set of compact subsets of S^*M. Each $\mathcal{P}_K$ is topologized in the same way as $\mathcal{P}$, and $\mathcal{P}_c$ is given the direct limit topology.

By analogy with the methods of the earlier sections, we will calculate the Hochschild and cyclic homologies of the algebras $\mathcal{P}$ and $\mathcal{P}_c$ (using completed tensor products as in [4, Chapter II]), by means of the spectral sequence associated to the natural filtrations:

$$F_m(\mathcal{P}) = \Psi^m(M)/\Psi^{-\infty}(M)$$

$$F_m(\mathcal{P}[u]) = \sum_{j=0}^{\infty} u^j F_{m+2j}(\mathcal{P}).$$

Theorem.

(1) *The Hochschild homology of the algebras $\mathcal{P}$ and $\mathcal{P}_c$ are equal to*

$$\mathrm{HH}_*(\mathcal{P}) \cong \mathrm{H}^{2n-*}(S^*M \times S^1)$$
$$\mathrm{HH}_*(\mathcal{P}_c) \cong \mathrm{H}_c^{2n-*}(S^*M \times S^1).$$

In particular, if $n > 1$, this is isomorphic to $\mathrm{H}^{n-*-1}(M) \oplus \mathrm{H}^{n-*}(M)$, respectively
$\mathrm{H}_c^{n-*-1}(M) \oplus \mathrm{H}_c^{n-*}(M)$.

(2) *The cyclic homology of the algebras $\mathcal{P}$ and $\mathcal{P}_c$ are equal to*

$$\mathrm{HC}_*(\mathcal{P}) \cong \mathrm{H}^{2n-*}(S^*M \times S^1)[u]$$
$$\mathrm{HC}_*(\mathcal{P}_c) \cong \mathrm{H}_c^{2n-*}(S^*M \times S^1)[u].$$

Proof. Since the proofs of these two assertions are rather similar, we will only present the details for the calculation of $\mathrm{HH}_*(\mathcal{P})$, following closely the calculation of $\mathrm{HH}_*(C^\infty(X)\{\hbar\})$ in Section 2. We grade the terms of the spectral sequence in such a way that E^1_{kl} consists of k-forms on $\dot{T}^*M$ which are homogeneous of degree l; since the differential d_i of the the E^i term of the spectral sequence is homogeneous, mapping E^i_{kl} to $E^i_{k-1,l-i}$, it follows that each term E^i inherits this grading.

The differential d_1 in the term E^1 of the spectral sequence is once more given by the operator δ. If α is the differential form on $\dot{T}^*M$ equal to $\iota(\omega)$, it follows that operator $\varepsilon(\alpha)$ maps E^1_{kl} to $E^1_{k+1,l+1}$; thus, for given kl, we may argue as before, using the homotopy

$$[\delta, \varepsilon(\alpha)] = \mathcal{L}_{\mathcal{R}} + (n-k),$$

that the term E^2_{kl} of the spectral sequence is only non-zero for $l = k - n$, and that $E^i_{kl} = E^\infty_{kl}$ if $i \geq 2$.

In fact, we have really shown more; considering the spectral sequence of the quotient complex $C^*(\mathcal{P})/F_mC^*(\mathcal{P})$, we see that E^2_{kl} is actually independent of m for m sufficiently large. But the complex $C^*(\mathcal{P})$ is actually the projective limit of the quotient complexes $C^*(\mathcal{P})/F_mC^*(\mathcal{P})$. Using induction on i and the Mittag-Leffler theorem of Grothendieck, as in Section 2 , we may deduce that the E^i_{kl} term of the spectral sequences of $C_*(\mathcal{P})$ and $C_*(\mathcal{P})/F_mC^*(\mathcal{P})$ are equal for large negative m.

Thus, we obtain the degeneracy of the spectral sequence for $C_*(\mathcal{P})$. The proof is finished once we observe that the homology of the algebra of differential forms on $\dot{T}^*M$ with coefficients in $\mathrm{gr}(\mathcal{P})$ with respect to the boundary operator δ is given by

$$\mathrm{H}_*(\Omega^*(\mathrm{gr}(\mathcal{P}))) \cong \mathrm{H}^{2n-*}(S^*M \times S^1). \quad \square$$

In particular, we see that if M is connected and $n > 1$, the Hochschild homology group $\mathrm{HH}_0(\mathcal{P}_c)$ is equal to $\mathrm{H}_c^{2n-1}(S^*M)$, so is one dimensional, and that every diffeomorphism of M, extended to a diffeomorphism of $S^*M \times S^1$, acts trivially on this group. Since a continuous trace on $\Psi_c^\infty(M)/\Psi_c^{-\infty}(M)$ is the same thing as a linear form on $\mathrm{HH}_0(\mathcal{P}_c)$, this shows that there is, up to a constant, a unique trace on this algebra (unless $n = 1$, in which case there are two, corresponding to the fact that in this case, S^*M has two components).

This trace on the algebra of compactly supported symbols, known as the residue and denoted $R(P)$, was first observed in the study of the KdV equation, with $M = \mathbb{R}$ (see, for example, [1], where the higher-dimensional case is also briefly touched upon). It has been applied by Guillemin and Wodzicki ([9],[19]) to the study of the residues of the zeta function of an elliptic pseudo-differential operator. They obtain, among other things, the following results:

Proposition.

(1) *The unique trace on* $\Psi_c^\infty(M)/\Psi_c^{-\infty}(M)$ *is given by the formula*

$$R(a(x,\xi)) = \int_{S^*M} a_{-n}(x,\xi)\, \iota(\mathcal{R})\omega^n,$$

where $a_i(x,\xi)$ *is the part of the complete symbol of* $a(x,\xi)$ *that is homogeneous of order* i.

(2) *Let* $A \in \Psi^m(M)$ $(m \geq 1)$ *be an elliptic operator with positive symbol, and let* $P \in \Psi^\infty(M)$ *be an arbitrary pseudo-differential operator. If* $\zeta(s)$ *denotes the zeta function* $\mathrm{Tr}(P.A^{-s})$*, analytically continued to the whole complex plane, then there is a constant c, depending only on* $\dim M$*, such that*

$$\mathrm{Res}_{s=0}\, \zeta(s) = cR(P).$$

Proof. (1) Let R_M be the generator of $\mathrm{HH}^0(\mathcal{P}_c)$ which takes the value 1 on the orientation class of $\mathrm{H}^{2n-1}(S^*M)$. For every open set U of M, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{P}_c(U) & \longrightarrow & \mathcal{P}_c(M) \\ {\scriptstyle R_U}\big\downarrow & & {\scriptstyle R_M}\big\downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

Therefore, using partitions of unity, we may reduce the computation of R to the case in which M is equal to $\mathbb{R}^n$. In this case, R is given by a sequence of distributions b_k on the cosphere bundle $S^*\mathbb{R}^n$, such that

$$R(a) = \sum_i (a_k, b_k)_{S^*\mathbb{R}^n}.$$

We will now consider the action of various diffeomorphisms of $\mathbb{R}^n$ on R. We start with the group of translations and rotations of $\mathbb{R}^n$ on S^*M; invariance of R under the action of this group on S^*M shows that the distributions b_k are independent of x and ξ, that is, are a multiple of integration over the cosphere bundle with repect to the volume form $\iota(\mathcal{R})\omega^n$. Next we consider the group of dilations of $\mathbb{R}^n$, which sends $a(x,\xi)$ to $a(\lambda x, \lambda^{-1}\xi)$. Since we know that $R(a(x,\xi)) = R(a(\lambda x, \lambda^{-1}\xi))$, it follows that $b_k = \lambda^{n+k} b_k$, so that $b_k = 0$ unless $k = -n$. This completes the calculation of the formula for R_M up to a constant; in fact, it can be shown that $c = 2\pi$.

(2) For the proof of this formula, see the articles of Guillemin and Wodzicki. □

Let us briefly mention the relationship between our description of the Hochschild homology of the algebra of the symbol algebra with Wodzicki's higher residue. Starting from the isomorphism of $\mathrm{HH}_*(\mathcal{P}_c)$ with $\mathrm{H}_c^{2n-*}(S^*M) \oplus \mathrm{H}_c^{2n-*-1}(S^*M)$, it follows by Poincaré duality that the Hochschild cohomology groups are given by the formula

$$\mathrm{HH}^*(\mathcal{P}_c) = \mathrm{H}^{*-1}(S^*M) \oplus \mathrm{H}^*(S^*M).$$

Hence, every m-form α on S^*M determines a Hochschild m-cochain on $\mathcal{P}_c$, that is, a $m+1$-multilinear functional on $\mathcal{P}_c$, whose image in the E^1 term of the spectral sequence is given by the current on $\dot{T}^*M$ given by

$$\beta \mapsto \begin{cases} \int_{S^*M} \iota(\mathcal{R})(\star\beta) \wedge \alpha & \text{if } \beta \text{ is homogeneous of degree } m-n \\ 0 & \text{otherwise} \end{cases}$$

(where β is a homogeneous differential form on $\dot{T}^*M$). Note that

$$\iota(\mathcal{R})(*f_0\, df_1 \wedge \ldots \wedge df_m) = f_0 \varepsilon^*(df_1) \ldots \varepsilon^*(df_m)(d\xi').$$

The higher residue defined by Wodzicki [20] is given by the following formula: if $\beta = f_0\, df_1 \wedge \ldots \wedge df_m$ is a homogeneous differential form on $\dot{T}^*M$, then

$$\mathrm{Res}_m(\beta) = \begin{cases} f_0\, \varepsilon^*(df_1) \ldots \varepsilon^*(df_m)(d\xi') & \text{if } \beta \text{ has degree } m-n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we may write the Hochschild cochain corresponding to the differential form α as

$$(T\alpha)(a_0, a_1, \ldots, a_m) = \int_{S^*M} \mathrm{Res}_m(a_0\, da_1 \wedge \ldots \wedge da_m) \wedge \alpha.$$

§5. Poincaré lemma for differential and pseudo-differential operators

We present in this section another approach to the Hochschild and cyclic homology of algebras of (pseudo)differential operators, based on a Poincaré lemma for these theories.

Before explaining this, let us recall the essence of Weil's well-known sheaf-theoretic proof [17] that for a smooth manifold, the Čech cohomology groups $\mathrm{H}^i(M,\mathbb{C})$ are computed by the de Rham complex

$$\dots \xrightarrow{d} \Omega^i(M) \xrightarrow{d} \Omega^{i+1}(M) \xrightarrow{d} \dots$$

(1) The space of differential forms $\Omega^i(M)$ is the space of sections of a *fine* sheaf Ω^i on M; therefore, the de Rham cohomology of M is equal to the *hypercohomology* of the complex of sheaves

$$\dots \xrightarrow{d} \Omega^i \xrightarrow{d} \Omega^{i+1} \xrightarrow{d} \dots$$

(2) The classical Poincaré lemma tells us that this complex of sheaves is a resolution of the constant sheaf $\mathbb{C}_M$; therefore, its hypercohomology is equal to the Čech cohomology of M, i. e. the cohomology of the sheaf $\mathbb{C}_M$.

(The same proof shows that the cohomology of the complex

$$\dots \xrightarrow{d} \Omega^i_c(M) \xrightarrow{d} \Omega^{i+1}_c(M) \xrightarrow{d} \dots$$

of compactly supported differential forms on M is equal to $\mathrm{H}^i_c(M,\mathbb{C})$.)

Now consider the algebra $\mathcal{D}(M)$ of differential operators on M. As in [3], we denote by $\mathcal{D}_M$ the sheaf of germs of differential operators on M. The Hochschild cohomology of this sheaf may be calculated as the hypercohomology of the complex of sheaves $\mathcal{C}^*$, where $\mathcal{C}^n$ is the sheaf associated to the presheaf $U \mapsto C_{-n}(\mathcal{D}(U))$. (Here, $C_*(\mathcal{D}(U))$ is the complex of Hochschild chains of the algebra $\mathcal{D}(U)$.) This completes the analogue of the first part of Weil's proof.

The analogue of the Poincaré lemma for differential operators is the following result:

Lemma. *If U is an open set in M diffeomorphic to the ball, then*

$$\mathrm{HH}_q(\mathcal{D}(U)) = \begin{cases} \mathbb{C} & \text{if } q = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

(This lemma, with U equal to the polynomial coefficient differential operators on an affine space, is due to Feigin and Tsygan [6]*.)*

Proof. In the spectral sequence of Section 1, the term $E^2_q = \mathrm{H}^{2n-q}(T^*U)$ equals 0 unless $q = 2n$. Hence, the spectral sequence degenerates. □

With this Poincaré lemma in hand, we see that the complex $\mathcal{C}^*$ is quasi-isomorphic to $\mathbb{C}_M[2n]$. We immediately obtain the following result.

Proposition. *The Hochschild homology of the algebra of differential operators on the manifold M (respectively, differential operators of compact support) is given by*

$$\mathrm{HH}_*(\mathcal{D}(M)) \cong \mathrm{H}^{2n-*}(M,\mathbb{C}),$$
$$\mathrm{HH}_*(\mathcal{D}_c(M)) \cong \mathrm{H}_c^{2n-*}(M,\mathbb{C}). \quad \square$$

This proposition gives another proof of the degeneracy of the spectral sequence of Section 1 in the case of differential operators, directly from the "Poincaré lemma" and a bit of sheaf theory.

We now turn to the calculation of the cyclic homology groups $\mathrm{HC}_*(\mathcal{D}(M))$. They are given by the hypercohomology of the complex of sheaves $\mathcal{C}^{\lambda,*}$, where $\mathcal{C}^{\lambda,n}$ is the sheaf associated to the presheaf $U \mapsto C^\lambda_{-n}(\mathcal{D}(U))$ (here, $C^\lambda_\cdot$ is the cyclic chain complex of Connes, [4, 15]). The Poincaré lemma for the complex $\mathcal{C}^{\lambda,*}$ is more complicated, since the cohomology sheaves of $\mathcal{C}^{\lambda,*}$ are:

$$\begin{cases} \mathbb{C}_M & \text{in degrees } -2n,\ -2n-2, \text{ etc.} \\ 0 & \text{in all other degrees.} \end{cases}$$

Lemma. *The complex $\mathcal{C}^{\lambda,*}$ is quasi-isomorphic to $\bigoplus_{j\geq 0} \mathbb{C}_M[2n+2j]$.*

Proof. We first show that $\mathbb{C}_M[2n]$ is, up to a quasi-isomorphism, a direct factor of the complex of sheaves $\mathcal{C}^{\lambda,*}$. On the one hand, there is a natural morphism $\mathcal{C}^{\lambda,*} \to \mathbb{C}_M[2n]$ since $-2n$ is the largest degree in which the complex $\mathcal{C}^{\lambda,*}$ has a non-zero cohomology sheaf. On the other hand, if $\mathcal{C}^*$ is the sheaf corresponding to the Hochschild complex, we have the natural injection of complexes $I : \mathcal{C}^* \to \mathcal{C}^{\lambda,*}$. The Poincaré lemma for $\mathcal{C}^*$ shows that $\mathcal{C}^*$ is quasi-isomorphic to $\mathbb{C}_M[2n]$; thus, I is a left inverse to the map $\mathbb{C}_M[2n] \to \mathcal{C}^{\lambda,*}$.

From the short exact sequence of complexes of sheaves [15]

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^* \arrow[r, "I"] \arrow[d, equal] & \mathcal{C}^{\lambda,*} \arrow[r, "S"] & \mathcal{C}^{\lambda,*}[2] \arrow[r] & 0 \\ & \mathbb{C}[2n] & & & \end{tikzcd}$$

we deduce that $\mathcal{C}^{\lambda,*} \cong \mathbb{C}_M[2n] \oplus \mathcal{C}^{\lambda,*}[2]$. Now, $\mathcal{C}^{\lambda,*}[2n]$ contains $\mathbb{C}_M[2n+2]$ as a direct factor; the lemma follows by induction. $\square$

From this Poincaré lemma, we immediately deduce:

Proposition. *The cyclic homology of the algebra of differential operators on the manifold M (respectively, differential operators of compact support) is given by*

$$\mathrm{HC}_*(\mathcal{D}(M)) = \mathrm{H}^{2n-*}(M,\mathbb{C})[u],$$
$$\mathrm{HC}_*(\mathcal{D}_c(M)) = \mathrm{H}_c^{2n-*}(M,\mathbb{C})[u].$$

Proof. We will illustrate the case of $\mathcal{D}(M)$:

$$\begin{aligned}\mathrm{HC}_*(\mathcal{D}(M)) = \mathbb{H}^{-*}(M, \mathcal{C}^{\lambda,*}) &\cong \bigoplus_{j\geq 0} \mathrm{H}^{-*}(M, \mathbb{C}_M[2n+2j]) \\ &\cong \bigoplus_{j\geq 0} \mathrm{H}^{2n+2j-*}(M, \mathbb{C}). \quad \square\end{aligned}$$

The same method gives the Hochschild and cyclic homology of the algebra of complex analytic differential operators on a Stein manifold, which we denote by $D^{\mathrm{an}}(M)$. If, in addition, M is an algebraic manifold over $\mathbb{C}$, let $D^{\mathrm{alg}}(M)$ denote the algebra of polynomial (i.e. regular) differential operators. The morphism of algebras $D^{\mathrm{alg}}(M) \to D^{\mathrm{an}}(M)$ is compatible with the filtrations, hence defines a map between the spectral sequences of Section 4 for these two algebras. At the E^2 term, we recover the natural map from the algebraic de Rham cohomology to the analytic de Rham cohomology, which Grothendieck has shown to be an isomorphism [7]. So, $D^{\mathrm{an}}(M)$ and $D^{\mathrm{alg}}(M)$ have the same Hochschild homology, and hence the same cyclic homology. Using the Lefschetz principle, we deduce the following result.

Proposition (Kassel and Mitschi [13]**).** *If M is a smooth affine algebraic variety, of pure dimension n over a field k of characteristic* 0*, then we have:*

$$\begin{aligned}\mathrm{HH}_*(D^{\mathrm{alg}}(M)) &= \mathrm{H}^{2n-*}_{DR}(M|k), \\ \mathrm{HC}_*(D^{\mathrm{alg}}(M)) &= \mathrm{H}^{2n-*}_{DR}(M|k)[u]. \quad \square\end{aligned}$$

Now we will apply the sheaf theoretic method to the Hochschild and cyclic homologies of the algebras of pseudo-differential symbols on M. Since M may be endowed with a real-analytic structure, we may replace the algebra of smooth symbols $\mathcal{P}$ considered in Section 4 by the sheaf of micro-differential operators $\mathcal{E}$ on S^*M, of Sato, Kashiwara and Kawai [16] (see also [12]). This is the algebra of sums $\sum_{i=-\infty}^{m} a_i(x,\xi)$ of analytic homogeneous functions satisfying certain growth conditions. A comparison of the spectral sequences associated to the standard filtrations, of the Hochschild homology of $\mathcal{P}$ and the Hochschild hyperhomology of $\mathcal{E}$, shows that these invariants of M are isomorphic, so we can concentrate on the case of $\mathcal{E}$.

Looking at the form of the main result of Section 4, we see that the would-be Poincaré lemma for pseudo-differential operators would have to occur on $S^*M \times S^1$, or the homotopy-equivalent space $S^*M \times \mathbb{C}^\times$. To geometrize this situation, we have to introduce a complexification $M \overset{i}{\hookrightarrow} M_{\mathbb{C}}$. This leads to the following commutative diagram.

$$\begin{tikzcd}
S^*M \times \mathbb{C}^\times \arrow[r] \arrow[d, "\pi"'] & \dot{T}^*M_{\mathbb{C}} \arrow[d, "\pi'"'] \\
S^*M \arrow[r, "i'"] \arrow[d] & \dot{T}^*M_{\mathbb{C}}/\mathbb{C}^\times \arrow[d] \\
M \arrow[r, "i"] & M_{\mathbb{C}}
\end{tikzcd}$$

(Here, i' is the composition of a closed immersion and an étale map of degree 2.) The point of this diagram is the geometrical fact that the $\mathbb{C}^\times$-principal bundle $\dot{T}^*M_{\mathbb{C}}$ over $\dot{T}^*M_{\mathbb{C}}/\mathbb{C}^\times$ trivializes canonically when pulled back to S^*M, once a Riemannian metric has been chosen.

Now consider the complex of sheaves on $\dot{T}^*M_{\mathbb{C}}$, the Hochschild complex $\mathcal{C}^*(\mathcal{E}_{\mathbb{R}})$ of the sheaf of algebras $\mathcal{E}_{\mathbb{R}}$, introduced by Sato, Kashiwara and Kawai [16]. We will not give a detailed discussion here of this sheaf, but will summarize the discussion of Kashiwara ([12, Chapter III, §2]), who gives a description of the stalks of $\mathcal{E}_{\mathbb{R}}$ in local coordinates.

We will describe the stalk of $\mathcal{E}_{\mathbb{R}}$ at the point $(0, dz_1)$ of $T^*\mathbb{C}^n$. Let D be a fixed polydisk at the origin of $\mathbb{C}^n$, and let V_j be the subsets of $\mathbb{C}^n \times \mathbb{C}^n$ consisting of pairs (z, w) satisfying

$$\begin{cases} a_1 \operatorname{Re}(w_1 - z_1) < \operatorname{Im}(w_1 - z_1), & \text{for } j = 1, \\ a_j |z_1 - w_1| < |z_j - w_j|, & \text{for } 2 \leq j \leq n, \end{cases}$$

where $(a_1, \dots, a_n)$ are chosen positive numbers. An element of the stalk at $(0, dz_1)$ is represented (for suitable D and a_j) by a holomorphic "kernel" $K(z, w)$ on the region $(D \times D) \cap_{j=1}^n V_j$, which is considered to equal 0 if, for some k, it extends to a holomorphic function on the set $(D \times D) \cap_{j \neq k} V_j$. Let us make the following comments on this definition:

(1) the stalk of $\mathcal{E}_{\mathbb{R}}$ has a description in terms of "local cohomology with support", similar to (but more complicated than) the description of the differential operators as cohomology with support;
(2) the multiplication on $\mathcal{E}_{\mathbb{R}}$ only exists modulo holomorphic kernels on $(D \times D) \cap_{j \neq k} V_j$.

We refer the reader to the book of Kashiwara for the definition of the product on the stalks of $\mathcal{E}_{\mathbb{R}}$. The outcome is that $\mathcal{E}_{\mathbb{R}}$ is a sheaf of algebras locally constant on the orbits of the $\mathbb{R}_+$-action on $\dot{T}^*M_{\mathbb{C}}$. Its most important property for us is the following Poincaré lemma.

Lemma. *If U is an open set in $\dot{T}^*M_{\mathbb{C}}$ diffeomorphic to the ball, then we have*

$$\mathrm{HH}_k(\mathcal{E}_{\mathbb{R}}(U)) = \begin{cases} \mathbb{C} & q = 2n \\ 0 & q \neq 2n \end{cases}$$

Proof. Since we are working on $\mathbb{C}^n$, we will identify algebras of microdifferential operators with their opposite algebras.

In the notation of [3, §2], the dual of the $\mathcal{E}(U \times U)$-module $\mathcal{E}(U)$ is isomorphic to $\mathcal{E}(U)[-2n]$ ([3, Lemma 2.2.1]). Since $\mathcal{E}_{\mathbb{R}}(U)$ is faithfully flat over $\mathcal{E}(U)$ ([16]), it follows that the dual of the $\mathcal{E}_{\mathbb{R}}(U \times U)$-module $\mathcal{E}_{\mathbb{R}}(U)$ is isomorphic to $\mathcal{E}_{\mathbb{R}}(U)[-2n]$.

We see from this that $\mathrm{HH}_*(\mathcal{E}_{\mathbb{R}}(U))$ is equal to $\mathrm{Ext}^{2n-*}_{\mathcal{E}_{\mathbb{R}}(U\times U)}(\mathcal{E}_{\mathbb{R}}(U), \mathcal{E}_{\mathbb{R}}(U))$ (see [3, §2]). But $\mathcal{E}(U)$ is a *holonomic* module over $\mathcal{E}(U \times U)$, with support on the diagonal, of multiplicity one. The proof of lemma is completed by application of Theorem 3.2.1 of [12]. $\square$

From this Poincaré lemma, we can compute the Hochschild homology of the ring $\mathcal{E}^\infty$ of microdifferential operators of order infinity, since $\mathcal{E}^\infty = \pi'_* \mathcal{E}_{\mathbb{R}}$. To finish our computation, we require yet one more

Lemma. *The inclusion $\mathcal{E} \subset \mathcal{E}^\infty$ induces an isomorphism on Hochschild homology (and hence, on cyclic homology as well, by Connes's spectral sequence linking these two theories).*

Proof. By expressing the Hochschild homology groups as Ext groups, as in the proof of the last lemma, we may reduce the proof to the following two statements.

(1) $\mathcal{E}_{\mathbb{C}^n}$ is a holonomic $\mathcal{E}_{\mathbb{C}^n\times\mathbb{C}^n}$-module with *regular singularities*—this follows from Theorem 5.4.1 of [16].

(2) If X is a complex manifold, $\mathcal{M}$ and $\mathcal{N}$ are two holonomic $\mathcal{E}_X$-modules with regular singularities, and $\mathcal{M}^\infty = \mathcal{E}^\infty \otimes_{\mathcal{E}} \mathcal{M}$, $\mathcal{N}^\infty = \mathcal{E}^\infty \otimes_{\mathcal{E}} \mathcal{N}$, then we have

$$\mathcal{E}xt^i_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}) \xrightarrow{\approx} \mathcal{E}xt^i_{\mathcal{E}_X^\infty}(\mathcal{M}^\infty, \mathcal{N}^\infty).$$

This is Theorem 6.1.3 of Kashiwara and Kawai's article [11], and is actually one of the deepest results of this article. $\square$

As a reward for all of this hard work, but without giving all the detailed sheaf-theory arguments required, we obtain:

Theorem. *If U is an open set in $(\dot{T}^*X)/\mathbb{C}^\times$, where X is a complex manifold, then the Hochschild and cyclic homology groups of $\mathcal{E}(U)$ (or, what amounts to the same thing, of $\mathcal{E}^\infty(U)$) are given by*

$$\mathrm{HH}_*(\mathcal{E}(U)) = \mathrm{H}^{2n-*}({\pi'}^{-1}(U))$$
$$\mathrm{HC}_*(\mathcal{E}(U)) = \mathrm{H}^{2n-*}({\pi'}^{-1}(U))[u]. \quad \square$$

Taking X to be the complexification $M_{\mathbb{C}}$ of a real analytic manifold M, we also obtain, by taking the limit over neighbourhoods of S^*M in $(\dot{T}^*M_{\mathbb{C}})/\mathbb{C}^\times$, the computation of the Hochschild and cyclic homologies of $\mathcal{P}$ that we did in Section 4 by a different method. The point of all this is simply that $\pi^{-1}S^*M$ is diffeomorphic to $S^*M \times \mathbb{C}^\times$.

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