THE BARGMANN REPRESENTATION, GENERALIZED DIRAC OPERATORS AND THE INDEX OF PSEUDODIFFERENTIAL OPERATORS ON $\mathbb{R}^n$

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ABSTRACT. Using the Bargmann representation, we show that the index of an elliptic pseudodifferential operator on $\mathbb{R}^n$ equals the index of a generalized Dirac operator on the cotangent bundle $T^*\mathbb{R}^n \cong \mathbb{C}^n$. We extend the local index theorem for Dirac operators to this setting, and thereby obtain a proof of Fedosov's index theorem.

This paper consists of two parts. In the first, we describe a new approach to the Atiyah-Singer index theorem on Euclidean space, which we hope will be of use in studying more general index problems. Let $\mathcal{L}$ be the trivial line bundle $\mathcal{L} = \mathbb{C}^n \times \mathbb{C}$ on $\mathbb{C}^n$, with connection one-form

$$\theta = \tfrac{1}{2}(z, d\bar{z}) - \tfrac{1}{2}(\bar{z}, dz) = i(\xi, dx) - i(x, d\xi),$$

where $z = x + i\xi$, and let D_0 be the Dirac operator on $\mathbb{C}^n$ twisted by $\mathcal{L}$. Let $a(x, \xi)$ be the symbol of an elliptic pseudodifferential operator on $\mathbb{R}^n$, which we think of as a function on $T^*\mathbb{R}^n \cong \mathbb{C}^n$. Let $\mathbb{C}^{1|1}$ be the $\mathbb{Z}/2$-graded vector space whose even and odd subspaces each have dimension 1. The vector bundle $\Lambda^{0,*}\mathbb{C}^n \otimes \mathbb{C}^{1|1} \otimes \mathcal{L}$ obtained by tensoring $\mathcal{L}$ with the trivial bundle with fibre $\Lambda^{0,*}\mathbb{C}^n \otimes \mathbb{C}^{1|1}$ is $\mathbb{Z}/2$-graded by the sum of the gradations on $\Lambda^{0,*}\mathbb{C}^n$ and $\mathbb{C}^{1|1}$. We prove that the index of the elliptic pseudodifferential operator obtained by quantizing the symbol a equals the index of the twisted Dirac operator $\begin{pmatrix} \mathsf{D}_0 & a^* \\ a & \mathsf{D}_0 \end{pmatrix}$ on the $\mathbb{Z}/2$-graded bundle $\Lambda^{0,*}\mathbb{C}^n \otimes \mathbb{C}^{1|1} \otimes \mathcal{L}$.

Our main tool is the Bargmann representation which shows, by an explicit formula, that the operation of projection onto the kernel of D_0 is a zeroth-order pseudodifferential operator on $\mathbb{C}^n$. We expect that the results of this paper likewise extend to the more general situation where $\mathbb{C}^n$ is replaced by the cotangent bundle T^*M of a Riemannian manifold M, although the proofs will be more complicated, since the analogue of the Bargmann kernel will have to be constructed by microlocal methods. We also expect that the results of this paper will lead to a new proof of Vergne's equivariant index theorem for transversally elliptic pseudodifferential operators on a Euclidean space carrying a linear representation of a compact Lie group [11].

The methods of the first part echo those of [8], in which we proved a closely related result in the setting of Toeplitz operators on CR manifolds.

In the second part of this paper, we prove a local Atiyah-Singer index theorem for generalized Dirac operators, or Dirac operators twisted by a superconnection. Our proof is almost identical to the proof of the local index theorem in [7] for Dirac operators twisted by a connection. We have not been successful in finding a proof which avoids the use of stochastic processes.

Note that proofs of similar results are sketched in Bismut [4] and Bismut-Cheeger [5]. They are also forced to use stochastic methods.

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Applying the local index theorem to calculate the index of the Dirac operator $\begin{pmatrix} \mathsf{D}_0 & a^* \\ a & \mathsf{D}_0 \end{pmatrix}$ on $\mathbb{C}^n$, we obtain a new proof of the index theorem in $\mathbb{R}^n$ (see Hörmander [10], Section 19.3, for the far more elementary proof of Fedosov).

In another paper, we will generalize the local index theorem of this paper to Dirac operators acting on infinite-dimensional vector bundles, in this way obtaining a generalization of Theorem 2.10 of Bismut-Freed [6] (who study the case where the base is the real line).

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1. The Bargmann representation and pseudodifferential operators

1.1. **Pseudodifferential operators and their symbols.** Let W be a vector space, and let $V = W \oplus W^*$ be its cotangent bundle, with symplectic form $\langle (x,\xi),(y,\eta)\rangle = \xi(y) - \eta(x)$.

If $s \geq 0$ and $m \in \mathbb{Z}$, let $\mathcal{S}^m_s(V) \subset C^\infty(V)$ be the class of symbols which satisfy the estimates

$$|\partial^\alpha a| \leq C(\alpha)\,(1 + |\cdot|)^{m - s|\alpha|},$$

where $|\cdot|$ is a norm function on V. The constants $C(\alpha)$ may be used to give $\mathcal{S}^m_s$ the structure of Fréchet space. When $s = 1$, we will write $\mathcal{S}^m$ instead of $\mathcal{S}^m_s$. For example, polynomials, and more generally, classical symbols, lie in $\mathcal{S}^m$. We refer the reader to Section 18.5 of Hörmander [10] for further details.

Recall the main technical result used in the study of pseudodifferential operators.

Proposition 1.1. *Let A be an element of $S^2(V)$ (the second symmetric power of V) whose imaginary part is positive definite. The operator $e^{iA(\partial,\partial)}$ is bounded on $\mathcal{S}^m_s$, and if $a \in \mathcal{S}^m_s$, there is an asymptotic expansion*

$$e^{iA(\partial,\partial)}a \sim \sum_{n=0}^\infty \frac{i^n}{n!} A(\partial,\partial)^n a,$$

where $A(\partial,\partial)^n a \in \mathcal{S}^{m-2ns}_s$.

Quantization is a procedure associating to a symbol $a \in \mathcal{S}^m_s(V)$ an unbounded operator a^c on $L^2(W)$: if a is a function of $W \subset V$ alone, its quantization is the operation of multiplication by a, while if a is a function of $W^* \subset V$ alone, it acts as a Fourier multiplier. Our conventions for Fourier multipliers differ from the usual one, in that we quantize the linear function ξ on $T^*\mathbb{R}$ to $i\partial_x$ and not $i^{-1}\partial_x$ as is usual: this will prove convenient when we turn to the Bargmann quantization.

There are a number of quantization procedures, two of the best known of which are the following. Here, we use Dirac's bra-ket notation for kernels: if K is an operator on W, we denote by $\langle x|K|y\rangle$ its Schwartz kernel, a distribution on $W \times W$.

(1) The classical quantization associates to the symbol $a \in \mathcal{S}^m_s$ the operator A with kernel

$$\langle x \mid a^c \mid y\rangle = (2\pi)^{-\dim(W)} \int_{W^*} a(x,\xi) e^{i\xi(y-x)}\, d\xi.$$

(2) The Weyl ordering, characterized by equivariance under conjugation by metaplectic transformations, associates to the symbol a the operator a^w with kernel

$$\langle x \mid a^w \mid y\rangle = (2\pi)^{-\dim(W)} \int_{W^*} a(\tfrac{x+y}{2}, \xi) e^{i\xi(y-x)}\, d\xi.$$

The classical symbol of a pseudodifferential operator P may be recovered by the formula

$$p(x,\xi) = (Pe^{i\xi(x-y)})(y)|_{y=x}. \tag{1}$$

These two quantizations, applied to the space of symbols $\mathcal{S}^m_s$, yield the same set of operators, which we denote by Ψ^k_s. The two quantizations are related by the formula

$$a^c = (e^{(\partial_x,\partial_\xi)/2i} a)^w. \tag{2}$$

Composition of operators induces a bounded bilinear map

$$\circ : \mathcal{S}^m_s \times \mathcal{S}^n_t \longrightarrow \mathcal{S}^{m+n}_{\min(s,t)},$$

characterized by the property that $a^w \cdot b^w = (a \circ b)^w$. Furthermore, if $a \in \mathcal{S}^m_s$ and $b \in \mathcal{S}^n_t$, then $a \circ b - ab \in \mathcal{S}^{m+n-s-t}_{\min(s,t)}$. It follows that if $a \in \mathcal{S}^m$, then

$$[p,a] = p \circ a - a \circ p \in \mathcal{S}^{m-1}_0.$$

1.2. **The Bargmann representation.** If W is a Euclidean vector space, its cotangent bundle $V = W \oplus W^*$ carries a complex structure on V, defined by the matrix

$$J = \begin{pmatrix} 0 & -F^* \\ F & 0 \end{pmatrix}$$

where $F : W \longrightarrow W^*$ is the isomorphism induced by the Euclidean structure. We will identify the element $(x,\xi) \in V$ with $x + iF\xi \in W \otimes \mathbb{C}$, which we will write simply as $x + i\xi$.

The vector space $V = W \oplus W^*$ of the last section carries a Euclidean structure, in which $|x+i\xi|^2 = |x|^2 + |\xi|^2$. Let $d\mu$ be the Gaussian measure on V

$$d\mu = \pi^{-n} e^{-|z|^2} d\,(\mathrm{vol}),$$

where $d\,(\mathrm{vol})$ is the Lebesgue measure of V, and n is the dimension of W.

Let (w,z) be the complex bilinear form on V

$$(x+i\xi, y+i\eta) = (x,y) - (\xi,\eta) + i(x,\eta) - i(\xi,y),$$

whose imaginary part is the symplectic form of V.

Let K be the operator on $L^2(d\mu)$ given by the integral

$$(Kf)(w) = \int_V e^{(w,\bar z)} f(z,\bar z)\, d\mu(z).$$

The following result is due to Bargmann [1]

Proposition 1.2. *The operator K is the projection onto the closed subspace $\mathcal{H} \subset L^2(d\mu)$ of holomorphic functions.*

Denote by $\Omega^{0,*}(d\mu)$ the space of L^2-sections of the bundle of antiholomorphic forms on V, with respect to the measure $d\mu$. The operator K extends to an operator on $\Omega^{0,*}(d\mu)$, in such a way that it commutes with exterior multiplication by constant $(0,1)$-forms. Clearly, we may identify the Hilbert space $\mathcal{H}$ with the kernel of $\bar\partial + \bar\partial^*$.

The unitary operator $U : \Omega^{0,*}(V) \longrightarrow \Omega^{0,*}(d\mu)$

$$f(z) \mapsto \pi^{n/2} e^{|z|^2/2} f(z).$$

identifies the Hilbert space $\Omega^{0,*}(d\mu)$ with the Hilbert space $\Omega^{0,*}(V)$ associated to the Lebesgue measure $d\,(\mathrm{vol})$ on V. Let P be the projection $P = U^{-1}KU$ on $L^2(V)$.

Proposition 1.3. *The operator P is a pseudodifferential operator of order zero, with classical symbol*

$$p(z,\zeta) = 2^n e^{-|z-i\zeta|^2/2} \in \mathcal{S}^0_0(T^*V).$$

Proof. Write $z = x+iy$, $w = u+iv$, and $\zeta = \xi + i\eta$. By Equation (1), the classical symbol of the operator P is given by the formula

$$\begin{aligned} p(x+iy,\xi+i\eta) &= \int_V e^{-|w|^2/2+(w,\bar{z})+|z|^2/2+i(\xi,u-x)+i(\eta,v-y)}\, d\mu(z)\big|_{z=w} \\ &= \pi^{-n} e^{i(\xi,u)-|u|^2/2+i(\eta,v)-|v|^2/2} \\ &\qquad \int_V e^{(u+iv-i\xi,x)-|x|^2/2+(v-iu-i\eta,y)-|y|^2/2}\, dx\, dy\big|_{x=u,y=v} \\ &= 2^n e^{i(\xi,u)-|u|^2/2+(u+iv-i\xi,u+iv-i\xi)^2/2}\big|_{x=u} \\ &\qquad \times e^{i(\eta,v)-|v|^2/2+(v-iu-i\eta,v-iu-i\eta)^2/2}\big|_{y=v} \\ &= 2^n e^{-|u+\eta|^2/2-|v-\xi|^2/2}\big|_{x=u,y=v}. \quad \square \end{aligned}$$

Corollary 1.4. *The Weyl and classical symbols of P are equal.*

Proof. The symbol p is invariant under the action of the operator

$$(\partial_x,\partial_\xi)+(\partial_y,\partial_\eta);$$

the corollary follows from Equation (2). $\square$

Let $Q : L^2(W) \to L^2(d\mu)$ be the operator with kernel

$$\langle z \mid Q \mid y \rangle = \pi^{-n/4} e^{-(z,z)/2+2^{1/2}(z,y)-(y,y)/2}.$$

It is easily shown that Q^*Q is the identity on $L^2(W)$, and that $QQ^* = K$ on $L^2(d\mu)$ (Bargmann [1]).

Definition 1.5. If a is a symbol in $\mathcal{S}^k_s(V)$, the Bargmann quantization of a is the operator $a^b = Q^*aQ$ on $L^2(W)$.

Proposition 1.6. *The Bargmann quantization a^b is a pseudodifferential operator with Weyl symbol $e^{-(\partial_x^2+\partial_\xi^2)/8}a$.*

Proof. Choose an orthonormal basis $\{x_i \mid 1 \le i \le n\}$ of linear forms on W, with dual basis $\{\xi_i \mid 1 \le i \le n\}$ of linear forms on W^*. We obtain the basis

$$z_i = x_i + i\xi_i \quad , \quad \bar{z}_i = x_i - i\xi_i$$

of linear forms on $V \otimes \mathbb{C}$.

Suppose that a is the monomial $\bar{z}^\alpha z^\beta$. The operator Q satisfies

$$Q \cdot \eta_i = z_i \cdot Q,$$

where

$$\eta_i = 2^{-1/2}(y_i - \partial_{y_i}).$$

From this, we see that $Q^*\bar{z}^\alpha z^\beta Q = (\eta^*)^\alpha \eta^\beta$. Thus, if a is polynomial, Q^*aQ is given by the formula

$$e^{(\eta^*,\partial_{\bar{z}})} e^{(\eta,\partial_z)}\big|_{z=\bar{z}=0} a(z,\bar{z}).$$

The Baker-Campbell-Hausdorff formula $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$ shows that

$$e^{(\eta^*,\partial_{\bar{z}})} e^{(\eta,\partial_z)} = e^{(y,\partial_x)+i(\partial_y,\partial_\xi)}\, e^{(\partial_x^2+\partial_\xi^2)/8}.$$

The proposition now follows for polynomial symbols from the fact that the Weyl quantization of a polynomial equals

$$e^{(y,\partial_x)+i(\partial_y,\partial_\xi)}\big|_{x=\xi=0}a(x,\xi).$$

We do not give the details of the extension of this proof to all symbols: it is merely a matter of expressing the above argument in terms of Gaussian oscillatory integrals. □

1.3. **A twisted Dirac operator on $\mathbb{C}^n$.** Let ε_i be the operation of exterior multiplication by $d\bar{z}_i$ on $\Omega^{0,*}(d\mu)$, and let ε_i^* be its adjoint. Thus,

$$[\varepsilon_j,\varepsilon_k^*]=2\delta_{jk} \quad , \quad [\varepsilon_j,\varepsilon_k]=[\varepsilon_j^*,\varepsilon_k^*]=0.$$

Define the Clifford action of V on $\Omega^{0,*}(d\mu)$ by

$$c(x_j)=2^{-1/2}(\varepsilon_j-\varepsilon_j^*) \quad , \quad c(\xi_j)=2^{-1/2}i(\varepsilon_j+\varepsilon_j^*).$$

Thus, $c(\bar{z}_j)=2^{1/2}\varepsilon_j$, while $c(z_j)=-2^{1/2}\varepsilon_j^*$.

Let $\mathcal{L}$ be the trivial holomorphic line bundle $V\times\mathbb{C}$ on V with metric

$$|1|^2_{\mathcal{L}}=\pi^{-n}e^{-|z|^2/2},$$

and connection one-form

$$\Theta=-\sum_{j=1}^n \bar{z}_j\,dz_j. \tag{3}$$

Proposition 1.7. *The operator $\bar\partial+\bar\partial^*$ equals $2^{-1/2}$ times the Dirac operator on W associated to the line bundle $\mathcal{L}$.*

Proof. Since $\bar\partial=\sum_{j=1}^n\varepsilon_j\partial_{\bar{z}_j}$, it follows that

$$\bar\partial^*=\sum_{j=1}^n\varepsilon_j^*(-\partial_{z_j}+\bar{z}_j)$$

and hence that

$$2^{1/2}(\bar\partial+\bar\partial^*)=\sum_{j=1}^n c(x_j)\frac{\partial}{\partial x_j}+\sum_{j=1}^n c(y_j)\frac{\partial}{\partial y_j}+2^{1/2}\sum_{j=1}^n\varepsilon_j^*\bar{z}_j.$$

The first two terms are the Dirac operator with respect to the flat connection d, while the third term is Cliford multiplication by the connection one-form Θ. □

Let $a\in\mathcal{S}^1(V)\otimes\mathrm{End}(E)$ be a symbol on V, where E is a Hermitian vector space. We say that a is elliptic if for some positive constant $\varepsilon>0$,

$$a^*a+1\geq\varepsilon|\cdot|^2,$$

where $|\cdot|$ is a norm function on V. It is an easy consequence of the pseudodifferential calculus that the operators a^c, a^w and a^b obtained by quantizing the symbol a are Fredholm from the Sobolev space $H^{2,1}(W)\otimes E$ to $L^2(W)\otimes E=H^{2,0}(W)\otimes E$, and that they all have the same index.

Let $\mathbb{C}^{1|1}$ be the graded vector space with one even basis element e_0 and one odd basis vector e_1, and let ε and ε^* be the matrices acting on $\mathbb{C}^{1|1}$,

$$\varepsilon=\begin{pmatrix}0&0\\1&0\end{pmatrix} \quad , \quad \varepsilon^*=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

One may think of $\mathbb{C}^{1|1}$ as the spinor space of $\mathbb{R}^2$.

The Hilbert space

$$\mathcal{H}=\Omega^{0,*}(d\mu)\otimes E\otimes\mathbb{C}^{1|1}$$

is the tensor product of two $\mathbb{Z}/2$-graded vector spaces, and hence is itself $\mathbb{Z}/2$-graded, by the subspaces

$$\begin{aligned}\mathcal{H}^+ &= \sum_{i \text{ even}} \Omega^{0,i}(d\mu) \otimes E \otimes \mathbb{C}e_0, \\ \mathcal{H}^- &= \sum_{i \text{ odd}} \Omega^{0,i}(d\mu) \otimes E \otimes \mathbb{C}e_1.\end{aligned}$$

The operator

$$(\bar\partial + \varepsilon a) + (\bar\partial + \varepsilon a)^* = \begin{pmatrix} \bar\partial + \bar\partial^* & a^* \\ a & \bar\partial + \bar\partial^* \end{pmatrix} \tag{4}$$

acts on $\mathcal{H}$, and is an odd operator, in the sense that it exchanges $\mathcal{H}^+$ and $\mathcal{H}^-$.

Definition 1.8. If M is an odd self-adjoint operator on $\Omega^{0,*}(d\mu) \otimes E \otimes \mathbb{C}^{1|1}$, the index of M is the integer

$$\operatorname{ind}(M) = \dim \ker_{\mathcal{H}^+}(M) - \dim \ker_{\mathcal{H}^-}(M).$$

Theorem 1.9. *If the symbol a is elliptic, the operator $\begin{pmatrix} \bar\partial + \bar\partial^* & a^* \\ a & \bar\partial + \bar\partial^* \end{pmatrix}$ is Fredholm on $\Omega^{0,*}(d\mu) \otimes E \otimes \mathbb{C}^{1|1}$, and its index equals the index of the pseudodifferential operator a^w on $L^2(W) \otimes E$.*

Proof. Conjugating by the isomorphism of Hilbert spaces U, we may transfer the problem to one involving operators on the $\mathbb{Z}/2$-graded Hilbert space $\Omega^{0,*}(V) \otimes E \otimes \mathbb{C}^{1|1}$. Denote by D_0 the operator $\mathsf{D}_0 = U^{-1}(\bar\partial + \bar\partial^*)U$ on $\Omega^{0,*}(V)$; it equals $2^{-1/2}$ times the Dirac operator on V associated to the trivial line bundle $V \times \mathbb{C}$ with connection one-form

$$\theta = \Theta + \tfrac{1}{2} \textstyle\sum_{j=1}^n (z_j \, d\bar z_j + \bar z_j \, dz_j) = \tfrac{1}{2} \sum_{j=1}^n (z_j \, d\bar z_j - \bar z_j \, dz_j).$$

The operator D_0^2 has Weyl symbol

$$\frac{1}{2}|z - i\zeta|^2 + (k - n/2) \in \mathcal{S}^2_1(T^*V),$$

where k is the operator of multiplication by k on $\Omega^{0,k}(V)$. The operator $P = U^{-1}KU$ is the projection onto the kernel of D_0.

Denote by A the operator $\varepsilon a + \varepsilon^* a^*$ acting on $\Omega^{0,*}(V) \otimes E \otimes \mathbb{C}^{1|1}$. Up to an error in $\mathcal{S}^1(T^*V)$, the symbol of the operator $(\mathsf{D}_0 + A)^2$ equals the sum of the symbols of D_0^2 and A^2. Thus, if a is an elliptic symbol in $\mathcal{S}^1_1(V)$, the operator $(\mathsf{D}_0 + A)^2$ has elliptic symbol in $\mathcal{S}^2_1(T^*V)$, and hence the operator $\mathsf{D}_0 + A$ is Fredholm.

Denote by $P^\perp$ the projection $I - P$. Since $A \in \Psi^1_1$ and $P \in \Psi^0_0$, the operator $PAP + P^\perp A P^\perp = (2P - 1)[A, P]$ is in Ψ^0_0. Thus, the operator

$$\mathsf{D}_{t,u} = t\mathsf{D}_0 + (1 - u)A + u(PAP + P^\perp A P^\perp)$$

is elliptic for all $(t, u) \in (0, \infty) \times [0, 1]$.

Lemma 1.10. *The function $f(t, u) = \operatorname{Str}(e^{-\mathsf{D}^2_{t,u}})$ is independent of $t \in (0, \infty)$ and $u \in [0, 1]$.*

Proof. Taking a derivative with respect to the t, we see that

$$\frac{\partial f(u,t)}{\partial t} = -\operatorname{Str}[\mathsf{D}_0 e^{-\mathsf{D}^2_{u,t}/2}, \mathsf{D}_{t,u} e^{-\mathsf{D}^2_{u,t}/2}],$$

which vanishes, being the supertrace of a graded commutator of two trace class operators. The vanishing of $\partial_u f(u, t)$ is proved in a similar fashion. □

Setting $t = 1$ and $u = 0$, we see that $f(t, u) = \mathrm{Str}(e^{-(\mathsf{D}_0+A)^2}) = \mathrm{ind}(\mathsf{D}_0+A)$. To complete the proof of the theorem, we will calculate $\lim_{t\longrightarrow\infty} f(t, 1)$.

Since the operator $\mathsf{D}_0 + PAP + P^\perp AP^\perp$ commutes with P, the supertrace of its heat kernel may be divided into supertraces over $\mathrm{im}(P)$ and $\mathrm{im}(P^\perp)$, the second of which decays exponentially fast as $t \longrightarrow \infty$:

$$\begin{aligned} f(t,1) &= \mathrm{Str}\big(e^{-(t\mathsf{D}_0+PAP+P^\perp AP^\perp)^2}\big) \\ &= \mathrm{Str}\big(Pe^{-(PAP)^2}\big) + \mathrm{Str}\big(P^\perp e^{-(t\mathsf{D}_0+P^\perp AP^\perp)^2}\big) \\ &= \mathrm{Str}\big(Pe^{-(PAP)^2}\big) + O(e^{-ct}). \end{aligned}$$

But $\mathrm{Str}\big(Pe^{-(PAP)^2}\big)$ is the index of the pseudodifferential operator $a^b = Q^*aQ$, which equals the index of a^w by Proposition 1.6. □

2. Generalized Dirac operators

2.1. Superconnections. Let us briefly recall Quillen's formalism of superconnections: for further details, see Sections 1.4 and 1.5 of [3].

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a $\mathbb{Z}/2$-graded vector bundle over a manifold M. The space of differential forms $\mathcal{A}^*(M, \mathcal{E})$ is $\mathbb{Z}/2$-graded, by the sum of the degree (modulo 2) as a differential form to the degree as a section of $\mathcal{E}$.

A superconnection is an operator $\mathbb{A}$ on $\mathcal{A}^*(M, \mathcal{E})$, odd with respect to the total $\mathbb{Z}/2$-grading, such that if $\alpha \in \mathcal{A}^*(M)$ and $\omega \in \mathcal{A}^*(M, \mathcal{E})$,

$$\mathbb{A}(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^{|\alpha|}\alpha \wedge \mathbb{A}\omega.$$

A superconnection $\mathbb{A}$ splits into a sum of operators

$$\mathbb{A}_{[k]} : \mathcal{A}^*(M, \mathcal{E}) \longrightarrow \mathcal{A}^{*+k}(M, \mathcal{E}),$$

such that $\mathbb{A}_{[1]}$ is a connection on the bundle $\mathcal{E}$, and $\mathbb{A}_{[k]} \in \mathcal{A}^k(M, \mathrm{End}(\mathcal{E}))$ for $k \neq 1$.

The curvature of a superconnection $\mathbb{A}$ is the operator $\mathbb{A}^2$: it is a local operator, that is, a differential form on M with values in the bundle of endomorphisms $\mathrm{End}(\mathcal{E})$.

There is a bundle map $\mathrm{Str} : \mathrm{End}(\mathcal{E}) \longrightarrow M \times \mathbb{C}$, equal on each fibre to the supertrace $\mathrm{Tr}_{\mathcal{E}^+} - \mathrm{Tr}_{\mathcal{E}^-}$. This induces a map $\mathrm{Str} : \mathcal{A}^*(M, \mathrm{End}(\mathcal{E})) \longrightarrow \mathcal{A}^*(M)$. The exponential $e^{\mathbb{A}^2}$ of the curvature $\mathbb{A}^2$ is an element of $\mathcal{A}^*(M, \mathrm{End}(\mathcal{E}))$. The Chern character of a superconnection $\mathbb{A}$ is defined by the formula

$$\mathrm{Ch}(\mathbb{A}) = \mathrm{Str}(e^{-\mathbb{A}^2}).$$

It is closed differential form of even degree.

2.2. Clifford modules. Let $\mathcal{V}$ be a Euclidean vector bundle over M, that is, an oriented real bundle with positive-definite metric. Let $C(\mathcal{V})$ be the associated bundle of Clifford algebras, generated by $\mathcal{V}$ subject to the relations

$$vw + wv = -2(v, w)$$

for sections $v, w \in \Gamma(M, \mathcal{V})$. Setting the degree of the subbundle $\mathcal{V} \subset C(\mathcal{V})$ to be odd, $C(\mathcal{V})$ is made into a bundle of $\mathbb{Z}/2$-graded algebras. There is a natural isomorphism between the bundles $\Lambda^*\mathcal{V}$ and $C(\mathcal{V})$.

Let $\{e_i | 1 \leq i \leq \mathrm{rank}(\mathcal{V})\}$ be an oriented orthonormal local frame of $\mathcal{V}$. Let $\omega_{\mathcal{V}}$ be the section of $C(\mathcal{V})$ defined by the local formula

$$\omega_{\mathcal{V}} = e_1 \dots e_{\mathrm{rank}(\mathcal{V})};$$

it is easily checked that this product is independent of the oriented orthonormal frame used in its definition, and hence defines a global section of $C(\mathcal{V})$. Note that $\omega_{\mathcal{V}}^2 = (-1)^{\mathrm{rank}(\mathcal{V})(\mathrm{rank}(V)+1)/2}$.

Definition 2.1. A $C(\mathcal{V})$-module is a graded representation $v \mapsto c(v)$ of $C(\mathcal{V})$ on a $\mathbb{Z}/2$-graded vector bundle $\mathcal{E}$.

If $\mathcal{V}$ is an even-dimensional spin-bundle, it has a spinor bundle $\mathcal{S}$; this is a $C(\mathcal{V})$-module such that $C(\mathcal{V}) \cong \operatorname{End}(\mathcal{S})$. (Any two such spinor bundles differ by a flat line bundle whose square is the trivial bundle, in other words, a $\mathbb{Z}/2$-bundle.) Any $C(\mathcal{V})$-module then $\mathcal{E}$ splits as a tensor product $\mathcal{S} \otimes \mathcal{W}$, where $\mathcal{W}$ is the super vector bundle $\operatorname{Hom}_{C(\mathcal{V})}(\mathcal{S}, \mathcal{E})$.

Let Γ denote the operator equal to ± 1 on $\mathcal{E}^{\pm}$. We will frequently make use of the fact that if $\mathcal{V}$ has odd rank, the operator

$$\sigma = i^{(\operatorname{rank}(\mathcal{V})+1)/2} \Gamma \, \omega_{\mathcal{V}}$$

is an odd operator with square $\sigma^2 = 1$, which commutes with $C(\mathcal{V})$, and hence defines an action of the Clifford algebra $C(\mathcal{V} \oplus \mathbb{R})$ on $\mathcal{E}$. This enables us to reduce the verification of results about Clifford modules to the case in which $\mathcal{V}$ has even rank.

Proposition 2.2. *Let* $\operatorname{End}_{\mathcal{V}}(\mathcal{E})$ *be the commutant (with respect to the graded commutator) of* $C(\mathcal{V})$ *in the bundle of algebras* $\operatorname{End}(\mathcal{E})$. *Then*

$$\operatorname{End}(\mathcal{E}) \cong C(\mathcal{V}) \otimes \operatorname{End}_{\mathcal{V}}(\mathcal{E}).$$

Proof. If $\mathcal{E}$ has even rank, this follows from the fact that locally, there is a Clifford module $\mathcal{S}$ (the spinor bundle) such that $C(\mathcal{V}) \cong \operatorname{End}(\mathcal{S})$. It follows that, again locally, we have a decomposition $\mathcal{E} \cong \mathcal{S} \otimes \mathcal{F}$, where the action of $C(\mathcal{V})$ on $\mathcal{F}$ is trivial, and $\mathcal{F} = \operatorname{Hom}_{\mathcal{V}}(\mathcal{S}, \mathcal{E})$. Thus, we see that $\operatorname{End}_{\mathcal{V}}(\mathcal{E}) \cong \operatorname{End}(\mathcal{F})$, and that

$$\operatorname{End}(\mathcal{E}) \cong \operatorname{End}(\mathcal{S} \otimes \mathcal{F}) \cong \operatorname{End}(\mathcal{S}) \otimes \operatorname{End}(\mathcal{F}) \cong C(\mathcal{S}) \otimes \operatorname{End}_{\mathcal{V}}(\mathcal{S}).$$

If $\mathcal{V}$ has odd rank, replace it by $\mathcal{V} \oplus \mathbb{R}$, as above. Since

$$\operatorname{End}_{\mathcal{V}}(\mathcal{E}) = \operatorname{End}_{\mathcal{V} \oplus \mathbb{R}}(\mathcal{E}) \oplus \sigma \operatorname{End}_{\mathcal{V} \oplus \mathbb{R}}(\mathcal{E}),$$

we see that

$$\begin{aligned} \operatorname{End}(\mathcal{E}) &\cong C(\mathcal{V} \oplus \mathbb{R}) \otimes \operatorname{End}_{\mathcal{V} \oplus \mathbb{R}}(\mathcal{E}) \\ &\cong \big(C(\mathcal{V}) \oplus \sigma C(\mathcal{V}) \big) \otimes \operatorname{End}_{\mathcal{V} \oplus \mathbb{R}}(\mathcal{E}) \\ &\cong C(\mathcal{V}) \otimes \operatorname{End}_{\mathcal{V}}(\mathcal{E}). \quad \square \end{aligned}$$

2.3. **Clifford superconnections.** Let $\nabla^{\mathcal{V}}$ be a connection on the Euclidean vector bundle $\mathcal{V}$ preserving the metric. This induces a connection on the Clifford bundle $C(\mathcal{V})$, compatible with the product.

Definition 2.3. A Clifford superconnection $\mathbb{A}$ on a module $\mathcal{E}$ over $C(\mathcal{V})$ is a superconnection on $\mathcal{E}$ such that if $v \in \Gamma(M, \mathcal{V})$ and $\omega \in \mathcal{A}^*(M, \mathcal{E})$,

$$\mathbb{A}(c(v)\omega) = c(\nabla^{\mathcal{V}} v)\omega - c(v)\mathbb{A}\omega.$$

If $\mathcal{V}$ is an even-dimensional spin-bundle with spinor bundle $\mathcal{S}$, then $\mathcal{S}$ inherits a connection $\nabla^{\mathcal{S}}$ from the connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}$. By Proposition 3.40 of [3], there is a bijective correspondence between Clifford superconnections $\mathbb{A}$ on $\mathcal{E}$ and superconnections $\mathbb{B}$ on $\mathcal{W} = \operatorname{Hom}_{C(\mathcal{V})}(\mathcal{S}, E)$, given by sending $\mathbb{B}$ to $\nabla^{\mathcal{S}} \otimes 1 + 1 \otimes \mathbb{B}$.

The following result is Proposition 3.43 of [3].

Proposition 2.4. *The curvature* $\mathbb{A}^2$ *of a Clifford superconnection* $\mathbb{A}$ *on* $\mathcal{E}$ *decomposes under the isomorphism* $\operatorname{End}(\mathcal{E}) \cong C(\mathcal{V}) \otimes \operatorname{End}_{\mathcal{V}}(\mathcal{E})$ *as*

$$\mathbb{A}^2 = c(R) + \mathbb{F}(\mathbb{A}),$$

where $c(R) \in \mathcal{A}^2(M, C(\mathcal{V})) \subset \mathcal{A}^2(M, \mathrm{End}(\mathcal{E}))$ *is the action of the curvature* R *of* $\mathcal{V}$ *on the bundle* $\mathcal{E}$*, given by the formula*

$$c(R)(\partial_i, \partial_j) = \frac{1}{2} \sum_{a<b} (R(\partial_i, \partial_j)e_a, e_b) c^a c^b,$$

and $\mathbb{F}(\mathbb{A}) \in \mathcal{A}^*(M, \mathrm{End}_{\mathcal{V}}(\mathcal{E}))$*.*

Proof. Since $\mathbb{A}$ is compatible with the Clifford action, we see that for $v \in \Gamma(M, \mathcal{V})$,

$$[\mathbb{A}^2, c(v)] = [\mathbb{A}, [\mathbb{A}, c(v)]] = [\mathbb{A}, c(\nabla^{\mathcal{V}} v)] = c\big((\nabla^{\mathcal{V}})^2 v\big) = c(Rv).$$

Since $c(R) = [c(R), c(v)]$, we see that $[\mathbb{A}^2, c(v)] = c(Rv) = [c(R), c(v)]$.

It follows that $\mathbb{F}(\mathbb{A}) = \mathbb{A}^2 - c(R)$ commutes with the operators $c(v)$, and hence lies in $\mathcal{A}(M, \mathrm{End}_{\mathcal{V}}(\mathcal{E}))$. □

We call $\mathbb{F}(\mathbb{A})$ the curvature of the Clifford superconnection $\mathbb{A}$; it is called the "twisting curvature" in [3]. If $\mathcal{V}$ is an even-dimensional spin bundle, with spinor bundle $\mathcal{S}$, the curvature of a Clifford superconnection $\mathbb{A}$ on a Clifford module $\mathcal{E} \cong \mathcal{S} \otimes \mathcal{W}$ equals the curvature of the corresponding superconnection on the supervector bundle $\mathcal{W}$.

Definition 2.5. If $\mathcal{E}$ is a $C(\mathcal{V})$-module, $\mathrm{Str}_{\mathcal{V}}$ is the supertrace on the bundle of algebras $\mathrm{End}_{\mathcal{V}}(\mathcal{E})$ given by the formula

$$\mathrm{Str}_{\mathcal{V}}(A) = (-4\pi)^{-\operatorname{rank}(\mathcal{V})/2} \, \mathrm{Str}_{\mathcal{E}}(\omega_{\mathcal{V}} A).$$

It takes values in the flat line bundle $\det(\mathcal{V})$.

The exponential $e^{-\mathbb{F}(\mathbb{A})}$ of $\mathbb{F}(\mathbb{A})$ is an element of $\mathcal{A}^*(M, \mathrm{End}_{\mathcal{V}}(\mathcal{E}))$. The Chern character of a Clifford superconnection $\mathbb{A}$ is defined by the formula

$$\mathrm{Ch}_{\mathcal{V}}(\mathbb{A}) = \mathrm{Str}_{\mathcal{V}}(e^{-\mathbb{F}(\mathbb{A})}).$$

It is an even (odd) closed differential form if $\operatorname{rank}(\mathcal{V})$ is even (odd). This Chern character is the relative Chern character of Section 4.1 of [3], though normalized in a fashion which is motivated by the calculations of [9].

The following proposition gives another formula for $\mathrm{Ch}_{\mathcal{V}}(\mathbb{A})$ which is sometimes more convenient.

Proposition 2.6.

$$\mathrm{Ch}_{\mathcal{V}}(\mathbb{A}) = \frac{\mathrm{Str}_{\mathcal{V}}(e^{-\mathbb{A}^2})}{\det^{1/2}(\cosh R/2)}$$

Proof. First, suppose that $\operatorname{rank}(\mathcal{V})$ is even. Let $c_{i_1} \dots c_{i_\ell}$, $i_1 < \dots < i_\ell$, be a basis element of the Clifford algebra $C(\mathcal{V})$. If $a \in \mathcal{A}^*(M, \mathrm{End}_{\mathcal{V}}(\mathcal{E}))$, then $\mathrm{Str}_{\mathcal{V}}(c_{i_1} \dots c_{i_\ell} a) = 0$ unless $k = 0$ (see [3], Proposition 3.21). The result now follows from Proposition 3.13 of [3], which shows that

$$e^{-c(R)} = \det{}^{1/2}(\cosh R/2) + \text{terms involving Clifford multiplication.}$$

If $\operatorname{rank}(\mathcal{V})$ is odd, then as above, $\mathcal{E}$ is a module for the Clifford bundle $C(\mathcal{V} \oplus \mathbb{R})$, and

$$\mathrm{Str}_{\mathcal{V}}(e^{-\mathbb{A}^2}) = i^{-(\operatorname{rank}(\mathcal{V})+1)/2} \, \mathrm{Tr}(\sigma e^{-c(R)} \, e^{-\mathbb{F}(\mathbb{A})}).$$

The only contribution to $\mathrm{Str}_{\mathcal{V}}(\exp(-\mathbb{A}^2))$ will come from the coefficient of σ in $e^{-\mathbb{F}(\mathbb{A})}$, multiplied by the coefficient of 1 in $e^{-c(R)}$, and once more the result follows. □

Define an involution on the bundle of algebras $C(\mathcal{V})$, by setting $v^* = -v$ for $v \in \Gamma(M, \mathcal{V})$. From now on, we restrict attention to $*$-modules over $C(\mathcal{V})$: a $*$-module is a $C(\mathcal{V})$-module $\mathcal{E}$, with Hermitian structures on $\mathcal{E}^{\pm}$, such that $c(v)$ is skew-adjoint for all $v \in \Gamma(M, \mathcal{V})$.

2.4. **Generalized Dirac operators.** Following Kasparov, we denote the cotangent bundle T^*M of a Riemannian manifold M by τ. Let ∇^τ be the Levi-Civita connection on τ. Denote the dimension of M by n. We will assume that M is oriented, allowing us to identify the line bundle $\det(\tau)$ with the trivial line bundle $M \times \mathbb{R}$.

Definition 2.7. Let $\mathbb{A}$ be a Clifford superconnection on the $C(\tau)$-module $\mathcal{E}$. The associated Dirac operator $\mathsf{D}^{\mathbb{A}}$ is the first-order differential operator on $\Gamma(M,\mathcal{E})$ given by composing the arrows in the following diagram:

$$\Gamma(M,\mathcal{E}) \xrightarrow{\mathbb{A}} \mathcal{A}^*(M,\mathcal{E}) \cong \Gamma(M, C(\tau)\otimes\mathcal{E}) \longrightarrow \Gamma(M,\mathcal{E}),$$

where the last map is given by the action of $C(\tau)$ on $\mathcal{E}$.

By Proposition 3.42 of [3], the assignment $\mathbb{A} \mapsto \mathsf{D}^{\mathbb{A}}$ is a bijection between Clifford superconnections $\mathbb{A}$ and first-order differential operators D such that

(1) $[\mathsf{D}, f] = c(df)$ for all $f \in C^\infty(M)$;
(2) D is odd.

Let $\mathbf{x}^i$ be a local coordinate system on M. Denote by $g^{ij} = (d\mathbf{x}^i, d\mathbf{x}^j)$ the Riemmanian metric in this local coordinate system, by r its scalar curvature, by Γ_{ijk} the Christoffel coefficients

$$\nabla^\tau_{\partial_i}\partial_j = g^{k\ell}\Gamma_{ijk}\partial_\ell$$

representing the Levi-Civita connection ∇^τ, and by $c(d\mathbf{x}^{i_1}\dots d\mathbf{x}^{i_\ell})$ the image of the differential form $d\mathbf{x}^{i_1}\dots d\mathbf{x}^{i_\ell}$ under the isomorphism $\Lambda^*\tau \cong C(\tau)$. The Clifford superconnection $\mathbb{A}$ may be written

$$\mathbb{A} = \sum_{i=1}^n d\mathbf{x}^i \otimes \big(\partial_i + \tfrac12 c(d\mathbf{x}^j d\mathbf{x}^k)\,\Gamma_{ijk}\big) + \textstyle\sum_{k=0}^n \sum_{i_1<\dots<i_k} d\mathbf{x}^{i_1}\dots d\mathbf{x}^{i_k} \otimes \omega_{i_1\dots i_k},$$

where $\omega_{i_1\dots i_k}$ is a section of $\mathrm{End}_\tau(\mathcal{E})$. We will denote the zero-form component $\mathbb{A}_{[0]}$ by $\mathcal{D}$, and the connection $\mathbb{A}_{[1]}$ by ∇.

The associated Dirac operator may be written

$$\mathsf{D}^{\mathbb{A}} = \sum_{i=1}^n c(dx^i)\big(\partial_i + \tfrac12 c(d\mathbf{x}^j d\mathbf{x}^k)\,\Gamma_{ijk}\big) + \textstyle\sum_{k=0}^n \sum_{i_1<\dots<i_k} c(d\mathbf{x}^{i_1}\dots d\mathbf{x}^{i_k})\omega_{i_1\dots i_k}.$$

The operator $\mathsf{D}^{\mathbb{A}}$ is self-adjoint if and only if the Clifford superconnection satisfies the condition

$$(\mathbb{A}_{[k]}x, y) = (-1)^{k(k+1)/2}(x, \mathbb{A}_{[k]}y)$$

for all $x, y \in \Gamma(M,\mathcal{E})$ (Proposition 3.44 of [3]). For $k = 0$, this says that $\mathcal{D} = \mathbb{A}_{[0]}$ is self-adjoint, while for $k = 1$, it states the compatibility of the connection $\mathbb{A}_{[1]}$ with the inner product on $\mathcal{E}$.

The following result generalizes Lichnerowicz's formula (Theorem 3.52 of [3]). Let $\alpha = \sum_{i=1}^n d\mathbf{x}^i\alpha_i$, where

$$\alpha_i = \sum_{k\ge2} \iota(\partial_i)\mathbb{A}_{[k]} \in \mathcal{A}^*(M, \Lambda^*\tau\otimes\mathrm{End}_\tau(\mathcal{E})).$$

Proposition 2.8. *If $\mathbb{A}$ is a Clifford superconnection, the operator $(\mathsf{D}^{\mathbb{A}})^2$ is given in local coordinates by the formula*

$$(\mathsf{D}^{\mathbb{A}})^2 = -\sum_{ij} g^{ij}\big(\nabla^\alpha_i\nabla^\alpha_j + \Gamma^k_{ij}\nabla^\alpha_k\big) + c(\mathbb{F}(\mathbb{A})) + \frac{r}{4} + P(\mathbb{A}, g),$$

where $\nabla^\alpha_i = \nabla_i + c(\alpha_i)$, and P is an invariant polynomial in the differential forms $\mathbb{A}_{[k]}$, $k \ge 2$, and Riemannian metric g on τ.

For example, if $\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}$, then the polynomial P is equal to

$$2g^{ij} c(d\mathbf{x}^k d\mathbf{x}^\ell)\, \omega_{ik}\, \omega_{j\ell} - g^{ij} g^{k\ell}\, \omega_{ik}\, \omega_{j\ell},$$

where $\mathbb{A}_{[2]} = \sum_{i<j} d\mathbf{x}^i d\mathbf{x}^j\, \omega_{ij}$.

2.5. **The heat kernel of a generalized Dirac operator.** In this section, following closely the treatment of [7], we prove a local index theorem for the Dirac operator associated to a Clifford superconnection $\mathbb{A}$ on a $C(\tau)$-module $\mathcal{E}$. The proof consists of two steps:

(1) given a point $x \in M$, we show that the asymptotics of the heat kernel of the Dirac operator $\mathsf{D}^{\mathbb{A}}$ around $x \in M$ are the same those of a heat equation on tangent space $T_x M$;
(2) having transferred the problem to the tangent space TxM, we use a rescaling argument to reduce the problem to one which can be solved explicitly.

In both of these steps, we use stochastic differential equations to make the necessary estimates. However, it is only in the second step that stochastic differential equations appear to us to be essential to the proof.

For the first step, we use the well-known fact that the asympotics of the heat kernel $\langle \exp_x \mathbf{x} | e^{-t(\mathsf{D}^{\mathbb{A}})^2} | x \rangle$ in the limit $(\mathbf{x}, t) \longrightarrow 0$ are local, in the sense that regions of M outside a ball $B_\delta(x)$ of radius $\delta > 0$ contribute an exponentially vanishing amount to the heat kernel inside this ball. For example, this follows immediately from the Feynman-Kac formula for the heat kernel as an integral with respect to a Brownian bridge measure:

$$\big\langle y \mid e^{-t(\mathsf{D}^{\mathbb{A}})^2} \mid x \big\rangle = \int_{\omega \in P_{x,y,t}(M)} \Phi(\omega, t)\, db_{x,y,t}.$$

Here, $P_{x,y,t}(M)$ is the space of all continuous paths $\omega : [0, t] \longrightarrow M$ such that $\omega_0 = x$ and $\omega_t = y$, $db_{x,y,t}$ is the Brownian bridge measure on $P_{x,y,t}(M)$ associated to the Riemmanian metric of M, and $\Phi(\omega, s) \in \operatorname{Hom}(\mathcal{E}_x, \mathcal{E}_{\omega_s})$ is the solution of the Stratanovich stochastic differential equation

$$D\Phi(\omega, s) \cdot \Phi(\omega, s)^{-1} = -c(\alpha(\omega_s)) \circ db \\ - \Big(c(\mathbb{F}(\mathbb{A}))(\omega_s) + \frac{r(\omega_s)}{4} + P(\mathbb{A}(\omega_s), g(\omega_s)) \Big)\, ds,$$

with initial condition $\Phi(\omega, 0) = I$. (Here, $D\Phi$ denotes the covariant stochastic differential with respect to the connection ∇ on the bundle $\mathcal{E}$.) For more details of this construction, see the appendix of [7].

Examination of this stochastic differential equation shows that $\Phi(\omega, t)$ is uniformly bounded by a constant depending only on the sum of supremum norms

$$\|R\| + \|F\| + \|\nabla \mathcal{D}\| + \sum_{k \geq 2} (\|\mathbb{A}_{[k]}\| + \|\nabla \mathbb{A}_{[k]}\|);$$

here, F is the curvature of the connection ∇. The supremum of $\mathcal{D}$ is not needed for this estimate, since it enters in the stochastic differential equation only through the sum

$$-\big(\mathcal{D}(\omega_s)^2 + \nabla \mathcal{D}(\omega_s) + \sum_{k \geq 2} [\mathbb{A}_{[k]}(\omega_s), \mathcal{D}(\omega_s))]\big)\, ds.$$

But $\mathcal{D}^2$ is a positive-definite section of $\operatorname{End}(\mathcal{E})$, so that

$$\mathcal{D}^2 + \nabla \mathcal{D} + \sum_{k \geq 2} [\mathbb{A}_{[k]}, \mathcal{D}] \geq -\|\nabla \mathcal{D}\| - \sum_{k \geq 2} \|\mathbb{A}_{[k]}\|^2,$$

allowing us to majorize $\Phi(\omega, t)$ without $\mathcal{D}$ being bounded. This is important since typically, $\mathcal{D}$ is indeed unbounded.

Thus, at the cost of a uniformly bounded error $O(e^{-at/\delta})$, where $a > 0$ depends only on the norm $\|R\|$ of the Riemannian metric, we may assume that the manifold M is diffeomorphic to the tangent space $V = T_xM$ at a point $x \in M$, that the Riemannian metric is a small perturbation of the Euclidean metric inside the ball $B_\delta(0) \subset V$, that the Clifford module $\mathcal{E}$ is a trivial bundle over V with fibre $E = \mathcal{E}_x$, and that the Clifford superconnection $\mathbb{A}$ equals the trivial connection d outside the ball $B_\delta(0)$. We may also assume that the exponential map from V to M by $\mathbf{x} \mapsto \exp_x \mathbf{x}$ is the identity map, and that parallel translation respect to the connection ∇ along the geodesic $t \mapsto t\mathbf{x}$ from 0 to $\mathbf{x}$ is the identity map of E, for all $\mathbf{x} \in V$.

We now turn to the second step of the proof. By means of the identifications

$$\begin{aligned} C^\infty(V, \mathrm{End}(E)) &\cong C^\infty(V, C(V) \otimes \mathrm{End}_{V^*}(E)) \cong C^\infty(V, \Lambda^*V^* \otimes \mathrm{End}_{V^*}(E)) \\ &\cong \mathcal{A}^*(V, \mathrm{End}_{V^*}(E)), \end{aligned}$$

we may think of the kernel $\langle \mathbf{x} | e^{-t(\mathsf{D}^{\mathbb{A}})^2} | 0\rangle$ as a differential form on V, denoted $\mathbf{k}_t(\mathbf{x}) \in \mathcal{A}^*(V, \mathrm{End}_{V^*}(E))$.

Let $\mathbf{k}_t(\mathbf{x}, \epsilon)$ denote the kernel analogous to $\mathbf{k}_t(\mathbf{x})$, except that the Clifford superconnection $\mathbb{A}$ is replaced by

$$\mathbb{A}(\epsilon) = \epsilon^{-1}\mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \epsilon\mathbb{A}_{[2]} + \epsilon^2\mathbb{A}_{[3]} + \dots .$$

Note that if we replace the superconnection $\mathbb{A}$ by $\mathbb{A}(\epsilon)$ in the first step, the estimates which we used are uniform in ϵ as $\epsilon \longrightarrow 0$.

The fundamental technique in the proof of the local index theorem of [7] is the rescaling of this kernel. If ω is a differential form on the vector space V, let $T_\epsilon\omega$ be the differential form

$$(T_\epsilon\omega)(\mathbf{x})_{[k]} = \epsilon^{n-k}\omega(\epsilon\mathbf{x})_{[k]}.$$

Let $\mathbf{k}_t^\epsilon(\mathbf{x})$ be the kernel $\mathbf{k}_t^\epsilon(\mathbf{x}) = T_\epsilon \mathbf{k}_{\epsilon^2 t}(\mathbf{x}, \epsilon)$, and let $\mathbf{k}_t^0(\mathbf{x})$ be the kernel

$$\begin{aligned} \mathbf{k}_t^0(\mathbf{x}) = (4\pi t)^{-n/2} \det{}^{1/2}\Big(\frac{tR(0)/2}{\sinh(tR(0)/2)}\Big) \\ \exp\Big(-\frac{1}{4t}\langle \mathbf{x} \mid \frac{tR(0)/2}{\tanh(tR(0)/2)} \mid \mathbf{x}\rangle - t\mathbb{F}(\mathbb{A})(0) + \iota(\mathcal{R})\alpha(0)\Big), \end{aligned}$$

where $R(0) \in \Lambda^2 V^* \otimes \mathrm{End}(V)$ is the curvature of M at $0 \in V$, and

$$\iota(\mathcal{R})\alpha(0) = \sum_{i=1}^n \mathbf{x}^i\, \alpha_i(0) \in \Lambda^*V^* \otimes \mathrm{End}_{V^*}(E)$$

is the pairing of the one-form $\alpha(0) \in V^* \otimes \Lambda^*V^* \otimes \mathrm{End}_{V^*}(E)$ at $0 \in V$ with the Euler vector field $\mathcal{R} = \mathbf{x}^i\partial_i$. We may now state our main result.

Theorem 2.9. *If $\mathbb{A}$ is a Clifford superconnection on a $C(\tau)$-module $\mathcal{E}$ over the Riemannian manifold M, denote by $f_{\mathbb{A}}$ the function on M obtained by adding the pointwise norms of the following geometric objects:*

(1) *the Riemannian curvature R of M and its covariant derivative;*
(2) *the curvature F of the connection $\nabla = \mathbb{A}_{[1]}$ and its covariant derivative;*
(3) *$\nabla\mathcal{D}$ and $\nabla^2\mathcal{D}$;*
(4) *$\mathbb{A}_{[k]}$, $\nabla\mathbb{A}_{[k]}$ and $\nabla^2\mathbb{A}_{[k]}$ for $k \geq 2$.*

Then for $\epsilon < 1$, $\delta > 0$ and $t < T$, we have the estimate

$$|\mathbf{k}_t^\epsilon(\mathbf{x}) - \mathbf{k}_t^0(\mathbf{x})| \leq c\,\epsilon\, t^{-n/2+1-\delta}\, e^{-|\mathbf{x}|^2/8t}\, \big\| e^{-t\mathcal{D}^2/2} \big\|,$$

where T and c are constants depending only on the dimension of M, δ and the supremum of $f_{\mathbb{A}}$ over M.

Remark 2.10. This result is quite different from the special case, proved in [7], where $\mathbb{A}$ is a connection, since the superconnection $\mathbb{A}$ is replaced by $\mathbb{A}(\epsilon)$ at the same time as t is rescaled to $\epsilon^2 t$. That this is necessary may be seen by considering the case in which M is a point: a $C(\tau)$-module is a $\mathbb{Z}/2$-graded vector space E, a Clifford superconnection $\mathbb{A}$ reduces to an odd endomorphism of E, and the heat kernel $\mathbf{k}_t$ equals $e^{-t\mathbb{A}^2}$. It is clear that if t is replaced by $\epsilon^2 t$, then $\mathbb{A}$ must be multiplied by ϵ^{-1} in order that $\mathbf{k}_t^\epsilon(\mathbb{A})$ have a limit.

Proof. The $\mathbb{Z}/2$-graded vector space $\Lambda^* V^*$ is a $C(V^*)$-module, with respect to the action $c(\alpha) = \varepsilon(\alpha) - \varepsilon^*(\alpha)$. Using this action, we may transfer the operator $(\mathsf{D}^{\mathbb{A}})^2$ to an operator $\mathbb{L}$ acting on $\mathcal{A}^*(V, \operatorname{End}_{V^*}(E))$, and $\mathbf{k}_t(\mathbf{x})$ may be characterized as solving the heat equation for $\mathbb{L}$ with initial condition

$$\lim_{t \longrightarrow 0} \mathbf{k}_t(\mathbf{x}) = \delta(\mathbf{x}).$$

Similarly, there is a family of operators $\mathbb{L}^\epsilon$ acting on $\mathcal{A}^*(V, \operatorname{End}_{V^*}(E))$, such that $\mathbf{k}_t^\epsilon(\mathbf{x})$ is the solution of the heat equation for $\mathbb{L}^\epsilon$ with the same initial condition as $\mathbf{k}_t(\mathbf{x})$. To define $\mathbb{L}^\epsilon$, we repeat the definition of $\mathbb{L}$, replacing the superconnection $\mathbb{A}$ by $\mathbb{A}(\epsilon)$, conjugating the resulting operator on $\mathcal{A}^*(V, \operatorname{End}_{V^*}(E))$ by T_ϵ, and finally multiplying by ϵ. To obtain an explicit formula for $\mathbb{L}^\epsilon$, introduce the rescaled Clifford action $c_\epsilon(\alpha) = \varepsilon(\alpha) - \epsilon^2 \varepsilon^*(\alpha)$, and the connection on the trivial bundle over V with fibre E given by the formula

$$\nabla^\epsilon = \sum_{i=1}^n d\mathbf{x}^i \otimes \Big(\partial_i + \frac{1}{2\epsilon} c_\epsilon(d\mathbf{x}^j d\mathbf{x}^k)\, \Gamma_{ijk}(\epsilon \mathbf{x}) + \epsilon \omega_i(\epsilon \mathbf{x}) + \alpha_i(\epsilon \mathbf{x}) \Big).$$

Then we see that

$$\begin{aligned} \mathbb{L}^\epsilon = - \sum_{ij} g^{ij}(\epsilon \mathbf{x}) \big(\nabla_i^\epsilon \nabla_j^\epsilon + \epsilon \Gamma_{ij}^k(\epsilon \mathbf{x}) \nabla_k^\epsilon \big) \\ + c_\epsilon(\mathbb{F}(\mathbb{A})(\epsilon \mathbf{x})) + \frac{\epsilon^2}{4} r(\epsilon \mathbf{x}) + \epsilon P(\mathbb{A}(\epsilon), g)(\epsilon \mathbf{x}), \end{aligned}$$

Generalizing the proof of Proposition 4.19 in [3], we see that $\mathbb{L}^\epsilon$ converges as $\epsilon \longrightarrow 0$ to the operator

$$\mathbb{L}^0 = - \sum_{i=1}^n \Big(\partial_i - \frac{1}{4} \sum_{j=1}^n R_{ij}(0) \mathbf{x}^j + \alpha_i(0) \Big)^2 + \mathbb{F}(\mathbb{A})(0).$$

Lemma 2.11. *The kernel $\mathbf{k}_t^0(\mathbf{x})$ is the solution of the heat equation for $\mathbb{L}^0$ with initial condition $\lim_{t \longrightarrow 0} \mathbf{k}_t^0(\mathbf{x}) = \delta(\mathbf{x})$.*

Proof. This follows the observation that

$$e^{\iota(\mathcal{R})\alpha(0)} \mathbb{L}^0 e^{-\iota(\mathcal{R})\alpha(0)} = - \sum_{i=1}^n \Big(\partial_i - \frac{1}{4} \sum_{j=1}^n R_{ij}(0) \mathbf{x}^j \Big)^2 + \mathbb{F}(\mathbb{A}).$$

It is shown in Theorem 4.20 of [3] that the heat equation for the operator on the right-hand side of this equation equals $e^{\iota(\mathcal{R})\alpha(0)} \mathbf{k}_t^0(\mathbf{x})$. □

The following estimate is a generalization of a theorem of [7].

Lemma 2.12. *Uniformly in $\mathbf{x}$ and small t,*

$$\big\| (\mathbb{L}^\epsilon - \mathbb{L}^0) \mathbf{k}_t^0(\mathbf{x}) \big\| \le c\, \epsilon\, t^{-n/2} e^{-|\mathbf{x}|^2/8t} \, \big\| e^{-t\mathcal{D}^2(0)/2} \big\|,$$

where the constant c depends only on $f_{\mathbb{A}}(0)$.

Proof. For multi-indices $\alpha, \beta \geq 0$, we have the estimate

$$\|\mathbf{x}^\alpha \partial^\beta \mathbf{k}_t^0(\mathbf{x})\| \leq c\, t^{(|\alpha|-|\beta|-n)/2}\, e^{-|\mathbf{x}|^2/8t}\, \|e^{-t\mathcal{D}^2(0)/2}\|,$$

where the constant c depends only on α, β, and the norms of $\Omega(0)$, $\alpha(0)$ and $\mathcal{F}(\mathbb{A})_{[k]}(0)$, $k > 0$. The lemma follows by a straightforward, if lengthy, estimation of the family of differential operators $\mathbb{L}^\epsilon - \mathbb{L}^0$. □

Duhamel's formula now shows that

$$\mathbf{k}_t^\epsilon(\mathbf{x}) - \mathbf{k}_t^0(\mathbf{x}) = \int_0^t \int_V \langle \mathbf{x} | e^{-s\mathbb{L}^\epsilon} | \mathbf{y} \rangle\, (\mathbb{L}^0 - \mathbb{L}^\epsilon) \mathbf{k}_{t-s}^0(\mathbf{y})\, d\mathbf{y}\, ds.$$

The proof of Theorem 2.9 is an immediate consequence of Appendix A of [7]. □

The following result extends the local index theorem of Patodi to generalized Dirac operators.

Corollary 2.13. *Let $\mathbb{A}$ be a Clifford superconnection such that*

(1) *$\mathbb{A}$ satisfies the conditions of Theorem 2.9;*
(2) *$\|e^{-\mathcal{D}^2}\|$ is integrable.*

Then the index of the generalized Dirac operator $\mathsf{D}^{\mathbb{A}}$ is given by the absolutely convergent integral

$$\operatorname{ind}(\mathsf{D}^{\mathbb{A}}) = \int_M \det{}^{1/2}\Big(\frac{R/2}{\sinh(R/2)}\Big)\, \mathrm{Ch}_\tau(\mathbb{A}).$$

Proof. This follows from Theorem 2.9 by the same argument as is employed in the proof of Theorem 4.2 of [3]. □

We may apply this corollary to the operator $(\bar\partial + \varepsilon a) + (\bar\partial + a)^*$ of Equation (4). This is proportional to the generalized Dirac operator associated to the Clifford superconnection

$$\mathbb{A} = d + \begin{pmatrix} \Theta & 2^{1/2} a^* \\ 2^{1/2} a & \Theta \end{pmatrix}$$

on the $C(\tau)$-bundle $\Lambda^{0,*}\tau \otimes E \otimes \mathbb{C}^{1|1}$ over V, where Θ is the connection one-form of Equation (3). It is easily seen that this superconnection verifies the conditions of Corollary 2.13: the curvature of the connection $d + \Theta$ is constant, while the essential hypothesis, that the functions $\|e^{-2a^*a}\|$ and $\|e^{-2aa^*}\|$ are absolutely convergent, is equivalent to the assumption that the symbol $a \in \mathcal{S}^1(V)$ is elliptic. Combining Theorem 2.13, which gives an explicit formula for the index of the generalized Dirac operator $\mathsf{D}^{\mathbb{A}}$, with Theorem 1.9, we obtain Fedosov's index theorem, in the form

$$\operatorname{ind}(a^w) = \int_V \mathrm{Ch}_\tau(\mathbb{A}).$$

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