Contemporary Mathematics
Volume **00**, 1997

Cyclic Homology and the Atiyah-Patodi-Singer Index Theorem

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Abstract. We apply the boundary pseudodifferential calculus of Melrose to study the Chern character in entire cyclic homology of the Dirac operator of a manifold with boundary.

Recently, Melrose has proved the Atiyah-Patodi-Singer index theorem using a calculus of pseudodifferential operators for manifolds with boundary [**11**]. In this article, we apply his method to study the Chern character of the Dirac operator of a manifold with boundary, working in the setting of entire cyclic cohomology.

Recall that if M is an n-dimensional spin-manifold with boundary, its Dirac operator D defines an element $[\mathsf{D}] \in K_n(M, \partial M)$ of the K-homology of the pair $(M, \partial M)$. (There is a more general construction using Clifford modules, but we prefer in this introduction to restrict ourselves to the simplest case.) It has been proved by Baum, Douglas and Taylor [**1**] and Melrose and Piazza [**12**] that in the long exact homology sequence

$$\to K_n(\partial M) \to K_n(M) \to K_n(M, \partial M) \xrightarrow{\partial} K_{n-1}(\partial M) \to ,$$

$[\mathsf{D}] \in K_n(M, \partial M)$ maps to the Dirac operator of the boundary $[\mathsf{D}_\partial] \in K_{n-1}(\partial M)$. The homology Chern character Ch_* maps this sequence into the long exact sequence

$$\to H_n(\partial M) \to H_n(M) \to H_n(M, \partial M) \xrightarrow{\partial} H_{n-1}(\partial M) \to \ .$$

In this article, we will prove an extension of the Atiyah-Patodi-Singer index theorem to entire cyclic cohomology, which realizes the formula $\partial \,\mathrm{Ch}_*[\mathsf{D}] = \mathrm{Ch}_*[\mathsf{D}_\partial]$ at the level of cyclic cochains instead of cohomology classes.

Let us describe Melrose's proof in outline. He introduces a calculus of pseudodifferential operators $\Psi_b^*(M)$, generated by the vector fields tangential to the boundary. Such vector fields are sections of a bundle ${}^bTM$ naturally associated to the manifold with boundary, called the b-tangent bundle. (This bundle is isomorphic

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1991 *Mathematics Subject Classification.* Primary 46L80; Secondary 35S15, 58G20.

Parts of this article were written at Oberwolfach, the Centre for Mathematical Analysis at the Australian National University, and Oxford University, and the author would like to acknowledge the assistance of the NSF, the SERC and the Sloan Foundation. This paper is in final form and no version of it will be submitted for publication elsewhere.

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to TM, although not naturally.) A metric on ${}^bTM$ is called a b-metric: these give M the structure of a manifold with an asymptotically cylindrical end. (In fact, we must suppose that the metric is exact, a technical condition which will be explained in Section 5.)

If M is a spin-manifold with a b-metric, and $\mathcal{S}$ is the associated bundle of spinors, its Dirac operator D lies in $\Psi_b^1(M,\mathcal{S})$ and its heat kernel $e^{t\mathsf{D}^2}$ lies in $\Psi_b^{-\infty}(M,\mathcal{S})$. However, owing to the presence of a boundary, operators in $\Psi_b^{-\infty}(M,\mathcal{S})$ are not automatically trace class. Melrose defines a renormalized trace ${}^b\mathrm{Tr}$ on $\Psi_b^{-\infty}(M,\mathcal{S})$, by subtracting a logarithmic divergence contributed by the boundary. The b-trace is not a trace — in the language of physics, it is anomalous.

Suppose that M is even-dimensional, and denote by

$$
{}^b\mathrm{Str}(K) = {}^b\mathrm{Tr}|_{\mathcal{S}^+}(K) - {}^b\mathrm{Tr}|_{\mathcal{S}^-}(K)
$$

the renormalized supertrace of $K \in \Psi_b^{-\infty}(M,\mathcal{S})$. If D_∂ is invertible, the operator D is Fredholm, and the limit

$$
\lim_{t\to\infty} {}^b\mathrm{Str}(e^{t\mathsf{D}^2}) = \mathrm{ind}(\mathsf{D}^+)
$$

equals the index of the operator $\mathsf{D}^+ : L^2(M,\mathcal{S}^+) \to L^2(M,\mathcal{S}^-)$. On the other hand, the local index theorem holds, in the form

$$
\lim_{t\to 0} {}^b\mathrm{Str}(e^{t\mathsf{D}^2}) = \int_M \hat{A}({}^bT^*M).
$$

The theorem is proved by interpolating between these two endpoints, just as in the proof of the local index theorem for Dirac operators on a closed manifold. If $K \in \Psi_b^{-\infty}(M,\mathcal{S})$, there is a simple formula for ${}^b\mathrm{Tr}[\mathsf{D},K]$ (see Section 5), from which we see that

$$
\frac{d}{dt}{}^b\mathrm{Str}(e^{t\mathsf{D}^2}) = \frac{1}{2}{}^b\mathrm{Str}[\mathsf{D},\mathsf{D}e^{t\mathsf{D}^2}] = \frac{1}{2i}\frac{1}{(\pi t)^{1/2}}{}^\partial\mathrm{Tr}\big(\mathsf{D}_\partial e^{t\mathsf{D}_\partial^2}\big).
$$

Since D_∂ is invertible, the integral

$$
\int_0^\infty {}^b\mathrm{Str}[\mathsf{D},e^{t\mathsf{D}^2}]\,dt = -\eta(\mathsf{D}_\partial)
$$

is minus the eta-invariant of D_∂. Combining these formulas, we obtain the Atiyah-Patodi-Singer index theorem:

$$
\mathrm{ind}(\mathsf{D}^+) = \int_M \hat{A}({}^bT^*M) - \frac{1}{2}\eta(\mathsf{D}_\partial).
$$

In this article, we will follow the steps outlined above to obtain a formula at the level of cyclic cochains, replacing the function ${}^b\mathrm{Str}(e^{t\mathsf{D}^2})$ by a cyclic cochain analogous to the Chern character defined by Jaffe-Lesniewski-Osterwalder [**8**], except that we replace the supertrace by the b-supertrace. This cochain is the multi-linear form on $C^\infty(M)$ defined by the formula

$$
(f_0,\dots,f_k) \mapsto \int_{\Delta^k} {}^b\mathrm{Str}\big(f_0 e^{\sigma_0\mathsf{D}_t^2}[\mathsf{D}_t,f_1]e^{\sigma_1\mathsf{D}_t^2}\dots[\mathsf{D}_t,f_k]e^{\sigma_k\mathsf{D}_t^2}\big)\,d\sigma,
$$

where $\mathsf{D}_t = t^{1/2}\mathsf{D}$ and Δ^k is the simplex

$$
\{(\sigma_0,\dots,\sigma_k)\in[0,1]^k \mid \sigma_0+\dots+\sigma_k = 1\},
$$

with Lebesgue measure $d\sigma$. However, unlike in the case of the JLO Chern character, the cochain which we obtain is not closed, since the b-supertrace is not a trace. By calculating its coboundary, we will obtain an index theorem which extends the Atiyah-Patodi-Singer theorem to the setting of cyclic cohomology. In our theorem, the role of the eta-invariant is played by a cyclic cochain which has been studied independently by Wu [**16**]. Wu calls it the higher eta-invariant: however, motivated by Theorem 7.1 and at the suggestion of J. Kaminker, we prefer to call it the total eta-invariant.

We have taken the opportunity to impose a series of conventions for Dirac operators and Clifford algebras, which have the effect of supressing extraneous constant factors in the formulas. Dirac operators are skew-symmetric, in order that the commutator $a \mapsto [\mathsf{D}, a]$ maps self-adjoint operators to self-adjoint operators. Consistent with this choice, we define the Clifford algebra by the relation

$$vw + wv = 2(v, w).$$

In the Appendix, we define the Clifford supertrace $\operatorname{Str}_{C(q)}(A)$ on Clifford modules, which allows us to discuss in a uniform way the even and odd dimensional cases; this is an extension of a trick of Quillen [**13**] for modules over the Clifford algebra $C(1)$.

I would like to thank Jerry Kaminker, Richard Melrose and Paolo Piazza for their helpful comments on this article.

1. Cyclic cochains

Cyclic cochains were introduced by Connes as a generalization, to non-commutative algebras, of the complex of currents on a manifold [**5**]. (They were introduced independently by Tsygan in an algebraic setting.) Just as the Chern character of a vector bundle with connection is a differential form, Connes defines the Chern character of a Fredholm module, representing a K-homology class, as a cyclic chain. In this section, we will describe the complex of cyclic cochains, and in the next, we will define the Chern character in the setting in which we need it, that of theta-summable Fredholm modules (those with trace-class heat kernel).

Definition 1.1. *If $\mathcal{A}$ is a topological algebra, the space of cyclic k-cochains $\mathsf{C}^k(\mathcal{A})$ on $\mathcal{A}$ is the space of continuous multilinear forms on $\mathcal{A} \times (\mathcal{A}/\mathbb{C})^k$.*

Note that cyclic cochains, as we understand them here, are not invariant under the action of the cyclic group: in the Connes-Tsygan (b, B)-complex, the role of the cyclicity condition is played by the B-operator (see Loday-Quillen [**10**] for more on this point).

Cyclic cochains generalize currents on a manifold, in the following sense: if μ is a k-current on the compact manifold M, we may form a k-cochain c_μ on $C^\infty(M)$ by the formula

$$c_\mu(a_0, \dots, a_k) = \frac{1}{k!} \int_\mu a_0 \, da_1 \dots da_k.$$

Since $d(1) = 0$, we see that $c_\mu(a_0, \dots, a_k) = 0$ if a_i is a multiple of 1 for some $1 \le i \le k$.

Incidentally, the reversal of variance which links the homology of a space M to the (cyclic) cohomology of $C^\infty(M)$ comes about because the functor from manifolds M to algebras $C^\infty(M)$ is contravariant — functions pull back.

Let $b : \mathsf{C}^k(\mathcal{A}) \to \mathsf{C}^{k+1}(\mathcal{A})$ be the operator

$$\begin{aligned}(bc)&(a_0, \dots, a_{k+1}) \\ &= \sum_{i=0}^{k} (-1)^{i+1} c(a_0, \dots, a_i a_{i+1}, \dots, a_{k+1}) + (-1)^k c(a_{k+1}a_0, \dots, a_k).\end{aligned}$$

Note that b satisfies the formula $b^2 = 0$. The following lemma gives one motivation for its introduction.

Lemma 1.2. *If μ is a current on a compact manifold M, then $bc_\mu = 0$.*

Proof. Leibniz's formula $d(a_1a_2) = (da_1)a_2 + a_1(da_2)$ shows that

$$(bc_\mu)(a_0, \dots, a_{k+1}) = \frac{(-1)^k}{k!} \int_\mu [a_{k+1}, a_0\, da_1 \dots da_k].$$

But this vanishes by commutativity of $C^\infty(M)$. □

We introduce another operator $B : \mathsf{C}^{k+1}(\mathcal{A}) \to \mathsf{C}^k(\mathcal{A})$, by the formula

$$(Bc)(a_0, \dots, a_k) = \sum_{i=0}^{k} (-1)^{ik} c(1, a_i, \dots, a_k, a_0, \dots, a_{i-1}).$$

It may be shown that B^2 and $bB+Bb$ vanish. The following lemma, due to Rinehart [**15**] and Connes [**5**], gives one motivation behind its introduction: it corresponds to the boundary operator on currents.

Lemma 1.3. *If μ is a k-current, then $Bc_\mu = c_{\delta\mu}$.*

Proof.

$$\begin{aligned}(Bc_\mu)(a_0, \dots, a_{k-1}) &= \sum_{i=0}^{k-1} (-1)^{i(k-1)} c_\mu(1, a_i, \dots, a_{k-1}, a_0, \dots, a_{i-1}) \\ &= \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^{i(k-1)} \int_\mu da_i \dots da_{k-1}\, da_0 \dots da_{i-1} \\ &= \frac{1}{(k-1)!} \int_\mu da_0 \dots da_{k-1} = \frac{1}{(k-1)!} \int_{\delta\mu} a_0\, da_1 \dots a_k \square\end{aligned}$$

The cohomology of the differential $b + B$ on the vector space $\bigoplus_{k=0}^\infty \mathsf{C}^k(\mathcal{A})$ is naturally $\mathbb{Z}/2$-graded, and not $\mathbb{Z}$-graded, because b and B have degree $+1$ and -1 respectively. This cohomology is called the the periodic cyclic cohomology of $\mathcal{A}$, written

$$\mathrm{HP}^*(\mathcal{A}) = \mathrm{HP}^+(\mathcal{A}) \oplus \mathrm{HP}^-(\mathcal{A}).$$

In this and a number of other ways, periodic cyclic cohomology is closer to K-homology than to ordinary homology (this may be seen more clearly in the study of the periodic cyclic homology of group actions). In any case, we have the following result, due to Connes [**5**].

PROPOSITION 1.4. *The map $\mu \mapsto c_\mu$ from the $\mathbb{Z}/2$-graded complex of currents on M to the complex of periodic cyclic cochains induces an isomorphism between the $\mathbb{Z}/2$-graded space* $\mathrm{HP}^*(C^\infty(M))$ *and*

$$\sum_{i \text{ even}} H_i(M) \oplus \sum_{i \text{ odd}} H_i(M).$$

The cyclic cochains defining periodic cyclic homology are finite sums $\sum_{k=0}^{N} c_k$, where $c_k \in \mathsf{C}^k(\mathcal{A})$. This is appropriate in the algebraic setting, but when $\mathcal{A}$ is a Fréchet algebra, it is useful to consider infinite sums $c = \sum_{k=0}^{\infty} c_k$ with bounds on the size of c_k as $k \to \infty$. The most useful such class of cyclic cochains is that of Connes's entire cyclic cochains [**6**]. If $\mathcal{A}$ is a Fréchet algebra, and $\|\cdot\|$ is a continuous seminorm on $\mathcal{A}$, we introduce a sequence of seminorms on the space of such sums,

$$\|c_0 + c_1 + \dots\|_n = \sum_k \Gamma(k/2)\, n^k \|c_k\|.$$

The space of entire cyclic cochains $\mathsf{C}^*_\omega(\mathcal{A})$ is the Fréchet space of sequences of cyclic cochains on which all of these seminorms are finite. The cohomology of $b + B$ on $\mathsf{C}^*_\omega(\mathcal{A})$ is called the entire cyclic cohomology of $\mathcal{A}$, written $\mathrm{HE}^*(\mathcal{A})$. Just as in the case of periodic cyclic cohomology, it is $\mathbb{Z}/2$-graded:

$$\mathrm{HE}^*(\mathcal{A}) = \mathrm{HE}^+(\mathcal{A}) \oplus \mathrm{HE}^-(\mathcal{A}).$$

It is known that the periodic cyclic cohomology $\mathrm{HP}^*(C^\infty(M))$ of the algebra $C^\infty(M)$ is a summand of the entire cyclic homology $\mathrm{HE}^*(C^\infty(M))$, and it is expected that they are equal, although this has not been proved except when M is one-dimensional.

2. Theta-summable Fredholm modules

Entire cyclic cocycles typically arise as the Chern character of theta-summable Fredholm modules. In this section, we recall some aspects of this theory (see also [**6**], [**7**], [**8**] and [**14**]). We use the theory of Hilbert modules over Clifford algebras, for which we refer to the appendix.

DEFINITION 2.1. *A degree q theta-summable Fredholm module $(\mathcal{H}, \mathsf{D})$ over a Fréchet algebra $\mathcal{A}$ consists of a continuous representation $\rho : \mathcal{A} \to \mathcal{L}_{C(q)}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert module over $C(q)$, and an odd skew-adjoint operator D on $\mathcal{H}$, such that*

1. *if $a \in \mathcal{A}$, the supercommutator $[\mathsf{D}, \rho(a)]$ is densely defined and extends to a bounded operator on $\mathcal{H}$, and the map $a \mapsto [\mathsf{D}, \rho(a)]$ is bounded from $\mathcal{A}$ to $\mathcal{L}_{C(q)}(\mathcal{H})$.*
2. *for each $t > 0$, the operators $e^{t\mathsf{D}^2}$ and $\mathsf{D}e^{t\mathsf{D}^2}$ are in $\mathcal{L}^1_{C(q)}(\mathcal{H})$.*

We will now define the Chern character of a Fredholm module $(\mathcal{H}, \mathsf{D})$: this is an entire cyclic cocycle, to be thought of as a generalized current. We will often write a instead of $\rho(a)$.

We define multilinear forms on the space of operators on $\mathcal{H}$ by integrating over the simplex Δ^k: if A_i, $0 \le i \le k$, are operators on $\mathcal{H}$,

$$\langle A_0, \dots, A_k \rangle = \int_{\Delta^k} \mathrm{Str}_{C(q)}(A_0 e^{\sigma_0 \mathsf{D}^2} \dots A_k e^{\sigma_k \mathsf{D}^2})\, d\sigma.$$

(For the definition of the Clifford supertrace $\mathrm{Str}_{C(q)}(A)$, see the Appendix.)

LEMMA 2.2. *If $I \subset \{1, \dots, k\}$, we have the estimate*

$$|\langle A_0, \dots, A_k \rangle| \le \frac{O(1)^k}{\Gamma(k - |I|/2 + 1)} \cdot \mathrm{Tr}\big(e^{\mathsf{D}^2/2}\big) \cdot \prod_{i \notin I} \|A_i\| \cdot \prod_{i \in I} \|A_i(1 - \mathsf{D}^2)^{-1/2}\|$$

PROOF. We use the Hölder inequality for the trace on a Hilbert space: if $\|A\|_p$ denotes the p-Schatten norm of an operator A, and $\sigma_0 + \ldots + \sigma_k = 1$, we have

$$|\,\mathrm{Str}_{C(q)}(B_0 \dots B_k)| \le O(1) \|B_0\|_{\sigma_0^{-1}} \dots \|B_k\|_{\sigma_k^{-1}}.$$

This shows that

$$|\langle A_0, \dots, A_k \rangle| \le O(1) \int_{\Delta^k} \|A_0 e^{\sigma_0 \mathsf{D}^2}\|_{\sigma_0^{-1}} \dots \|A_k e^{\sigma_k \mathsf{D}^2}\|_{\sigma_k^{-1}} \, d\sigma.$$

We now observe that for positive s,

$$\begin{aligned} \|A e^{\sigma \mathsf{D}^2}\|_{\sigma^{-1}} &\le \|A(1 - \mathsf{D}^2)^{-s/2}\| \cdot \|(1 - \mathsf{D}^2)^{s/2} e^{\sigma \mathsf{D}^2/2}\| \cdot \|e^{\sigma \mathsf{D}^2/2}\|_{\sigma^{-1}} \\ &\le O(1) \left(\frac{s}{\sigma}\right)^{s/2} \cdot \|A(1 - \mathsf{D}^2)^{-s/2}\| \cdot \mathrm{Tr}\big(e^{\mathsf{D}^2/2}\big)^{\sigma}. \end{aligned}$$

Apply this with $s = 1$ for $i \in I$, and $s = 0$ for $i \notin I$. The proof is completed by noting that

$$\int_{\Delta^k} \prod_{i \in I} \sigma_i^{-1/2} \, d\sigma = \frac{\pi^{|I|/2}}{\Gamma(k - |I|/2 + 1)}. \square$$

The Jaffe-Lesniewski-Osterwalder Chern character [**8**] of a theta-summable Fredholm module $(\mathcal{H}, \mathsf{D})$, is the entire cyclic cochain $\mathrm{Ch}^*(\mathsf{D}) \in \mathsf{C}^*_\omega(\mathcal{A})$ on $\mathcal{A}$ defined by the formula

$$\mathrm{Ch}^k(\mathsf{D})(a_0, \dots, a_k) = \langle a_0, [\mathsf{D}, a_1], \dots, [\mathsf{D}, a_k] \rangle.$$

This cochain is closed:

$$(b + B)\, \mathrm{Ch}^*(\mathsf{D}) = 0.$$

The component $\mathrm{Ch}^k(\mathsf{D})$ vanishes unless k and the degree q of $(\mathcal{H}, \mathsf{D})$ have the same parity: thus, the Chern character is in $\mathsf{C}^+_\omega(\mathcal{A})$ if q is even, and in $\mathsf{C}^-_\omega(\mathcal{A})$ if q is odd.

Note the close similarity between the definitions of the cochain c_μ associated to a current μ and of $\mathrm{Ch}^*(\mathsf{D})$: instead of da_i, we have $[\mathsf{D}, a_i]$, while the integral is replaced by the supertrace over $\mathcal{H}$, and the factor $1/k!$ equals the volume of the simplex Δ^k used in the definition of $\langle A_0, \dots, A_k \rangle$.

The basic example of a theta-summable Fredholm module of degree 0 is the Dirac operator on a compact even-dimensional spin-manifold M, with $\mathcal{A} = C^\infty(M)$ the algebra of differentiable functions. The Hilbert spaces $\mathcal{H}^\pm$ are the spaces of square-summable sections of the half-spinor bundles $\mathcal{S}^\pm$, and the operator D is the Dirac operator. The commutator $[\mathsf{D}, a]$ is just the operator $c(da)$ of Clifford multiplication by the one-form da. It is proved in Block-Fox [**3**] that the cyclic cocycle $\mathrm{Ch}^*(t^{1/2}\mathsf{D})$ converges as $t \to 0$ to the current Poincaré dual to the $\hat{A}$-genus of the manifold M:

$$\lim_{t \to 0} \mathrm{Ch}^k(t^{1/2}\mathsf{D})(a_0, \dots, a_k)) = (2\pi i)^{-n/2} \int_M a_0 \, da_1 \dots da_k \wedge \det{}^{1/2} \left(\frac{R/2}{\sinh R/2} \right).$$

We will use a version of the Chern character parametrized by an open subset U of $\mathbb{R}^n$ — it is a closed element of $\Omega^*(U, \mathsf{C}^*_\omega(\mathcal{A}))$, the differential forms on U with values in the entire cyclic cochains on $\mathcal{A}$. This is a $\mathbb{Z}/2$-graded complex with differential $d + b + B$; the operators d, b and B have bidegrees $(1, 0)$, $(0, 1)$ and $(0, -1)$ respectively.

Definition 2.3. *A degree q family of theta-summable Fredholm modules*

$$(\mathcal{H}, \mathsf{D} = (\mathsf{D}_u)_{u\in U})$$

over a Fréchet algebra $\mathcal{A}$ consists of a continuous representation $\rho : \mathcal{A} \longrightarrow \mathcal{L}_{C(q)}(\mathcal{H})$ on a Hilbert module $\mathcal{H}$ over $C(q)$, and a family of odd skew-adjoint operators $\mathsf{D}_u : \mathcal{H} \longrightarrow \mathcal{H}$, parametrized by $u \in U$, such that

1. *if ∂ is a differential operator on U, the map $u \mapsto \partial\mathsf{D}_u(1 - \mathsf{D}_u^2)^{-1/2}$ is a continuous map from U to $\mathcal{L}_{C(q)}(\mathcal{H})$;*
2. *if $a \in \mathcal{A}$, the supercommutator $[\mathsf{D}_u, \rho(a)]$ is densely defined and extends to a family of bounded operators on $\mathcal{H}$, and the map $a \mapsto [\mathsf{D}_u, \rho(a)]$ is bounded from $\mathcal{A}$ to $C^\infty(U, \mathcal{L}_{C(q)}(\mathcal{H}))$;*
3. *for each $t > 0$, and for u in a compact subset $K \subset U$, the operators $e^{t\mathsf{D}_u^2}$ lie in a bounded subset of $\mathcal{L}^1_{C(q)}(\mathcal{H})$.*

If D is a family $(\mathsf{D}_u)_{u\in U}$ of Fredholm modules parametrized by U and A_i, $0 \le i \le k$, are operators, $\langle A_0, \dots, A_k\rangle \in C^\infty(U)$ is the degree 0 component of the differential form $\langle\!\langle A_0, \dots, A_k\rangle\!\rangle$, which is defined by the formula

$$\langle\!\langle A_0, \dots, A_k\rangle\!\rangle = \int_{\Delta^k} \operatorname{Str}_{C(q)}(A_0 e^{\sigma_0(d\mathsf{D}+\mathsf{D}^2)} \dots A_k e^{\sigma_k(d\mathsf{D}+\mathsf{D}^2)})\, d\sigma.$$

Here, $e^{t(d\mathsf{D}+\mathsf{D}^2)}$ is given by the formula

$$e^{t(d\mathsf{D}+\mathsf{D}^2)} = \sum_{k=0}^{\infty} t^k \int_{\Delta^k} e^{\sigma_0 t\mathsf{D}^2} d\mathsf{D} e^{\sigma_1 t\mathsf{D}^2} \dots d\mathsf{D} e^{\sigma_k t\mathsf{D}^2}\, d\sigma,$$

where $d\mathsf{D}$ is the one-form $d\mathsf{D} = \sum_{i=1}^n du^i \partial_i \mathsf{D}$. The following easy lemma allows us to extend the estimates of Lemma 2.2 to the multilinear forms $\langle\!\langle A_0, \dots, A_k\rangle\!\rangle$.

Lemma 2.5. *The multilinear forms $\langle\!\langle A_0, \dots, A_k\rangle\!\rangle$ are given by the formula*

$$\langle\!\langle A_0, \dots, A_k\rangle\!\rangle = \sum_{1\le\alpha_1,\dots,\alpha_m\le n}\ \sum_{0\le i_1\le\dots\le i_m\le k} \langle A_0, \dots, A_{i_1}, du^{\alpha_1}\partial_{\alpha_1}\mathsf{D}, A_{i_1+1}, \dots, A_{i_m}, du^{\alpha_m}\partial_{\alpha_m}\mathsf{D}, A_{i_m+1}, \dots, A_k\rangle.$$

In particular, if $U \subset \mathbb{R}$, so that D_u depends on a single parameter u,

$$\langle\!\langle A_0, \dots, A_k\rangle\!\rangle = \langle A_0, \dots, A_k\rangle + \sum_{i=0}^{k} \langle A_0, \dots, A_i, d\mathsf{D}_u, A_{i+1}, \dots, A_k\rangle.$$

If $\mathsf{D} = (\mathsf{D}_u)_{u\in U}$ is a family of Fredholm modules, its Chern character is the differential form on U with values in entire cyclic cochains on $\mathcal{A}$, defined by the formula

$$\operatorname{Ch}^k(\mathsf{D})(a_0, \dots, a_k) = \langle\!\langle a_0, [\mathsf{D}, a_1], \dots, [\mathsf{D}, a_k]\rangle\!\rangle.$$

The following homotopy formula, similar to the Chern-Simons transgression formula of Chern-Weil theory, may be proved

$$(d+b+B)\operatorname{Ch}^*(\mathsf{D})=0.$$

3. The index theorem for theta-summable Fredholm modules

In this section, we recall the abstract index theorem for a theta-summable Fredholm module $(\mathcal{H},\mathsf{D})$ of degree 0 (see [**6**] and [**7**]). Consider an idempotent $p=(p_{ij})\in M_r(\mathcal{A})$, which represents the class $[p]\in K_0(\mathcal{A})$. Associated to such an idempotent is its Chern character, the even entire cyclic chain defined by the formula

$$\operatorname{Ch}_*(p)=\sum_i p_{ii}+\sum_{\ell=1}^{\infty}(-1)^\ell\frac{(2\ell)!}{\ell!}\sum_{i_0\dots i_{2\ell}}\bigl(p_{i_0i_1}-\tfrac12\delta_{i_0i_1},p_{i_1i_2},\dots,p_{i_{2\ell}i_0}\bigr).$$

This Chern character is closed, $(b+B)\operatorname{Ch}_*(p)=0$ (see Section 1 of Getzler-Szenes [**7**]).

Given an idempotent p and a Fredholm module $(\mathcal{H},\mathsf{D})$ of degree 0 over $\mathcal{A}$, we define a Hilbert space $\mathcal{H}_p$, equal to the image of the idempotent $\rho(p)$ acting on $\mathcal{H}\otimes\mathbb{C}^r$, and a Fredholm operator

$$\mathsf{D}_p=p\mathsf{D}p:\mathcal{H}_p\to\mathcal{H}_p,$$

(where we write p instead of $\rho(p)$). The index pairing between the Fredholm module D and the idempotent p is defined to be the index of the operator $\mathsf{D}_p^+:\mathcal{H}_p^+\to\mathcal{H}_p^-$. Its value only depends on the equivalence class of p in $K_0(\mathcal{A})$, and we denote it by $\langle\mathsf{D},p\rangle$. The following abstract index theorem gives a formula for this index, in terms of the Chern character $\operatorname{Ch}^*(\mathsf{D})$ of the theta-summable Fredholm module $(\mathcal{H},\mathsf{D})$.

Theorem 3.1. *If p is an idempotent in $M_r(\mathcal{A})$, the index pairing with a Fredholm module $(\mathcal{H},\mathsf{D})$ of degree 0 is given by the formula*

$$\langle\mathsf{D},p\rangle=(\operatorname{Ch}^*(\mathsf{D}),\operatorname{Ch}_*(p)).$$

Proof. Following Connes [**6**], we consider the family of Fredholm modules parametrized by $u\in\mathbb{R}$,

$$\begin{aligned}\mathsf{D}_u&=(1-u)\mathsf{D}+u(p\mathsf{D}p+(1-p)\mathsf{D}(1-p))\\&=\mathsf{D}+u(2p-1)[\mathsf{D},p].\end{aligned}$$

This family is introduced because it interpolates between D and the operator $p\mathsf{D}p+(1-p)\mathsf{D}(1-p)$, which commutes with p.

Now, observe that both $\langle\mathsf{D}_u,p\rangle$ and $(\operatorname{Ch}^*(\mathsf{D}_u),\operatorname{Ch}_*(p))$ are independent of u. For the index pairing $\langle\mathsf{D}_u,p\rangle$, this follows from homotopy invariance of the index, while for $(\operatorname{Ch}^*(\mathsf{D}_u),\operatorname{Ch}_*(p))$, it follows from the fact that both $\operatorname{Ch}^*(\mathsf{D})$ and $\operatorname{Ch}_*(p)$ are closed:

$$\begin{aligned}&d\bigl(\operatorname{Ch}^*(\mathsf{D}_u),\operatorname{Ch}_*(p)\bigr)\\&\qquad=\bigl((d+b+B)\operatorname{Ch}^*(\mathsf{D}_u),\operatorname{Ch}_*(p)\bigr)-\bigl(\operatorname{Ch}^*(\mathsf{D}_u),(b+B)\operatorname{Ch}_*(p)\bigr)=0\end{aligned}$$

Thus, it suffices to prove the theorem under the assumption that D commutes with p. But then

$$\begin{aligned}(\mathrm{Ch}^*(\mathsf{D}), \mathrm{Ch}_*(p)) &= \mathrm{Str}_{\mathcal{H}}\big(p\, e^{(p\mathsf{D}p+(1-p)\mathsf{D}(1-p))^2}\big) \\ &= \mathrm{Str}_{\mathcal{H}_p}(e^{(p\mathsf{D}p)^2}).\end{aligned}$$

This equals the index of the operator $p\mathsf{D}^+p$, by the McKean-Singer formula. □

It is interesting to see what the above index theorem says for the Fredholm module associated to the Dirac operator on an even-dimensional spin-manifold. An idempotent $p \in M_r(C^\infty(M))$ determines a vector bundle $\mathrm{im}(p)$ over M, with connection

$$pdp + (1-p)d(1-p) = d + (2p-1)dp,$$

and curvature

$$F = (d + (2p-1)dp)^2 = (dp)^2.$$

If μ is a 2ℓ-current on M, we see that

$$\begin{aligned}(c_\mu, \mathrm{Ch}_*(p)) &= \frac{(-1)^\ell}{(2\ell)!}\frac{(2\ell)!}{\ell!}\int_\mu \mathrm{Tr}\left(p(dp)^{2\ell}\right) - \tfrac{1}{2}\mathrm{Tr}\left((dp)^{2\ell}\right) \\ &= \frac{(-1)^\ell}{\ell!}\int_\mu \mathrm{Tr}\left(p(dp)^{2\ell}\right) = \int_\mu \mathrm{Tr}\left(pe^{-F}\right),\end{aligned}$$

and we recover the usual Chern character associated to the bundle $\mathrm{im}(p)$. The index pairing $\langle \mathsf{D}, p\rangle$ is invariant under replacement of D by $t^{1/2}\mathsf{D}$; sending $t \longrightarrow 0$, we see that

$$(\mathrm{Ch}^*(\mathsf{D}), \mathrm{Ch}_*(p)) = (2\pi i)^{-n/2}\int_M \mathrm{Tr}\left(pe^{-F}\right) \det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right),$$

which is the Atiyah-Singer index theorem.

There is another proof of Theorem 3.1, which is more difficult, but avoids mention of the operators b and B of cyclic cohomology: we give it because an extension of the same technique will be used in the study of the total eta-invariant in Section 7.

We introduce an auxilliary parameter $s \in [0,1]$ and an auxilliary Clifford variable σ, and consider the family of Fredholm modules on $[0,1] \times \mathbb{R}$

$$\tilde{\mathsf{D}} = \mathsf{D}_u + is\sigma(p - \tfrac{1}{2}).$$

As the character of a superconnection, the differential form $\mathrm{Str}_{C(1)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big) \in \Omega^*([0,1]\times\mathbb{R})$ is closed; this may be seen either as a special case of the formula $(d+b+B)\,\mathrm{Ch}^*(\mathsf{D}) = 0$ combined with the fact that

$$\mathrm{Str}_{C(1)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big) = \big(Ch_*(\tilde{\mathsf{D}}), \mathrm{Ch}_*(1)\big),$$

or directly, by using the formula

$$d\,\mathrm{Str}_{C(1)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big) = \mathrm{Str}_{C(1)}\big[d+\tilde{\mathsf{D}}, e^{(d+\tilde{\mathsf{D}})^2}\big] = 0.$$

Let Γ_u be the contour in $[0,1]\times\mathbb{R}$ which goes from $s=-\infty$ to $s=\infty$ along the line of constant u. We see that the integral

$$a(u)=\int_{\Gamma_u}\operatorname{Str}_{C(1)}\bigl(e^{(d+\tilde{\mathsf{D}})^2}\bigr)$$

is independent of u. We will prove Theorem 3.1 by evaluating the integral at $u=0$ and at $u=1$.

First, we prove that

$$a(0)=\int_{\Gamma_0}\operatorname{Str}_{C(1)}\bigl(e^{(d+\tilde{\mathsf{D}})^2}\bigr)=i\bigl(\operatorname{Ch}^*(\mathsf{D}),\operatorname{Ch}_*(p)-\tfrac{1}{2}\operatorname{rk}(p)\operatorname{Ch}_*(1)\bigr).$$

Since $(d+\tilde{\mathsf{D}})^2$ is given by the formula

$$\mathsf{D}_u^2-s^2/4-i(1-u)s\sigma[\mathsf{D},p]+ids\,\sigma(p-1/2)+du\,(2p-1)[\mathsf{D},p],$$

we see that

$$\int_{\Gamma_0}\operatorname{Str}_{C(1)}\bigl(e^{(d+\tilde{\mathsf{D}})^2}\bigr)=\int_{\Gamma_0}\operatorname{Str}_{C(1)}\left(e^{\mathsf{D}^2-s^2/4-is\sigma[\mathsf{D},p]+ids\,\sigma(p-\frac{1}{2})}\right).$$

Expanding in powers of s, and only keeping terms with one factor of ds and an odd number of factors of σ, we see that this equals the sum

$$\sum_{\ell=0}^{\infty}\frac{i(-1)^\ell}{(4\pi)^{1/2}}\int_{-\infty}^{\infty}s^{2\ell}e^{-s^2/4}\,ds\times\sum_{i=0}^{2\ell}\langle 1,\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{i\text{ times}},p-\tfrac{1}{2},\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{2\ell-i\text{ times}}\rangle$$

LEMMA 3.2.
$$\frac{1}{(4\pi)^{1/2}}\int_{-\infty}^{\infty}s^{2\ell}e^{-s^2/4}\,ds=\frac{(2\ell)!}{\ell!}$$

PROOF. Completing the square, we see that

$$\int_{-\infty}^{\infty}e^{-s^2/4+as}\,ds=e^{a^2}\int_{-\infty}^{\infty}e^{-s^2/4}\,ds.$$

Expanding in a power series in a, we obtain the lemma. □

Using the identity

$$\langle A_0,\dots,A_k\rangle=\sum_{i=0}^{k}(-1)^{\eta_i(\eta_k-\eta_i)}\langle 1,A_i,\dots,A_k,A_0,\dots,A_{i-1}\rangle,$$

where $\eta_i=|A_0|+\dots+|A_i|$, we see that

$$\sum_{i=0}^{2\ell}\langle 1,\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{i\text{ times}},p-\tfrac{1}{2},\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{2\ell-i\text{ times}}\rangle=\langle p-\tfrac{1}{2},\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{2\ell\text{ times}}\rangle.$$

The formula for $a(0)$ follows by combining these formulas with Lemma 3.2. Note that the contribution $-\frac{1}{2}\operatorname{rk}(p)(\operatorname{Ch}^*(\mathsf{D}),\operatorname{Ch}_*(p))$ comes from the fact that the leading term of $\operatorname{Ch}_*(p)$ is $\operatorname{Tr}(p)$ and not $\operatorname{Tr}(p-\frac{1}{2})$).

We now evalute $a(1)$. From the explicit formula for $(d+\tilde{\mathsf{D}})^2$, it follows that

$$\int_{\Gamma_1} \mathrm{Str}_{C(1)}\left(e^{(d+\tilde{\mathsf{D}})^2}\right) = \frac{i}{(4\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/4}\, ds \times \mathrm{Tr}\big((p-\tfrac{1}{2})e^{(p\mathsf{D}p+(1-p)\mathsf{D}(1-p))^2}\big)$$
$$= i\big(\mathrm{Str}\big(p\cdot e^{(p\mathsf{D}p+(1-p)\mathsf{D}(1-p))^2}\big) - \tfrac{1}{2}\,\mathrm{rk}(p)\,\mathrm{Str}\big(e^{\mathsf{D}^2}\big)\big).$$

In this way, we obtain our second proof of Theorem 3.1: the unwanted term $\frac{1}{2}\,\mathrm{rk}(p)\,(\mathrm{Ch}^*(\mathsf{D}),\mathrm{Ch}_*(1))$ cancels from both sides.

4. The Melrose b-calculus

In this section, we briefly review Melrose's b-calculus of pseudodifferential operators for a manifold with boundary M.

The starting point of the calculus is the fact that the vector fields tangential to the boundary of M form a Lie algebra: these are the b-vector fields. The b-vector fields are the smooth sections of a vector bundle ${}^bTM$, which is isomorphic to the ordinary tangent bundle TM, although not naturally.

The dual of the b-tangent bundle is the b-cotangent bundle ${}^bT^*M$. We also have the b-differential forms

$$\Omega_b^k(M,\mathcal{E}) = \Gamma(M,\mathcal{E}\otimes\Lambda^k({}^bT^*M)).$$

Reflecting the fact that the b-vector fields form a Lie algebra, there is an exterior differential ${}^bd$ on $\Omega_b^*(M)$.

DEFINITION 4.1. *A b-connection on a bundle $\mathcal{E}$ is an operator*

$${}^b\nabla : \Omega_b^*(M,\mathcal{E}) \to \Omega_b^{*+1}(M,\mathcal{E}),$$

such that if $\alpha\in\Omega_b^k(M)$ and $\omega\in\Omega_b^\ell(M,\mathcal{E})$, then

$${}^b\nabla(\alpha\wedge\omega) = {}^bd\alpha\wedge\omega + (-1)^k\alpha\wedge{}^b\nabla\omega.$$

The curvature ${}^b\nabla^2$ of a b-connection is an element of $\Omega_b^2(M,\mathrm{End}(\mathcal{E}))$.

Let x be a defining function for the boundary, that is, x vanishes on the boundary and dx is positive when evaluated on an inward pointing normal vector to the boundary. The restriction of the b-vector field $x\partial_x$ to the boundary is independent of the defining function x. When restricted to the boundary, the b-tangent bundle fits into a short-exact sequence:

$$0\to \mathrm{span}(x\partial_x) \to {}^bTM|_{\partial M} \to T(\partial M) \to 0.$$

We also have the dual short-exact sequence

$$0\to T^*(\partial M) \to {}^bT^*M|_{\partial M} \to M\times\mathbb{R} \to 0.$$

These sequences may be split by the choice of a defining function x for the boundary. This determines a b-cotangent vector $x^{-1}dx \in \Omega_b^1(M)$, whose restriction to the boundary ν gives the desired splitting. This splitting only depends on the one-jet of x at the boundary, and we call it a conormal structure on the b-manifold M.

LEMMA 4.2. *The space of conormal structures is an affine space modelled on the space of exact one-forms on the boundary.*

Proof. Given two defining functions x and $x' = e^\varphi x$ for the boundary, we have

$$\frac{dx'}{x'} = \frac{d(e^\varphi x)}{e^\varphi x} = \frac{dx}{x} + d\varphi. \square$$

Given a conormal structure, the restriction of a b-differential form ω to the boundary of M has a unique decomposition $\omega = \omega_0 + \omega_1\nu$, with $\omega_i \in \Omega^*(\partial M)$. The residue $\operatorname{Res}_\nu(\omega)$ is defined by the formula

$$\operatorname{Res}_\nu(\omega_0 + \omega_1\nu) = \int_{\partial M} \omega_1.$$

Suppose that M is oriented. If ω is a b-differential form, we define its ν-integral by the formula

$$\int_\nu \omega = \lim_{\varepsilon\to 0}\Big(\int_{x\ge\varepsilon} \omega + \log\varepsilon \operatorname{Res}_\nu(\omega)\Big).$$

The ν-integral only depends on the conormal structure ν, and not on the defining function x. There is an analogue of the residue and the ν-integral for densities.

The b-differential operators are sums of operators of the form $D = fX_1 \dots X_k$, where X_i are b-vector fields. A b-vector field X defines a smooth function $\sigma(X)$ on ${}^bT^*M$, linear along the fibres, called its symbol. Associated to a b-differential operator is the leading symbol

$$\sigma(D) = f\,\sigma(X_1)\dots\sigma(X_k) \in C^\infty({}^bT^*M).$$

There is an algebra of b-pseudodifferential operators, bearing the same relationship to the b-differential operators that the classical pseudodifferential operators on a closed manifold bears to the classical differential operators. We will only recall those parts of their calculus which are needed for our construction of the Chern character.

Denote the space of b-pseudodifferential operators of order k on a vector bundle $\mathcal{E}$ by $\Psi^k_b(M,\mathcal{E})$. There is a homomorphism of filtered algebras

$$A \mapsto A_\partial : \Psi^k_b(M,\mathcal{E}) \to \Psi^k(\partial M,\mathcal{E})$$

defined as follows. Given a section $s \in \Gamma(\partial M,\mathcal{E})$, we extend it to a section $\tilde{s} \in \Gamma(M,\mathcal{E})$ over the interior, and then define

$$A_\partial s = A\tilde{s}|_{\partial M}.$$

This is well-defined, since elements of $\Psi^k_b(M,\mathcal{E})$ map the space of sections which vanish on the boundary $\Gamma_0(M,\mathcal{E})$ to itself. For example, if X is a b-vector field, then X_∂ is the restriction of X to the boundary.

Let ν be a conormal structure on M, and let x be an associated defining function for the boundary ∂M. The indicial family of a b-pseudodifferential operator $A \in \Psi^k_b(M,\mathcal{E})$ is the map from $\mathbb{R}$ to $\Psi^k(\partial M,\mathcal{E})$ defined by

$$I_\nu(A,\lambda) = \big(x^{-i\lambda}\cdot A\cdot x^{i\lambda}\big)_\partial.$$

It is easily shown that the indicial family only depends on the conormal structure ν determined by x, and not on x itself. We will only need the indicial family for b-differential operators, when it is polynomial in λ, and for b-smoothing operators, for which it is a morphism of algebras

$$I_\nu : \Psi^{-\infty}_b(M,\mathcal{E}) \to \mathcal{S}(\mathbb{R},\Psi^{-\infty}(\partial M,\mathcal{E}))$$

to the algebra of rapidly decreasing functions with values in smoothing operators on the boundary.

The restriction of the kernel of an element $K \in \Psi_b^{-\infty}(M,\mathcal{E})$ to the diagonal, denoted $\langle x,y|K|x,y\rangle$, is a section of the bundle ${}^b\Omega \otimes \mathrm{End}(\mathcal{E})$ of b-densities with values in $\mathrm{End}(\mathcal{E})$. If $\mathrm{tr}\langle x,y|K|x,y\rangle$ is the b-density obtained by taking the trace of this kernel point by point, we have the formula

$$\mathrm{Res}_\nu \, \mathrm{tr}\langle x,y|K|x,y\rangle = \int_{-\infty}^{\infty} {}^{\partial}\mathrm{Tr}(I_\nu(K,\lambda))\, d\lambda.$$

We extend the trace to all of $K \in \Psi_b^{-\infty}(M,\mathcal{E})$ by using the ν-integral:

$${}^b\mathrm{Tr}_\nu(K) = \int_\nu \mathrm{tr}\langle x,y|K|x,y\rangle.$$

The algebra $\mathcal{S}(\mathbb{R},\Psi_b^{-\infty}(\partial M,\mathcal{E}))$ has, up to a scalar factor, a unique central extension, defined by the Kac-Moody cocycle

$$c(K_0(\lambda),K_1(\lambda)) = \int_{-\infty}^{\infty} {}^{\partial}\mathrm{Tr}\left(\frac{dK_0(\lambda)}{d\lambda}K_1(\lambda)\right) d\lambda,$$

where ${}^{\partial}\mathrm{Tr}$ is the trace on $\Gamma(\partial M,\mathcal{E})$. Although the b-trace is not a trace, its "anomaly" is Morita equivalent to the Kac-Moody cocycle, suitably normalized, as is shown by the following formula of Melrose [**11**].

PROPOSITION 4.3. *If $A \in \Psi_b^k(M,\mathcal{E})$ and $K \in \Psi_b^{-\infty}(M,\mathcal{E})$, the b-trace of the commutator $[A,K]$ is given by the formula*

$${}^b\mathrm{Tr}[A,K] = \frac{1}{2\pi i}\int_{-\infty}^{\infty} {}^{\partial}\mathrm{Tr}\left(\frac{dI_\nu(A,\lambda)}{d\lambda}I_\nu(K,\lambda)\right) d\lambda.$$

5. Riemannian b-metrics and Dirac operators

Let us now turn to the theory of Riemannian geometry on b-manifolds.

DEFINITION 5.1. *A b-metric on a manifold M with boundary is a metric on the b-tangent bundle ${}^bTM$ such that $(x\partial_x, x\partial_x) = 1 + O(x)$.*

Given a b-metric, there is a unique Levi-Civita b-connection

$${}^b\nabla : \Omega_b^k(M,{}^bTM) \to \Omega_b^{k+1}(M,{}^bTM),$$

that is, a torsion-free b-connection such that for any b-vector fields X, Y and Z,

$$Z(X,Y) = ({}^b\nabla_Z X, Y) + (X, {}^b\nabla_Z Y).$$

It is given by exactly the same formula as in the absence of a boundary:

$$\begin{aligned} 2\big({}^b\nabla_X Y, Z\big) &= X(Y,Z) + Y(X,Z) - Z(X,Y) \\ &\qquad + (X,[Y,Z]) + (Y,[X,Z]) - (Z,[X,Y]). \end{aligned}$$

Given a conormal structure ν on M, the restriction of the b-metric g to the boundary ∂M decomposes as

$$g = \nu\otimes\nu + \nu\otimes\alpha + \alpha\otimes\nu + h,$$

where $\alpha \in \Omega^1(\partial M)$, and h is a Riemannian metric on ∂M. We say that the b-metric is exact if α is an exact one-form. Adding α to the conormal structure ν, we obtain a new conormal structure ν' such that

$$g = \nu' \otimes \nu' + h',$$

where $h' = h - \alpha \otimes \alpha$. Thus, an exact b-metric determines a conormal structure ν and a metric on the boundary ∂M. When we work with an exact b-metric, we will always use the indicial family and b-trace appropriate to this conormal structure. Thus, we will write ${}^b\mathrm{Tr}$ and $I(A,\lambda)$ instead of ${}^b\mathrm{Tr}_\nu$ and $I_\nu(A,\lambda)$.

The following lemma is an easy consequence of the formula for the Levi-Civita b-connection.

Lemma 5.2. *Let g be an exact b-metric on M, and let ν be the associated conormal structure. Then the Levi-Civita b-connection associated to g, when restricted to ∂M, preserves the splitting ${}^bTM \cong T(\partial M) \oplus \mathrm{span}(x\partial_x)$, inducing the Levi-Civita connection associated to the metric h on $T(\partial M)$, and the trivial connection on the line bundle spanned by $x\partial/\partial x$.*

Associated to a b-metric g is the bundle $C({}^bT^*M)$ of Clifford algebras, whose fibre at $z \in M$ is the Clifford algebra generated by vectors $v \in {}^bT^*_zM$ subject to the relations $vw + wv = 2g_z(v,w)$.

Definition 5.3. *A degree q Clifford module $\mathcal{E}$ over M is a $\mathbb{Z}/2$-graded Hermitian vector bundle over M, with commuting graded $*$-actions of the Clifford algebra $C(q)$ and the bundle $C({}^bT^*M)$.*

A Clifford superconnection ${}^b\mathbb{A}$ on a degree q Clifford module $\mathcal{E}$ over M is a superconnection ${}^b\mathbb{A} : \Omega^*_b(M,\mathcal{E}) \to \Omega^*_b(M,\mathcal{E})$ commuting with the action of $C(q)$, and such that if $\alpha \in \Omega^1_b(M)$, $\big[{}^b\mathbb{A}, c(\alpha)\big] = c\big({}^b\nabla\alpha\big)$.

If M is a spin-manifold, the spinor bundle leads to an example of a Clifford module. The even and odd dimensional cases must be treated a little differently:

1. if $\dim(M) = 2\ell$ is even, then the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ associated to ${}^bT^*M$ is $\mathbb{Z}/2$-graded, the action of $C({}^bT^*M)$ on $\mathcal{S}$ respects the grading, and $\mathcal{S}$ is a degree 0 Clifford module;
2. if $\dim(M) = 2\ell+1$ is odd, let $\mathcal{S}$ to be the spinor bundle associated to the even-dimensional spin-bundle ${}^bT^*M \oplus \mathbb{R}$. Since

$$C({}^bT^*M \oplus \mathbb{R}) \cong C({}^bT^*M) \otimes C(1),$$

the bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is a degree 1 Clifford module, with action of $\alpha \in \Gamma(M, C({}^bT^*M))$ given by $\begin{pmatrix} 0 & c(\alpha) \\ c(\alpha) & 0 \end{pmatrix}$ and the generator $e_1 \in C(1)$ acting by $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

In both cases, the Levi-Civita connection induces a Clifford superconnection on $\mathcal{S}$.

Restricted to the boundary, the Clifford bundle $C({}^bT^*M)$ splits into a tensor product

$$C({}^bT^*M)|_{\partial M} \cong C(T^*(\partial M)) \otimes C(1),$$

where $C(1)$ is the Clifford algebra generated by $c(\nu)$. This shows that the restriction of a degree q Clifford module $\mathcal{E}$ over M to the boundary is a degree $q+1$ Clifford module over ∂M, with $e_{q+1} = c(\nu)$. If the b-metric on M is exact, then ${}^b\nabla c(\nu) = 0$, and Lemma 5.2 shows that any Clifford superconnection ${}^b\mathbb{A}$ on $\mathcal{E}$ restricts to a Clifford superconnection on $\mathcal{E}|_{\partial M}$.

There is a canonical isomorphism of bundles $\Lambda^*({}^bT^*M) \cong C({}^bT^*M)$, defined by sending a differential form $f_0\,df_1\dots df_k$ to the antisymmetrization

$$\frac{1}{k!}\sum_{\pi\in S_k} f_0\, c(df_{\pi(1)})\dots c(df_{\pi(k)}).$$

We use this to define the action of a differential form on a Clifford module. If $\mathcal{E}$ is a Clifford module of degree q on M, with Clifford superconnection ${}^b\mathbb{A}$, we define the associated Dirac operator D, by the composition

$$\Gamma(M,\mathcal{E}) \xrightarrow{{}^b\mathbb{A}} \Gamma(M,\Lambda^*({}^bT^*M)\otimes\mathcal{E}) \xrightarrow{\text{Clifford multiplication}} \Gamma(M,\mathcal{E}).$$

This gives a Fredholm module of degree q over the algebra $C^\infty(M)$: the Hilbert space $\mathcal{H}$ is the space of L^2-sections of $\mathcal{E}$, the algebra $C^\infty(M)$ acts by pointwise multiplication, and the Fredholm operator is D.

Let $\mathcal{E}$ be a Clifford module on M with Clifford superconnection ${}^b\mathbb{A}$. Denote by D_∂ the Dirac operator associated to the restriction of $\mathcal{E}$ and ${}^b\mathbb{A}$ to the boundary ∂M.

PROPOSITION 5.4. *If (M,g) is an exact Riemannian b-manifold, then*

$$I(\mathsf{D},\lambda) = \mathsf{D}_\partial + i\lambda c(\nu),\ \textit{and}\ I(\mathsf{D}^2,\lambda) = \mathsf{D}_\partial^2 - \lambda^2.$$

PROOF. Around a point p in the boundary ∂M, choose an orthonormal frame $\{e^1,\dots,e^{n-1}\}$ for $T^*(\partial M)$, and a frame for the bundle $\mathcal{E}$. Then ${}^b\mathbb{A}$ decomposes as follows:

$${}^b\mathbb{A} = {}^bd + \sum_{\alpha\subset\{1,\dots,n-1\}} e^\alpha\left(\omega_\alpha + \nu\,\tilde\omega_\alpha\right),$$

where ω_α and $\tilde\omega_\alpha$ are sections of $\operatorname{End}(\mathcal{E})$. From this, we see that if x is a defining function compatible with the conormal structure ν,

$$x^{-i\lambda}\,{}^b\mathbb{A}x^{i\lambda} = {}^bd + i\lambda\nu + \sum_{\alpha\subset\{1,\dots,n-1\}} e^\alpha\left(\omega_\alpha + \nu\,\tilde\omega_\alpha\right).$$

Since ${}^b\mathbb{A}$ is a Clifford b-superconnection, $[{}^b\mathbb{A}, c(\nu)] = c({}^b\nabla\nu)$, and since M carries an exact metric, we see that this vanishes on the boundary; in other words, for each α, $\tilde\omega_\alpha$ vanishes on the boundary. Thus,

$$I({}^b\mathbb{A},\lambda) = {}^b\mathbb{A}|_\partial + i\lambda\nu,$$

from which the formula for $I(\mathsf{D},\lambda)$ follows, on applying Clifford multiplication to both sides. The formula for $I(\mathsf{D}^2,\lambda)$ follows from the formula for $I(\mathsf{D},\lambda)$, since

$$c(\nu)\mathsf{D}_\partial + \mathsf{D}_\partial c(\nu) = 0. \square$$

COROLLARY 5.5. *If $K\in\Psi_b^{-\infty}(M,\mathcal{S})$, then*

$${}^b\mathrm{Str}_{C(q)}[\mathsf{D},K] = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} {}^\partial\mathrm{Str}_{C(q+1)}(I(K,\lambda))\,d\lambda.$$

Proof. Proposition 5.4 shows that $\dfrac{dI(\mathsf{D},\lambda)}{d\lambda} = ie_{q+1}$, so by Proposition 3.3

$$
\begin{aligned}
&{}^b\mathrm{Str}_{C(q)}[\mathsf{D},K] \\
&\quad = \frac{1}{2\pi(4\pi)^{q/2}} \int_{-\infty}^{\infty} \Big(\mathrm{Tr}\,|_{\Gamma(\partial M,\mathcal{E}^+)} - \mathrm{Tr}\,|_{\Gamma(\partial M,\mathcal{E}^-)} \Big)(e_1 \dots e_{q+1} I(K,\lambda))\, d\lambda \\
&\quad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} {}^{\partial}\mathrm{Str}_{C(q+1)}(I(K,\lambda))\, d\lambda. \square
\end{aligned}
$$

6. The Chern character of Dirac operators on b-manifolds

In this section, we extend the definition of the Chern character of a Dirac operator to the setting of b-manifolds, replacing the supertrace Str by ^{b}Str in all of the formulas. The Chern character thus defined is not closed, but we will show that its boundary is the Chern character of the Dirac operator on the boundary.

Let ${}^b\mathbb{A}_u$ be a differentiable family of Clifford superconnections on $\mathcal{E}$ parametrized by an open subset U of $\mathbb{R}^n$, let D_u be the associated family of Dirac operators, and consider the family of Dirac operators $\mathsf{D}_{t,u} = t^{1/2}\mathsf{D}_u$ parametrized by $(t,u) \in (0,\infty) \times U$.

Proposition 6.1. *The $\Omega^*((0,\infty)\times U)$-valued multilinear form on $\Psi_b^1(M,\mathcal{E})$*

$$
\langle\!\langle A_0,\dots,A_k \rangle\!\rangle_b = \int_{\Delta^k} {}^b\mathrm{Str}(A_0 e^{\sigma_0(d\mathsf{D}+\mathsf{D}^2)} \dots A_k e^{\sigma_k(d\mathsf{D}+\mathsf{D}^2)})\, d\sigma,
$$

is continuous.

Proof. Denoting by e_{ij} the matrix with one non-vanishing entry, in the i-th row and j-th column, let P be the $(k+1)\times(k+1)$-matrix of b-pseudodifferential operators

$$
P = e_{k,0}A_0,
$$

and let Q be the $(k+1)\times(k+1)$-matrix of b-pseudodifferential operators

$$
Q = \sum_{i=0}^{k} e_{i,i}\, d\mathsf{D} + \sum_{i=1}^{k} e_{i-1,i} A_i .
$$

Observe that

$$
\exp(\mathsf{D}^2+Q) = \sum_{0\le i\le j\le k} e_{i,j} \int_{\Delta^{j-i}} e^{\sigma_0(d\mathsf{D}+\mathsf{D}^2)} A_{i+1} \dots A_j e^{\sigma_{j-i}(d\mathsf{D}+\mathsf{D}^2)},
$$

and hence that $\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_b = {}^b\mathrm{Str}(P\exp(\mathsf{D}^2+Q))$. But D^2+Q is elliptic in the b-calculus (indeed, it has the same symbol as D^2), so that the matrix entries of $\exp(\mathsf{D}^2+Q)$ lie in $\Psi_b^{-\infty}(M,\mathcal{E})$. This shows that the matrix entries of $P\exp(\mathsf{D}^2+Q)$ lie in $\Psi_b^{-\infty}(M,\mathcal{E})$, and hence that it has well-defined b-supertrace. From this representation, the continuity is evident. $\square$

The Chern character of the family of b-Dirac operators $\mathsf{D}_{t,u}$ on a manifold with boundary M is the cochain $\mathrm{Ch}^*(\mathsf{D})$ on $C^\infty(M)$, with values in $\Omega^*((0,\infty)\times U)$, defined by the formula

$$
\mathrm{Ch}^k(\mathsf{D})(a_0,\dots,a_k) = \langle\!\langle a_0,[\mathsf{D},a_1],\dots,[\mathsf{D},a_k]\rangle\!\rangle_b.
$$

This is an entire cyclic cochain, as may be shown by careful estimation using the method of Proposition 6.1. The following result is our generalization of the Atiyah-Patodi-Singer index theorem.

THEOREM 6.2. $(d+b+B)\operatorname{Ch}^*(\mathsf{D}) = \operatorname{Ch}^*(\mathsf{D}_\partial)$

Before proving this, we need a number of formulas which we collect in a lemma. In the following, denote by $\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_\partial$ the multilinear form associated to the Dirac operator on the boundary ∂M. Let $\Phi_b(M,\mathcal{E})$ be the subalgebra of $\Psi_b(M,\mathcal{E})$ consisting of b-pseudodifferential operators A such that the indicial family $I(A,\lambda)$ of A is independent of λ and commutes with the actions of $C(q)$ and $c(\nu)$.

LEMMA 6.3. *Let A_i be b-pseudodifferential operators in $\Phi_b(M,\mathcal{E})$.*

1. *If $\varepsilon_i = (|A_0|+\dots+|A_{i-1}|)(|A_i|+\dots+|A_k|)$, then*

$$\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_b = (-1)^{\varepsilon_i}\langle\!\langle A_i,\dots,A_k,A_0,\dots,A_{i-1}\rangle\!\rangle_b$$

2.

$$\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_b = \sum_{i=0}^k (-1)^{\varepsilon_i}\langle\!\langle 1,A_i,\dots,A_k,A_0,\dots,A_{i-1}\rangle\!\rangle_b$$

3.

$$\begin{aligned} d\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_b + \sum_{i=0}^k (-1)^{|A_0|+\dots+|A_{i-1}|}\langle\!\langle A_0,\dots,[\mathsf{D},A_i],\dots,A_k\rangle\!\rangle_b \\ = \langle\!\langle (A_0)_\partial,\dots,(A_k)_\partial\rangle\!\rangle_\partial \end{aligned}$$

4.

$$\begin{aligned} &\langle\!\langle A_0,\dots,[d\mathsf{D}+\mathsf{D}^2,A_i],\dots,A_k\rangle\!\rangle_b \\ &\qquad = \langle\!\langle A_0,\dots,A_iA_{i+1},\dots,A_k\rangle\!\rangle_b - \langle\!\langle A_0,\dots,A_{i-1}A_i,\dots,A_k\rangle\!\rangle_b \end{aligned}$$

PROOF. In the proof, we use the formula

$$I(d\mathsf{D}+\mathsf{D}^2,\lambda) = d\mathsf{D}_\partial + \mathsf{D}_\partial^2 - \lambda^2 - \frac{ic(\nu)\lambda\,dt}{2t^{1/2}},$$

which follows from Proposition 5.4.

The cyclic symmetry in (1) follows from the fact that

$$\begin{aligned} &\langle\!\langle A_k,A_0,\dots,A_{k-1}\rangle\!\rangle_b - (-1)^{|A_k|(|A_0|+\dots+|A_{k-1}|)}\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_b \\ &\qquad = \int_{\Delta^k} {}^b\mathrm{Str}_{C(q)}\big[A_k e^{\sigma_k(d\mathsf{D}+\mathsf{D}^2)}, A_0 e^{\sigma_0(d\mathsf{D}+\mathsf{D}^2)}\dots A_{k-1}e^{\sigma_{k-1}(d\mathsf{D}+\mathsf{D}^2)}\big]. \end{aligned}$$

Applying Proposition 4.3, and using the independence of $I(A_k,\lambda) = (A_k)_\partial$ on λ and the fact that $I(A_i,\lambda)$ commutes with $c(\nu)$ and with $C(q)$, we see that this is proportional to the integral over $(\lambda;\sigma_0,\dots,\sigma_k)\in\mathbb{R}\times\Delta^k$ of

$$^\partial\mathrm{Str}_{C(q)}\Big(F(\lambda)\,(A_k)_\partial\, e^{\sigma_k(d\mathsf{D}_\partial+\mathsf{D}_\partial^2)}(A_0)_\partial\, e^{\sigma_0(d\mathsf{D}_\partial+\mathsf{D}_\partial^2)}\dots(A_{k-1})_\partial\, e^{\sigma_{k-1}(d\mathsf{D}_\partial+\mathsf{D}_\partial^2)}\Big),$$

where

$$\begin{aligned} F(\lambda) &= \sigma_k\Big(-2\lambda - \frac{ic(\nu)\,dt}{2t^{1/2}}\Big)\exp\Big(-\lambda^2 - \frac{ic(\nu)\,dt}{2t^{1/2}}\Big) \\ &= \sigma_k\Big(-2\lambda - (1-2\lambda^2)\frac{ic(\nu)\,dt}{2t^{1/2}}\Big)e^{-\lambda^2}. \end{aligned}$$

But the integral of $F(\lambda)$ over λ vanishes, proving the result.

To prove (2), we start from the equation

$$\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_b = \int_{[0,1]\times\Delta^k} {}^b\mathrm{Str}(A_0 e^{\sigma_0(d\mathsf{D}+\mathsf{D}^2)}\dots A_k e^{\sigma_k(d\mathsf{D}+\mathsf{D}^2)})\,ds\,d\sigma.$$

We divide the region of integration into $k+1$ pieces

$$R_i = \{\sigma_0+\dots+\sigma_i \le s \le \sigma_0+\dots+\sigma_{i+1}\}.$$

Each of these regions R_i is a simplex, which contributes the term

$$\langle\!\langle A_0,\dots,A_i,1,A_{i+1},\dots,A_k\rangle\!\rangle_b,$$

to the sum. Part (2) now follows by applying (1) to each term.

Part (3) follows from Corollary 5.5, which shows that

$$\begin{aligned}
d\langle\!\langle A_0,\dots,A_k\rangle\!\rangle_b &+ \sum_{i=0}^k (-1)^{|A_0|+\dots+|A_{i-1}|}\langle\!\langle A_0,\dots,[\mathsf{D},A_i],\dots,A_k\rangle\!\rangle_b \\
&= {}^b\mathrm{Str}_{C(q)}[\mathsf{D},A_0 e^{\sigma_0(d\mathsf{D}+\mathsf{D}^2)}\dots A_k e^{\sigma_k(d\mathsf{D}+\mathsf{D}^2)}] \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{\Delta^k} {}^\partial\mathrm{Str}_{C(q+1)}\Big((A_0)_\partial\, e^{\sigma_0 I(d\mathsf{D}+\mathsf{D}^2,\lambda)}\dots(A_k)_\partial\, e^{\sigma_k I(d\mathsf{D}+\mathsf{D}^2,\lambda)}\Big)\,d\lambda \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\lambda^2}\,d\lambda \times \langle\!\langle (A_0)_\partial,\dots,(A_k)_\partial\rangle\!\rangle_\partial \\
&\quad + \frac{idt}{(4\pi t)^{1/2}}\int_{-\infty}^{\infty}\lambda\, e^{-\lambda^2}\,d\lambda \times \langle\!\langle c(\nu)\,(A_0)_\partial,\dots,(A_k)_\partial\rangle\!\rangle_\partial.
\end{aligned}$$

On integration, the second term vanishes, while the first gains a factor of $\sqrt{\pi}$.

To prove part (4) we insert the formula

$$[e^{\sigma_i(d\mathsf{D}+\mathsf{D}^2)},A_i] = \int_0^{\sigma_i} e^{(\sigma_i-\sigma)(d\mathsf{D}+\mathsf{D}^2)}[d\mathsf{D}+\mathsf{D}^2,A_i]e^{\sigma(d\mathsf{D}+\mathsf{D}^2)}\,d\sigma.$$

into the definition of $\langle\!\langle A_0,\dots,[d\mathsf{D}+\mathsf{D}^2,A_i],\dots,A_k\rangle\!\rangle_b$. □

Proof of Theorem 6.2. If $a\in C^\infty(M)$, then $I(a,\lambda)=a_\partial$ and

$$I([\mathsf{D},a],\lambda) = [\mathsf{D}_\partial + ic(\nu)\lambda, a_\partial] = [\mathsf{D}_\partial,a_\partial].$$

Applying part (3) of the lemma with $A_0=a_0$ and $A_i=[\mathsf{D},a_i]$ for $1\le i<k$ gives

$$\begin{aligned}
\mathrm{Ch}^{k-1}(\mathsf{D}_\partial)(a_0,\dots,a_{k-1}) &= d\langle\!\langle a_0,[\mathsf{D},a_1],\dots,[\mathsf{D},a_{k-1}]\rangle\!\rangle_b \\
&\quad + \langle\!\langle [\mathsf{D},a_0],\dots,[\mathsf{D},a_{k-1}]\rangle\!\rangle_b \\
&\quad + \sum_{i=1}^k (-1)^{i-1}\langle\!\langle a_0,[\mathsf{D},a_1],\dots,[\mathsf{D}^2,a_i],\dots,[\mathsf{D},a_{k-1}]\rangle\!\rangle_b.
\end{aligned}$$

By part (2) of the lemma, the second of the terms on the right-hand side equals

$$(B\,\mathrm{Ch}^k(\mathsf{D}))(a_0,\dots,a_{k-1}).$$

By part (4) of the lemma, we see that

$$\begin{aligned}(-1)^{i-1}&\langle\!\langle a_0,[\mathsf{D},a_1],\dots,[\mathsf{D}^2,a_i],\dots,[\mathsf{D},a_{k-1}]\rangle\!\rangle_b\\ &= (-1)^i\langle\!\langle a_0,[\mathsf{D},a_1],\dots,[\mathsf{D},a_{i-1}]a_i,[\mathsf{D},a_{i+1}],\dots,[\mathsf{D},a_k]\rangle\!\rangle_b\\ &+ (-1)^{i-1}\langle\!\langle a_0,[\mathsf{D},a_1],\dots,[\mathsf{D},a_{i-1}],a_i[\mathsf{D},a_{i+1}],\dots,[\mathsf{D},a_k]\rangle\!\rangle_b\end{aligned}$$

Adding all of this up and using the fact that $[\mathsf{D},a_ia_{i+1}]=[\mathsf{D},a_i]a_{i+1}+a_i[\mathsf{D},a_{i+1}]$, we easily see that these terms conspire to give $b\,\mathrm{Ch}^{k-2}(\mathsf{D},t)$ evaluated on the chain $(a_0,\dots,a_{k-1})$. □

7. The Atiyah-Patodi-Singer theorem for twisted Dirac operators

In this section, we will relate the main result of this paper, Theorem 6.2, to the Atiyah-Patodi-Singer index theorem, obtaining an analogue of Theorem 4.1 of Wu [**16**] in the setting of the b-calculus.

If $(\mathcal{H},\mathsf{D})$ is a theta-summable Fredholm module, let

$$\mathrm{Ch}^*(\mathsf{D},t)\in\Omega^*((0,\infty),\mathsf{C}^*_\omega(\mathcal{A}))$$

be the Chern character associated to the family $\mathsf{D}_t=t^{1/2}\mathsf{D}$; the zero-form component of $\mathrm{Ch}^*(\mathsf{D},t)$ is the Chern character of D_t, while the one-form component is the secondary characteristic class $\widetilde{\mathrm{Ch}}^*(\mathsf{D}_t,\dot{\mathsf{D}}_t)$ of Getzler-Szenes [**7**]. Denoting the multilinear form associated to D_t by $\langle A_0,\dots,A_k\rangle_t$, we easily obtain the explicit formula

$$\begin{aligned}\mathrm{Ch}^k(\mathsf{D},t)(a_0,\dots,a_k)&=\langle a_0,[\mathsf{D}_t,a_1],\dots,[\mathsf{D}_t,a_k]\rangle_t\\ &+\sum_{i=0}^k\frac{(-1)^i dt}{2t}\langle a_0,[\mathsf{D}_t,a_1],\dots,[\mathsf{D}_t,a_i],\mathsf{D}_t,[\mathsf{D}_t,a_{i+1}],\dots,[\mathsf{D}_t,a_k]\rangle_t.\end{aligned}$$

Note that the first (second) term vanishes unless $k+q$ is even (odd).

If the operator D is sufficiently well-behaved, we can integrate $\mathrm{Ch}^*(\mathsf{D},t)$ over the interval $(0,\infty)$, obtaining the total eta-invariant of Wu,

$$\eta^*(\mathsf{D})=\int_0^\infty \mathrm{Ch}^*(\mathsf{D},t),$$

which is an even (odd) cyclic cochain if $(\mathcal{H},\mathsf{D})$ has odd (even) degree.

In the rest of this section, we will restrict attention to the following situation. Let N be a compact odd-dimensional manifold (which we will eventually take to be ∂M, where M is an even-dimensional b-manifold). Let $\mathcal{E}$ be an ungraded Clifford module over N with Clifford connection ${}^b\nabla^{\mathcal{E}}$, and let $\mathcal{D}$ be the associated Dirac operator. Associate to $\mathcal{D}$ a Fredholm operator on the $C(1)$-module $\Gamma(M,\mathcal{E}\oplus\mathcal{E})$

$$\mathsf{D}=\begin{pmatrix}0&\mathcal{D}\\ \mathcal{D}&0\end{pmatrix}.$$

The convergence of the integral defining $\eta^*(\mathsf{D})$ as $t\to 0$ is guaranteed by a generalization of the calculation showing that the residue of the eta-function of a Dirac operator at $s=0$ vanishes (see Bismut-Freed [**2**]), provided D is associated to a Clifford connection ${}^b\nabla^{\mathcal{E}}$, as against a general Clifford superconnection. (Alternatively, the pseudodifferential calculus used to prove the local index theorem for Dirac operators [**3**] yields the same result.) The convergence of the integral as

$t \longrightarrow \infty$ is controlled by the lowest eigenvalue λ of $|\mathsf{D}|$: thus, the total eta-invariant is not an entire cyclic cochain, but has finite radius of convergence proportional to λ. We now have the following analogue of Theorem 3.1, which has been proved under the hypothesis that $N = \partial M$ is a boundary and the idempotent p extends to the interior of M by Wu [**16**].

THEOREM 7.1. *Let λ be the lowest eigenvalue of $|\mathcal{D}|$. If p is an idempotent in $M_r(C^\infty(N))$ and $\eta^*(\mathsf{D})$ is the total eta-invariant of D, then the pairing*

$$(\eta^*(\mathsf{D}), \mathrm{Ch}_*(p))$$

is convergent if the idempotent p satisfies the estimate $|dp| < \lambda$, and equals $-\frac{1}{2}$ times the eta-invariant of the Dirac operator $p\mathcal{D}p$.

PROOF. The fact that the pairing of $\eta^*(\mathsf{D})$ with $\mathrm{Ch}_*(p)$ is finite of $|dp|$ is smaller than the lowest eigenvalue of $|\mathcal{D}|$ is Theorem 3.1 of Wu [**16**]. We will calculate $(\eta^*(\mathsf{D}), \mathrm{Ch}_*(p))$ by a modification of our second proof of Theorem 3.1.

As in that proof, we consider a family of Fredholm modules on $[0,1]\times\mathbb{R}\times[0,\infty)$, parametrized by (u,s,t),

$$\tilde{\mathsf{D}} = t^{1/2}\mathsf{D}_u + is\sigma(p - \tfrac{1}{2}).$$

Thus, we are now working in the Clifford module $\Gamma(M, \mathcal{E}\oplus\mathcal{E}) \oplus \Gamma(M, \mathcal{E}\oplus\mathcal{E})$ over the Clifford algebra $C(2)$ generated by e_1 and σ.

If $\pi : [0,1]\times\mathbb{R}\times[0,\infty) \longrightarrow [0,1]\times\mathbb{R}$ is the projection which sends (u,s,t) to (u,s), we see that

$$d\big(\pi_* \operatorname{Str}_{C(2)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big)\big) = \lim_{t\longrightarrow 0} \operatorname{Str}_{C(2)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big) \in \Omega^*([0,1]\times\mathbb{R}).$$

This may be shown to vanish, by the method of Bismut-Freed [**2**] or the asymptotic pseudodifferential calculus [**3**]. Thus, the integral

$$a(u) = \int_{\Gamma_u} \pi_* \operatorname{Str}_{C(2)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big)$$

is independent of u. We will prove Theorem 7.1 by evaluating the integral at $u=0$ and at $u=1$.

First, we prove that

$$a(0) = \int_{\Gamma_0} \pi_* \operatorname{Str}_{C(2)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big) = i\big(\eta^*(\mathsf{D}), \mathrm{Ch}_*(p) - \tfrac{1}{2}\operatorname{rk}(p)\,\mathrm{Ch}_*(1)\big).$$

Since $(d+\tilde{\mathsf{D}})^2$ is given by the formula

$$t\mathsf{D}_u^2 - s^2/4 - i(1-u)t^{1/2}s\sigma[\mathsf{D},p] + ids\,\sigma(p-\tfrac{1}{2}) + t^{1/2}du\,(2p-1)[\mathsf{D},p] + \tfrac{1}{2}t^{-1/2}dt\,\mathsf{D},$$

we see that

$$\begin{aligned}\int_{\Gamma_0} &\pi_* \operatorname{Str}_{C(2)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big) \\ &= \int_{\Gamma_0}\int_0^\infty \operatorname{Str}_{C(2)}\big(e^{t\mathsf{D}^2 - s^2/4 - it^{1/2}s\sigma[\mathsf{D},p] + ids\,\sigma(p-\frac{1}{2}) + dt\,\mathsf{D}/2t^{1/2}}\big).\end{aligned}$$

Expanding in powers of s, and only keeping terms with one factor of ds, one factor of dt, and an odd number of factors of σ, we see that this equals the sum

$$-i\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2(4\pi)^{1/2}}\int_{-\infty}^{\infty}s^{2\ell}e^{-s^2/4}\,ds\times\int_0^\infty t^{\ell-1/2}$$
$$\sum_{i=0}^{2\ell}\sum_{j=0}^{2\ell-i}\Big\{\langle 1,\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{i\text{ times}},p-\tfrac12,\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{j\text{ times}},dt\,\mathsf{D},\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{2\ell-i-j\text{ times}}\rangle_t$$
$$+\langle 1,\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{i\text{ times}},dt\,\mathsf{D},\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{j\text{ times}},p-\tfrac12,\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{2\ell-i-j\text{ times}}\rangle_t\Big\}.$$

This in turn equals

$$\frac{i}{2}\sum_{\ell=0}^{\infty}\frac{(2\ell)!}{\ell!}\int_0^\infty t^{\ell-1/2}\sum_{i=0}^{2\ell}(-1)^{\ell+i}\langle p-\tfrac12,\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{i\text{ times}},\mathsf{D},\underbrace{[\mathsf{D},p],\dots,[\mathsf{D},p]}_{2\ell-i\text{ times}}\rangle_t$$
$$=i\big(\eta^*(\mathsf{D}),\mathrm{Ch}_*(p)-\tfrac12\,\mathrm{rk}(p)\,\mathrm{Ch}_*(1)\big).$$

We now evalute $a(1)$. From the explicit formula for $(d+\tilde{\mathsf{D}})^2$, it follows that

$$\int_{\Gamma_1}\mathrm{Str}_{C(2)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big)=\frac{1}{2i(4\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-s^2/4}\,ds\int_0^\infty\mathrm{Str}_{C(1)}\big((p-\tfrac12)\mathsf{D}_1e^{t\mathsf{D}_1^2}\big)\,\frac{dt}{t^{1/2}},$$

where $\mathsf{D}_1=p\mathsf{D}p+(1-p)\mathsf{D}(1-p)$. The Clifford supertrace is given by the formula

$$\begin{aligned}\mathrm{Str}_{C(1)}\big((p-\tfrac12)\mathsf{D}_1e^{t\mathsf{D}_1^2}\big)&=\frac{1}{(4\pi)^{1/2}}\,\mathrm{Str}\Big\{(p-\tfrac12)\begin{pmatrix}0&-i\\ i&0\end{pmatrix}\begin{pmatrix}0&\mathcal{D}_1e^{t\mathcal{D}_1^2}\\ \mathcal{D}_1e^{t\mathcal{D}_1^2}&0\end{pmatrix}\Big\}\\&=\frac{1}{(4\pi)^{1/2}}\,\mathrm{Str}\begin{pmatrix}-i(p-\tfrac12)\mathcal{D}_1e^{t\mathcal{D}_1^2}&0\\0&i(p-\tfrac12)\mathcal{D}_1e^{t\mathcal{D}_1^2}\end{pmatrix}\\&=-i\pi^{-1/2}\,\mathrm{Tr}\big((p-\tfrac12)\mathcal{D}_1e^{t\mathcal{D}_1^2}\big).\end{aligned}$$

Thus, we see that

$$\begin{aligned}\int_{\Gamma_1}\mathrm{Str}_{C(2)}\big(e^{(d+\tilde{\mathsf{D}})^2}\big)&=\frac{1}{2i\pi^{1/2}}\int_0^\infty\mathrm{Tr}\big((p-\tfrac12)\mathcal{D}_1e^{t\mathcal{D}_1^2}\big)\frac{dt}{t^{1/2}}\\&=-\tfrac12\eta(\mathcal{D}_p)-\tfrac{i}{2}\,\mathrm{rk}(p)(\eta^*(\mathsf{D}),\mathrm{Ch}_*(1)).\end{aligned}$$

The unwanted term $\frac{i}{2}\,\mathrm{rk}(p)(\eta^*(\mathsf{D}),\mathrm{Ch}_*(1))$ cancels from both $a(0)$ and $a(1)$, proving the theorem. ∎

The following theorem generalizes the Atiyah-Patodi-Singer index theorem, allowing us to twist the operator by a bundle which is sufficiently flat on the boundary. Taking $p=1$ in the theorem, we recover the original Atiyah-Patodi-Singer index theorem, in which D is untwisted.

THEOREM 7.2. *Let D be a Dirac operator on a degree 0 Clifford module $\mathcal{E}$ over a compact manifold M associated to a Clifford connection ${}^b\nabla^{\mathcal{E}}$. Let λ be the lowest eigenvalue of $|\mathsf{D}_\partial|$. Let p be an idempotent in $M_r(C^\infty(M))$ such that $|dp_\partial|<\lambda$, where p_∂ is the restriction of p to the boundary. Then the operator $p\mathsf{D}p$ is Fredholm* *on $\Gamma(M,\mathcal{E}\otimes\operatorname{im}(p))$, and the following higher Atiyah-Patodi-Singer index theorem holds:*

$$\langle \mathsf{D},p\rangle = (2\pi i)^{-n/2}\int_\nu \operatorname{Tr}\left(pe^{-F}\right)\,\det{}^{1/2}\left(\frac{{}^bR/2}{\sinh{}^bR/2}\right) + (\eta^*(\mathsf{D}_\partial),\operatorname{Ch}_*(p_\partial)).$$

(The integral here is the ν-integral introduced in the Section 5.)

Proof. Form the family of Dirac operators D_u on $\mathcal{E}\otimes\mathbb{C}^r$, parametrized by $u\in[0,1]$,

$$\mathsf{D}_u = (1-u)\mathsf{D} + u\big(p\mathsf{D}p + (1-p)\mathsf{D}(1-p)\big).$$

Let $\alpha\in\Omega^*((0,\infty)\times\mathbb{C})$ be the differential form given by the formula

$$\alpha = (\operatorname{Ch}^*(\mathsf{D}_u,t),\operatorname{Ch}_*(p)).$$

Then by Theorem 6.2,

$$d\alpha = (\operatorname{Ch}^*(\mathsf{D}_{\partial,u},t),\operatorname{Ch}_*(p_\partial)),$$

where $\mathsf{D}_{\partial.u} = (1-u)\mathsf{D}_\partial + u\big(p_\partial\mathsf{D}_\partial p_\partial + (1-p_\partial)\mathsf{D}_\partial(1-p_\partial)\big)$. The fundamental theorem of calculus shows that

$$\alpha(t_2,0)-\alpha(t_1,0) = \Big(\int_{t_1}^{t_2}\operatorname{Ch}^*(\mathsf{D}_\partial,t),\operatorname{Ch}_*(p_\partial)\Big),$$

and hence that

$$\lim_{t\longrightarrow\infty}\alpha(t,0) - \lim_{t\longrightarrow 0}\alpha(t,0) = (\eta^*(\mathsf{D}_\partial),\operatorname{Ch}_*(p_\partial)),$$

where the right-hand side is defined by Theorem 7.1.

A generalization of the methods of Block and Fox [**3**] (see also Chapter 8 of Melrose [**11**]) shows that

$$\lim_{t\longrightarrow 0}\operatorname{Ch}^k(\mathsf{D},t)(a_0,\dots,a_k) = (2\pi i)^{-n/2}\int_\nu a_0\,da_1\dots da_k\wedge\det{}^{1/2}\left(\frac{{}^bR/2}{\sinh{}^bR/2}\right),$$

where ${}^bR$ is the curvature of the Levi-Civita b-connection ${}^b\nabla$. This shows that

$$\lim_{t\longrightarrow 0}\alpha(t,0) = (2\pi i)^{-n/2}\int_\nu\operatorname{Tr}\left(pe^{-F}\right)\,\det{}^{1/2}\left(\frac{{}^bR/2}{\sinh{}^bR/2}\right).$$

Thus, it remains to show that $\alpha(\infty,0)$ equals the index of the operator $p\mathsf{D}^+p$.

By the hypothesis on $|dp_\partial|$, the operator $\mathsf{D}_{\partial,u} = \mathsf{D}_\partial + u(2p_\partial-1)c(dp_\partial)$ satisfies the lower bound

$$|\mathsf{D}_{\partial,u}| > \lambda - |(2p_\partial-1)c(dp_\partial)| > 0,$$

since $|(2p_\partial-1)c(dp_\partial)| = |dp_\partial|$. Thus, the operators D_u are Fredholm for $u\in[0,1]$.

Melrose's analogue of the McKean-Singer formula shows that for all $t>0$,

$$\langle\mathsf{D},p\rangle = \lim_{t\longrightarrow\infty}{}^b\operatorname{Str}(pe^{t\mathsf{D}_1^2}) = \lim_{t\longrightarrow\infty}\alpha(t,1).$$

The proof will be completed by the following lemma, which shows that

$$\lim_{t\longrightarrow\infty}\alpha(t,0) = \langle\mathsf{D},p\rangle.$$

Lemma 7.3. $\alpha(t,1)-\alpha(t,0) = O(e^{-(\lambda-|dp_\partial|)^2t/2})$

PROOF. Since $[\mathsf{D}_{\partial,u}, p_\partial] = (1-u)c(dp_\partial)$, we see that $\alpha(t,1) - \alpha(t,0)$ is given by the sum

$$\sum_{\ell=0}^{\infty} \frac{(2\ell)!}{\ell!} t^{\ell+1/2} \sum_{i=0}^{2\ell} (-1)^{i+\ell} \int_{u\in[0,1]} (1-u)^{2\ell} \langle p_\partial - \tfrac{1}{2}, c(dp_\partial), \dots, c((2p_\partial - 1)dp_\partial), \dots, c(dp_\partial)\rangle_{t,u}.$$

Now, the operator $\mathsf{D}^2_{\partial,u}$ is bounded below by $(\lambda - |dp_\partial|)^2$, and its heat kernel is uniformly trace-class, so the term indexed by ℓ may be bounded by

$$\frac{(2\ell)!}{(\ell!)^2} (t^{1/2}|dp_\partial|)^{2\ell+1} e^{-3(\lambda-|dp_\partial|)^2 t/4} \sum_{i=0}^{2\ell} \frac{1}{(2\ell+2)!} \int_0^1 (1-u)^{2\ell}\, du = \frac{(t^{1/2}|dp_\partial|)^{2\ell+1}}{(\ell!)^2} e^{-3(\lambda-|dp_\partial|)^2 t/4}.$$

Summing over ℓ, the result follows. □

Appendix. Clifford modules

In this appendix, we discuss some of the properties of Hilbert modules over Clifford algebras used in this article.

If q is a natural number, let $C(q)$ be the Clifford algebra on generators e_i, $1 \le i \le q$, such that

$$e_i e_j + e_j e_i = 2\delta_{ij}.$$

The 2^q vectors $e_I = e_{I_1} \dots e_{I_j}$ form a basis of $C(q)$, where I ranges over subsets $1 \le I_1 < \dots < I_j \le q$ of $\{1, \dots, q\}$. There is a canonical invariant inner product on $C(q)$, for which the vectors e_I form an orthonormal basis. The embedding of $C(q)$ in $\mathrm{End}(C(q))$ defines a norm $\|\cdot\|_{C(q)}$ on $C(q)$, for which it is a C^*-algebra; the conjugation is given by $e_i^* = e_i$. The algebra $C(q)$ has a $\mathbb{Z}/2$-grading defined by taking e_i odd.

DEFINITION A.1. *A Hilbert module for $C(q)$ is a graded $*$-representation of $C(q)$ on a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.*

This definition of a Hilbert module is not the same as the usual one (Kasparov [**9**]): however, it is equivalent to it, as we will now explain.

Recall that if $\mathcal{B}$ is a C^*-algebra, a Hilbert module $\mathcal{H}$ over $\mathcal{B}$ is a vector space $\mathcal{H}$, together with a bilinear map

$$\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{B}$$

with the following properties:

1. if $v, w \in \mathcal{H}$ and $a, b \in \mathcal{B}$, then $\langle w, v\rangle = \langle v, w\rangle^*$ and $\langle av, bw\rangle = a\langle v, w\rangle b^*$;
2. the norm $\|\langle v, v\rangle\|_{\mathcal{B}}^{1/2}$ on $\mathcal{H}$ is complete.

LEMMA A.2. *If $\mathcal{H}$ is a Hilbert module for the Clifford algebra $C(q)$ in the sense of Definition A.1, then it is a Hilbert module for $C(q)$ in the usual sense.*

Proof. If $v, w \in C(q)$, we define

$$\langle v, w\rangle = \sum_{I\subset\{1,\dots,n\}} (v, e_I\cdot w)e_I.$$

It is easily verified that this satisfies the formulas

$$\langle w, v\rangle = \langle v, w\rangle^*,\ \langle v, e_i\cdot w\rangle = \langle v, w\rangle e_i, \text{ and } \langle e_i\cdot v, w\rangle = e_i\langle v, w\rangle.$$

Furthermore, the norm $v \mapsto \|\langle v, v\rangle\|^{1/2}_{C(q)}$ on $\mathcal{H}$ is equivalent to its Hilbert norm, so that $\mathcal{H}$ is complete with respect to it. □

We denote by $\mathcal{L}_{C(q)}(\mathcal{H})$ the commutant of $C(q)$ in the algebra of bounded operators on $\mathcal{H}$, and by $\mathcal{L}^1_{C(q)}(\mathcal{H})$ the commutant of $C(q)$ in the algebra of trace-class operators on $\mathcal{H}$.

The following definition agrees, up to a factor of $(2\pi)^{-q/2}$, with the relative supertrace of Berline-Getzler-Vergne [**4**], and extends Quillen's definition of the supertrace [**13**] on $\mathcal{L}^1_{C(1)}(\mathcal{H})$.

Definition A.3. *If $\mathcal{H}$ is a Hilbert module over $C(q)$, the Clifford supertrace $\operatorname{Str}_{C(q)}$ is the supertrace on the algebra of trace class operators $A \in \mathcal{L}^1_{C(q)}(\mathcal{H})$, given by the formula*

$$\operatorname{Str}_{C(q)}(A) = (4\pi)^{-q/2}\operatorname{Tr}|_{\mathcal{H}^+}(e_1\dots e_qA) - (4\pi)^{-q/2}\operatorname{Tr}|_{\mathcal{H}^-}(e_1\dots e_qA).$$

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