SYMMETRIES OF THE N-BODY PROBLEM

JOSH FLECKNER, EVAN HUANG, BRENNAN JACKSON, JEN TANG

Abstract. write this last

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1. Introduction

The n-body problem is a well-known problem in celestial and classical mechanics, where we solve for the trajectories of n bodies interacting under their mutual gravitational attraction given their initial positions and velocities. A frequently-studied example is the 3-body system of the Sun, Moon, and Earth.

This document is a product of the 2023 Dynamics REU at Northwestern University, and in the following paragraphs, we will explore two questions in dynamical systems motivated by the n-body problem. The first question is presented in Section 2 (Part I), where we search for a novel proof for the non-integrability of the planar 3-body problem. The second question is presented in Section 3 (Part II), where we develop an analogue of Floquet theory for systems with quasi-periodic flows.

Before we dive into the specifics of the two questions, we will present some prerequisite knowledge relevant to both topics about Hamiltonian systems, linearization of autonomous differential equations, and Floquent theory. In writing this document, we assume that the reader is familiar with undergraduate real analysis, ordinary differential equations, abstract linear algebra, and high-school-level classical mechanics.

1.1. Hamiltonian Systems.

In this section, we will define and introduce the properties of a Hamiltonian system.

Definition 1.1. Let n be a positive integer. A *Hamiltonian system* on $\mathbb{R}^{2n}$ is specified by a smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$, called a *Hamiltonian*. For a Hamiltonian system on $\mathbb{R}^{2n}$, the coordinates on $\mathbb{R}^{2n}$ are denoted by $q_1, \ldots, q_n, p_1, \ldots, p_n$. It is common for $q_1, \ldots, q_n$ to denote positions of n particles in a physical system, and for $p_1, \ldots, p_n$ to denote the corresponding momenta. We often refer to $\mathbb{R}^{2n}$ as the *phase space* of the Hamiltonian system.

A Hamiltonian arising from a physical system usually takes the form

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = K(p_1, \ldots, p_n) + U(q_1, \ldots, q_n). \tag{1.2}$$

Here, K is the kinetic energy and U is the potential energy.

Example 1.3. For example, the Hamiltonian of a single free particle ($n = 1$) of mass m constrained to move along a line is given by

$$H(q, p) = \frac{p^2}{2m}.$$

Hamiltonian systems evolve with time according to *Hamilton's equations*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n. \tag{1.4}$$

Solutions in the form of $\varphi_H^t : \mathbb{R} \to \mathbb{R}^{2n}$ of (1.4) are sometimes called *trajectories* of H. If $q_1, \ldots, q_n$ are Cartesian coordinates on phase space and the Hamiltonian is of the form (1.2) with the usual kinetic energy

$$K(p_1, \ldots, p_n) = \frac{p_1^2}{2m_1} + \cdots + \frac{p_n^2}{2m_n},$$

then Hamilton's equations read

$$\dot{q}_i = \frac{p_i}{m_i}, \quad \dot{p}_i = -\frac{\partial U}{\partial q_i}, \quad i = 1, \ldots, n.$$

This is simply Newton's second law for a conservative force with potential U.

Another way of thinking about Hamilton's equations is with vector fields.

Definition 1.5. Given any smooth function $F : \mathbb{R}^{2n} \to \mathbb{R}$, the *Hamiltonian vector field* X_F on $\mathbb{R}^{2n}$ induced by F is

$$X_F = \left(\frac{\partial F}{\partial p_1}, \dots, \frac{\partial F}{\partial p_n}, -\frac{\partial F}{\partial q_1}, \dots, -\frac{\partial F}{\partial q_n} \right)^T. \tag{1.6}$$

If

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is the *standard symplectic matrix* of dimension $2n$, then

$$X_F = J\nabla F, \tag{1.7}$$

where ∇F is the ordinary gradient on $\mathbb{R}^{2n}$. The operator $J\nabla$ is often called the *symplectic gradient*. From the Hamiltonian vector field X_F, we can write down the autonomous differential equation

$$\dot{x} = X_F(x),$$

where $x = (q_1, \dots, q_n, p_1, \dots, p_n)^T$ are the coordinates on $\mathbb{R}^{2n}$. With every Hamiltonian vector field, there is an associated *flow* $\varphi_F^t(w)$, defined as a smooth function $\varphi_F : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying

$$\dot{\varphi}_F^t(w) = X_F(\varphi_F^t(w)), \quad \varphi_F^0(w) = w. \tag{1.8}$$

By uniqueness of solutions of ordinary differential equations, the flow φ_F^t satisfies the group property $\varphi_F^t \circ \varphi^s(w) = \varphi_F^{t+s}(w)$.

With this notation in mind, Hamilton's equations can be written concisely as

$$\dot{x} = X_H(x). \tag{1.9}$$

The flow $\varphi_H^t(w)$ is precisely the solution of Hamilton's equations (1.4) with initial condition w.

Example 1.10. The *angular momentum* of k particles in the plane $\mathbb{R}^2$ is given by

$$L(q_1, \dots, q_k, p_1, \dots, p_k) = \sum_{i=1}^{k} (q_{k1}p_{k2} - q_{k2}p_{k1}).$$

Here, $q_1, \dots, q_k \in \mathbb{R}^2$ are the positions of the particles and $p_1, \dots, p_k \in \mathbb{R}^2$ are the corresponding linear momenta. Let us find the flow $\varphi_L^t(w)$ induced by L on the phase space $\mathbb{R}^{4k}$. The Hamiltonian vector field X_L is given by

$$X_L = J\nabla L = (-q_{12}, q_{11}, \dots, -q_{k2}, q_{k1}, -p_{12}, p_{11}, \dots, -p_{k2}, p_{k1})^T$$

The autonomous differential equation associated to this vector field is $\dot{x} = X_L(x)$. Written out in components, we see that

$$\begin{aligned} \dot{q}_{11} &= -q_{12} \quad \dot{p}_{11} = -p_{12} \\ \dot{q}_{12} &= q_{11} \quad \dot{p}_{12} = p_{11} \\ &\vdots \\ \dot{q}_{k1} &= -q_{k2} \quad \dot{p}_{k1} = -p_{k2} \\ \dot{q}_{k2} &= q_{k1} \quad \dot{p}_{k2} = p_{k1}. \end{aligned}$$

This differential equation is really just $2k$ identical copies of the two-dimensional equation

$$\dot{\xi} = -\eta$$
$$\dot{\eta} = \xi,$$

which has solution

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}.$$

Hence the flow $\varphi_L^t(w)$ is given by

$$\varphi_L^t(w) = \begin{pmatrix} \mathcal{R}_t & & \\ & \ddots & \\ & & \mathcal{R}_t \end{pmatrix} w, \quad \mathcal{R}_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

This can be written more compactly as $\varphi_L^t(w) = R_t w$, with $R_t = \mathcal{R}_t^{\oplus 2k}$.

1.2. **Linearization of autonomous differential equations.** Consider the following autonomous differential equation

$$\dot{x} = f(x), \tag{1.11}$$

where $x \in \mathbb{R}^k$. This equation determines a flow $\varphi^t(w)$ on $\mathbb{R}^k$, which satisfies

$$\dot{\varphi}^t(w) = f(\varphi^t(w)), \quad \varphi^0(w) = w.$$

For each $i = 1, \ldots, k$, we have that

$$\frac{\partial \dot{\varphi}^t}{\partial w_i}(w) = \frac{\partial}{\partial w_i} f(\varphi^t(w))$$
$$\frac{d}{dt}\frac{\partial \varphi^t}{\partial w_i}(w) = Df(\varphi^t(w))\frac{\partial \varphi^t}{\partial w_i}(w).$$

This shows that the column vectors

$$\frac{\partial \varphi^t}{\partial w_i}(w), \quad i = 1, \ldots, k$$

all satisfy the *variational equation*

$$\dot{y} = Df(\varphi^t(w))y. \tag{1.12}$$

The variational equation arises as the linearization of the autonomous equation (1.11) along the flow $\varphi^t(w)$.

Let $Y(t, w)$ be the k-by-k matrix with column vectors

$$\frac{\partial \varphi^t}{\partial w_1}(w), \ldots, \frac{\partial \varphi^t}{\partial w_k}(w).$$

We may write this more concisely as

$$Y(t, w) = D\varphi^t(w). \tag{1.13}$$

Note that $Y(t, w)$ also satisfies the variational equation (1.12). Moreover, $Y(0, w) = I_k$ because $\varphi^0(w) = w$.

1.3. **Floquet theory and the monodromy matrix.** *Floquet theory* describes the structure of solutions to linear differential equations with periodic coefficients of the form

$$\dot{z} = A(t)z, \quad A(t+T) = A(t), \quad z \in \mathbb{R}^k. \tag{1.14}$$

Definition 1.15. The *principal fundamental matrix* of (1.14) is the k-by-k matrix $Z(t)$ satisfying

$$\dot{Z} = A(t)Z, \quad Z(0) = I_k.$$

From the principal fundamental matrix $Z(t)$, we get the *monodromy matrix* of (1.14):

$$M = Z(T).$$

The eigenvalues of the monodromy matrix are called *characteristic multipliers* (or sometimes *Floquet multipliers*) and play an important role in the stability of (1.14). For more details, see [6], Section 3.5.

When does a situation like (1.14) arise? Consider again the autonomous differential equation (1.11) with its flow $\varphi^t(w)$ and the associated matrix $Y(t, w)$. Suppose $w_0 \in \mathbb{R}^k$ is such that $\varphi^T(w_0) = w_0$ for some $T > 0$. Then for all t,

$$\varphi^{t+T}(w_0) = \varphi^t \circ \varphi^T(w_0) = \varphi^t(w_0),$$

so the flow $\varphi^t(w_0)$ is T-periodic. It follows that the matrix $Df(\varphi^t(w_0))$ appearing in the variational equation (1.12) is T-periodic, which puts us in the setting of Floquet theory. From the previous section, $Y(t, w) = D\varphi^t(w)$ is the principal fundamental matrix solution of (1.12), and hence $Y(T, w_0) = D\varphi^T(w_0)$ is the monodromy matrix M. In summary, we have proved:

Theorem 1.16. *Let $\dot{x} = f(x)$ be an autonomous differential equation with $x \in \mathbb{R}^k$, and let $\varphi^t(w)$ be the corresponding flow on $\mathbb{R}^k$. Suppose there is a $w_0 \in \mathbb{R}^k$ such that $\varphi^t(w_0)$ is T-periodic for some $T > 0$. Then the variational equation*

$$\dot{y} = Df(\varphi_t(w_0))y$$

has T-periodic coefficients, and the monodromy matrix is given by $M = D\varphi^T(w_0)$.

2. Part I: Non-integrability of the planar 3-body problem

2.1. **Motivation.** As a system of differential equations, our highest hope for the N-body problem is to find the general solution, i.e. the associated flow in its entirety. This problem is solved for $N \leq 2$, but for $N \geq 3$, the problem is in fact non-integrable. We will define integrability more rigorously in the next subsection, but this essentially means we don't have enough information to solve the systems of equations generally, and the best we can do is to find particular solutions (i.e. $\varphi_H^t(w_0)$ where w_0 is a fixed point in phase space).

Poincaré provided the first proof of the non-integrability of the 3-body problem using complex contour integrals, and in this section, we incorporate numerical methods and construct an alternative proof of the same fact in the planar subcase, which means we assume the solution lies in a two-dimensional plane. With further work, we hope to write an analogous proof for the full 3-body problem and extend our methods to apply more generally. Possible ideas for a more general proof and partial results are detailed in the last subsections.

Proving whether a Hamiltonian system is integrable or not is a fundamental problem in classical and celestial mechanics. If we find that a system is integrable,

then we try to construct as many first integrals as there are degrees of freedom to solve the system generally. If we show that a system is non-integrable, then we know that finding a general solution is not possible and we should study the system via some other methods. As a result, proving the non-integrability of the 3-body problem is a classic problem. Poincaré's original proof is rather lengthy and computationally heavy; therefore, in this document, we hope to provide a more accessible proof with some help from modern computational technology.

2.2. **Poisson bracket and first integrals.** Let H be a Hamiltonian system on the phase space $\mathbb{R}^{2n}$.

Definition 2.1. A smooth function $F : \mathbb{R}^{2n} \to \mathbb{R}$ is said to be a *first integral* of H if F is constant on all trajectories of H, i.e. $\frac{d}{dt}F(\varphi(t)) = 0$ for all solutions φ.

For any such $\varphi = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, we see that

$$\frac{d}{dt}F(\varphi(t)) = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right) = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right). \tag{2.2}$$

Definition 2.3. The quantity on the right side of (2.2) is called the *Poisson bracket* of H and F. For any two smooth functions $F_1, F_2 : \mathbb{R}^{2n} \to \mathbb{R}$, their Poisson bracket is defined as follows:

$$\{F_1, F_2\} = \sum_{i=1}^{n}\left(\frac{\partial F_1}{\partial q_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F_1}{\partial p_i}\frac{\partial F_2}{\partial q_i}\right). \tag{2.4}$$

Using (2.2), we see that F is a first integral of H if and only if their Poisson bracket vanishes: $\{F, H\} = 0$.

Let $C^\infty(\mathbb{R}^{2n})$ denote the $\mathbb{R}$-algebra of smooth functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$. If $f, g, h \in C^\infty(\mathbb{R}^{2n})$ and $c \in \mathbb{R}$, it can be checked that the Poisson bracket has the following properties:

$$\{f, g\} = -\{g, f\} \text{ (antisymmetry)}$$
$$\{f + cg, h\} = \{f, h\} + c\{g, h\} \text{ (linearity)}$$
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \text{ (Jacobi identity)}$$
$$\{f, gh\} = g\{f, h\} + \{f, g\}h \text{ (product rule)}.$$

The first three properties imply that the Poisson bracket endows $C^\infty(\mathbb{R}^{2n})$ (thought of as a real vector space) with the structure of a *Lie algebra*, and the last property is useful in computations. Together, these four properties show that the Poisson bracket endows $C^\infty(\mathbb{R}^{2n})$ with the structure of a *Poisson algebra*.

Note that the coordinate functions $q_1, \ldots, q_n, p_1, \ldots, p_n$ satisfy the *canonical Poisson bracket relations*:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}. \tag{2.5}$$

If F and G are first integrals of H, then the Jacobi identity implies that $\{F, G\}$ is also a first integral of H. Moreover, the antisymmetry property implies that $\{H, H\} = 0$, so that the Hamiltonian is always a first integral.

There is a relationship between the Poisson bracket on $C^\infty(\mathbb{R}^{2n})$ and the Lie bracket for vector fields on $\mathbb{R}^{2n}$: if $F, G \in C^\infty(\mathbb{R}^{2n})$ (they don't need to be first integrals) and X_F, X_G are the associated Hamiltonian vector fields, then

$$-[X_F, X_G] = (DX_F)X_G - (DX_G)X_F = X_{\{F,G\}}. \tag{2.6}$$

This can be verified directly in coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ (alternatively, see [1], Section 39 C).

2.3. **Proof sketch of non-integrability of the planar 3-body system.** In this subsection, we will first discuss what it means for a Hamiltonian system to be integrable, then provide sketch of our alternative proof.

If we fix the initial condition w_0, then the image of any smooth trajectory $\varphi_H^t(w_0)$ will be a one dimensional smooth submanifold of phase space. We call this image the *orbit* of the point w_0, and fulling solving a Hamiltonian system means finding these orbits generally. First integrals help us locate the orbits, and here's how:

First integrals are conserved physical quantities, which means that any particular orbit must lie in a level set of any first integral of the system. As a result, if we have multiple first integrals in a Hamiltonian system, we know that one particular trajectory must lie in the intersection of the level sets of all first integrals of the system. If these level sets intersect in a nice way, for every first integral we have, we can successfully knock down the dimension of the set in which our orbit lies by one. Then, in theory, if we can find $2n-1$ first integrals with nice level set intersection, we can reduce the potential trajectory from the phase space of $\mathbb{R}^{2n}$ to just a one dimensional submanifold, which means we would have solved the system. This is what it roughly means for a Hamiltonian system to be integrable.

The technical term for "nice level set intersection" is transversal intersection, which is implied by *functional independence* of the set of first integrals. We refer the interested readers to [2], Chapters 5 and 6 for more details.

Definition 2.7. A set of functions $F_1, ..., F_k : \mathbb{R}^m \to \mathbb{R}$ are *functionally independent* on $A \subset \mathbb{R}^m$ if at all $p \in A$, the vectors $dF_1(p), \ldots, dF_k(p)$ are linearly independent.

Functional independence is a rather foundational assumption, so with future sets of first integrals, we will in general assume that they are functionally independent.

If in addition to functional independence, we assume the set of first integrals are *in involution*, which simply means their pairwise Poisson brackets are zero, we obtain a stronger result named the Liouville-Arnold Theorem (see [1], Sections 49-50). This theorem essentially says that with the additional condition of in involution, a Hamiltonian system can be solvable with only n first integrals. This gives another way for a Hamiltonian system to be integrable, called *Liouville integrable* or *completely integrable*.

In summary, for a Hamiltonian system to be integrable, we need either $2n-1$ first integrals or n first integrals with some additional conditions. Therefore, to prove that a system is non-integrable, we simply need to prove an upper bound for the number of first integrals. This is essentially the goal of our proof, which will be sketched out in the next paragraph.

We first start with a periodic solution of the planar 3-body problem and reduce the differential equations to the case where the center of mass is fixed. We then show that in this set up, there exists two known first integral which are functionally independent, namely the Hamiltonian itself of the total energy and the angular momentum. We will prove that for every additional first integral of our system, we will obtain an additional eigenvector of eigenvalue one in the monodromy matrix associated with our periodic solution. We will then numerically approximate this monodromy matrix using Mathematica to show that the its eigenspace for eigenvalue one is only 2-dimensional, making it impossible for our system to have a third

additional first integral. This will then show that in our 8-dimensional phase space, we only have two first integrals, making the system non-integrable.

2.4. **The planar 3-body problem.** Consider three particles of mass m_1, m_2, m_3 with positions $q_1, q_2, q_3 \in \mathbb{R}^2$ and linear momenta $p_1, p_2, p_3 \in \mathbb{R}^2$. We fix the center of mass at the origin, so that $m_1q_1 + m_2q_2 + m_3q_3 = 0$. Solving for q_3 gives

$$q_3 = -\frac{m_1q_1 + m_2q_2}{m_3}.$$

Using this, we see that the kinetic energy takes the form

$$\begin{aligned} T &= \frac{1}{2}m_1|\dot{q}_1|^2 + \frac{1}{2}m_2|\dot{q}_2|^2 + \frac{1}{2}m_3|\dot{q}_3|^2 \\ &= \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_3}\right)|\dot{q}_1|^2 + \frac{1}{2}\left(m_2 + \frac{m_2^2}{m_3}\right)|\dot{q}_2|^2 + \frac{m_1m_2}{m_3}\dot{q}_1 \cdot \dot{q}_2 \end{aligned}$$

This determines the Lagrangian $\mathcal{L} = T - U$ for the system, and the generalized momenta are given by

$$\begin{aligned} p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{q}_1} = \left(m_1 + \frac{m_1^2}{m_3}\right)\dot{q}_1 + \frac{m_1m_2}{m_3}\dot{q}_2 \\ p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{q}_2} = \left(m_2 + \frac{m_2^2}{m_3}\right)\dot{q}_2 + \frac{m_1m_2}{m_3}\dot{q}_1 \end{aligned}$$

Solving for $\dot{q}_1, \dot{q}_2$ in terms of p_1, p_2 and inserting this into T, we see that

$$T(p_1, p_2) = \frac{m_2 + m_3}{2m_1m_0}|p_1|^2 + \frac{m_1 + m_3}{2m_2m_0}|p_2|^2 - \frac{1}{m_0}p_1 \cdot p_2,$$

where $m_0 = m_1 + m_2 + m_3$ is the total mass. With this, the Hamiltonian for the planar 3-body system with fixed center of mass is given by

$$H(q_1, q_2, p_1, p_2) = T(p_1, p_2) + U(q_1, q_2),$$

where

$$\begin{aligned} U(q_1, q_2) &= -\frac{m_1m_2}{|q_1 - q_2|} - \frac{m_1m_3}{|q_1 - q_3|} - \frac{m_2m_3}{|q_2 - q_3|} \\ &= -\frac{m_1m_2}{|q_1 - q_2|} - \frac{m_1m_3^2}{|(m_1 + m_3)q_1 + m_2q_2|} - \frac{m_2m_3^2}{|m_1q_1 + (m_2 + m_3)q_2|} \end{aligned}$$

is the potential energy. The phase space for this Hamiltonian has dimension 8. For more details on Lagrangians and Hamiltonians in classical mechanics, see [5], Chapters 7 and 13.

Given any values of the masses m_1, m_2, m_3, there is a periodic solution of Hamilton's equations where the masses rotate around their center of mass, which is fixed at the origin, and the masses form the vertices of an equilateral triangle. For more on these types of configurations, see [3], Chapter 2 (and especially Proposition 2.8.6).

We give the explicit form of this periodic orbit: set $r^3 = m_0/2$ and $\omega = \sqrt{2}$. Define $a_1, a_2 \in \mathbb{R}^2$ by

$$a_1 = \frac{r}{\sqrt{3}} \begin{pmatrix} \frac{\sqrt{3}}{2}\frac{m_2 - m_3}{m_0} \\ 1 - \frac{1}{m_0}\left(m_1 - \frac{1}{2}(m_2 + m_3)\right) \end{pmatrix}$$
$$a_2 = \frac{r}{\sqrt{3}} \begin{pmatrix} -\frac{\sqrt{3}}{2}\frac{m_1 + 2m_3}{m_0} \\ -\frac{1}{2} - \frac{1}{m_0}\left(m_1 - \frac{1}{2}(m_2 + m_3)\right) \end{pmatrix},$$

which are two vertices of an equilateral triangle of side length r in the plane. If we think of the particle at position a_i as having mass m_i, then this arrangement of particles has its center of mass at the origin (so the third vertex a_3 can be found by solving $m_1a_1 + m_2a_2 + m_3a_3 = 0$). Our periodic solution of Hamilton's equations then takes the form

$$q_i(t) = \exp(-\omega J_2 t)a_i, \quad p_i(t) = -m_i\omega J_2 \exp(-\omega J_2 t)a_i, \quad i = 1, 2. \tag{2.8}$$

Here,

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \exp(-\omega J_2 t) = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}.$$

Call this solution $\gamma(t)$, and note that $\gamma(t)$ is T-periodic with $T = 2\pi/\omega$.

By choosing units where the total mass of the system is 1, we may assume that $m_0 = 1$. This simplifies the form of the periodic solution, and the q_1, q_2 components of $\gamma(t)$ are explicitly given by

$$q_{11}(t) = \frac{r}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}(m_2 - m_3)\cos\omega t - \frac{3}{2}(m_2 + m_3)\sin\omega t\right]$$
$$q_{12}(t) = \frac{r}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}(m_2 - m_3)\sin\omega t + \frac{3}{2}(m_2 + m_3)\cos\omega t\right]$$
$$q_{21}(t) = \frac{r}{\sqrt{3}}\left[-\frac{\sqrt{3}}{2}(m_1 + 2m_3)\cos\omega t + \frac{3}{2}m_1\sin\omega t\right]$$
$$q_{22}(t) = -\frac{r}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}(m_1 + 2m_3)\sin\omega t + \frac{3}{2}m_1\cos\omega t\right]$$

Using computational software, such as Mathematica, one can compute the components $DX_H(\gamma(t)) = J\mathrm{Hess}(H)(\gamma(t))$ appearing in the variational equation (2.15). If $Y(t)$ is the principal fundamental matrix solution to

$$\dot{y} = J\mathrm{Hess}(H)(\gamma(t))y, \tag{2.9}$$

then one can compute the monodromy matrix $M = Y(T) = Y(2\pi/\omega)$ by numerically solving (2.9). At this point, we are free to choose any values of m_1, m_2, m_3 that sum to 1, so we work with equal masses $m_1 = m_2 = m_3 = 1/3$ for simplicity. Under these conditions, M has eigenvalues

$$85.0197, 85.0197, 1, 1, 1, 1, 0.011762, 0.011762.$$

The first integrals of energy (the Hamiltonian H) and angular momentum

$$L = q_1 \wedge p_1 + q_2 \wedge p_2 = q_{11}p_{12} - q_{12}p_{11} + q_{21}p_{22} - q_{22}p_{21}$$

explain the four eigenvalues of 1. The eigenspace of M corresponding to the eigenvalue of 1 has dimension 2, which is to be expected (using the results of the following

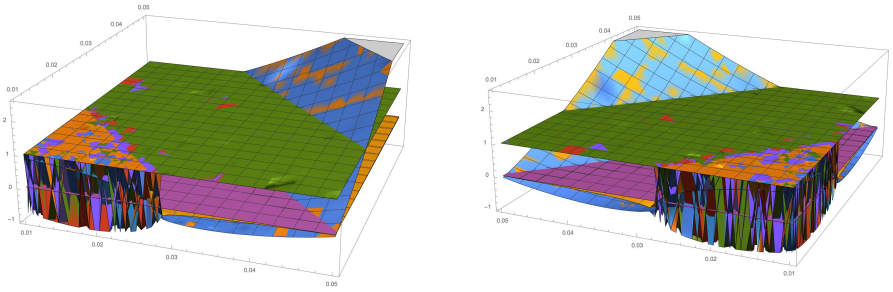

FIGURE 1. Eigenvalues of M.

sections) - there are two functionally independent first integrals. Furthermore, this shows that there cannot be a third first integral of this Hamiltonian system that is functionally independent with the energy and angular momentum. With this, we have proven:

Theorem 2.10. *The planar 3-body problem with fixed center of mass cannot have more than two functionally independent first integrals. Two such functionally independent first integrals are the energy and angular momentum. Consequently, the planar 3-body problem (without fixing the center of mass) cannot have more than four functionally independent first integrals. Four such functionally independent first integrals are the energy, angular momentum, and two components of linear momentum.*

As two of the masses get smaller, we expect to see some stability in the linearized system (2.9) (think of the Lagrange points in the Sun-Earth-Moon system). This corresponds to the eigenvalues of M having no real part. However, when two of the masses are small enough, we see wild behavior in the real part of the eigenvalues of M. Figure 1 shows a plot of the real parts of the eigenvalues of M as a function of m_1 and m_2, in the regime where $0.1 \leq m_1, m_2 \leq 0.05$. More can be said about this behavior once we understand the symplectic structure on M.

The code used for the computations in this section is attached at the end of the write-up.

2.5. **First integrals and left eigenvectors.** Let us return to the setting of an autonomous differential equation (1.11) with flow $\varphi^t(w)$. A *first integral* of (1.11) is a function $F : \mathbb{R}^k \to \mathbb{R}$ that is constant on the flow. More precisely,

$$F(\varphi^t(w)) = F(\varphi^0(w)) = F(w)$$

for all t. It follows that for each $i = 1, \ldots, k$,

$$\begin{aligned} \frac{\partial}{\partial w_i} F(w) &= \frac{\partial}{\partial w_i} F(\varphi^t(w)) \\ &= \sum_{j=1}^{k} \frac{\partial F}{\partial w_j}(\varphi^t(w)) \frac{\partial \varphi_j^t}{\partial w_i}(w) \\ &= dF(\varphi^t(w)) \frac{\partial \varphi^t}{\partial w_i}(w). \end{aligned} \tag{2.11}$$

Here, $\frac{\partial \varphi^t}{\partial w_i}$ is a column vector as before and dF is the row vector

$$dF = \left(\frac{\partial F}{\partial w_1}, \dots, \frac{\partial F}{\partial w_k} \right).$$

Since (2.11) holds for each $i = 1, \dots, k$, we see that

$$dF(w) = dF(\varphi^t(w)) D\varphi^t(w). \tag{2.12}$$

Suppose we have a periodic orbit $\varphi^t(w_0)$, so that $\varphi^{t+T}(w_0) = \varphi^t(w_0)$ for all t. Evaluating (2.12) at $t = T$, $w = w_0$ and recalling that $D\varphi^T(w_0) = M$ is the monodromy matrix gives:

$$dF(w_0) = dF(\varphi^T(w_0)) D\varphi^T(w_0) = dF(w_0)M. \tag{2.13}$$

We have proved:

Proposition 2.14. *Consider the autonomous differential equation (1.11) with flow φ^t, and let F be a first integral of (1.11). Suppose $\varphi^t(w_0)$ is T-periodic, and let $M = D\varphi^T(w_0)$ be the monodromy matrix of the variational equation*

$$\dot{y} = Df(\varphi^t(w_0))y.$$

Then $dF(w_0) = dF(w_0)M$. In particular, if $dF(w_0) \neq 0$, then $dF(w_0)$ is a left eigenvector of M with eigenvalue 1.

2.6. **First integrals of Hamiltonian systems and right eigenvectors.** Let $(\mathbb{R}^{2n}, H)$ be a Hamiltonian system, and recall that Hamilton's equations can be written as an autonomous differential equation (1.9) on phase space. On the flow $\varphi_H^t(w)$, the variational equation reads

$$\dot{y} = DX_H(\varphi_H^t(w))y. \tag{2.15}$$

Let $F : \mathbb{R}^{2n} \to \mathbb{R}$ be a first integral of H, so that $\{F, H\} = 0$. The relationship (2.6) between the Poisson and Lie brackets then implies that

$$[X_F, X_H] = (DX_F)X_H - (DX_H)X_F = 0.$$

Hence $(DX_F)X_H = (DX_H)X_F$. It is worth mentioning that

$$\begin{aligned} \frac{d}{dt}(X_F(\varphi_H^t(w))) &= DX_F(\varphi_H^t(w))\dot{\varphi}_H^t(w) \\ &= DX_F(\varphi_H^t(w))X_H(\varphi_H^t(w)) \\ &= DX_H(\varphi_H^t(w))X_F(\varphi_H^t(w)), \end{aligned}$$

which means that $X_F(\varphi_H^t(w))$ satisfies the variational equation (2.15).

It is well-known from differential geometry (see [2], Theorem 9.44 or [1], Section 39 E) that two vector fields commute if and only if their flows commute. Alternatively, this can be deduced from the Frobenius integrability theorem. Since $[X_F, X_H] = 0$, their flows commute:

$$\varphi_F^s \circ \varphi_H^t(w) = \varphi_H^t \circ \varphi_F^s(w).$$

Differentiating both sides with respect to s gives

$$X_F(\varphi_F^s \circ \varphi_H^t(w)) = (D\varphi_H^t)(\varphi_F^s(w))X_F(\varphi_F^s(w)),$$

and evaluating at $s = 0$ gives

$$X_F(\varphi_H^t(w)) = (D\varphi_H^t)(w)X_F(w). \tag{2.16}$$

If $w_0 \in \mathbb{R}^{2n}$ is such that the trajectory $\varphi_H^t(w_0)$ is T-periodic, then $M = D\varphi_H^T(w_0)$ is the monodromy matrix of the T-periodic system (2.15), and evaluating (2.16) at $t = T$ and $w = w_0$ shows that

$$X_F(w_0) = MX_F(w_0).$$

In conclusion, we obtain:

Proposition 2.17. *Let $(\mathbb{R}^{2n}, H)$ be a Hamiltonian system and let X_H be the Hamiltonian vector field associated to H. Let $\varphi_H^t(w)$ be the flow corresponding to Hamilton's equation (1.9), and suppose $\varphi_H^t(w_0)$ is T-periodic. Let $M = D\varphi_H^T(w_0)$ be the monodromy matrix of (2.15). If F is a first integral of H, then $X_F(w_0) = MX_F(w_0)$. In particular, if $X_F(w_0) \neq 0$, then $X_F(w_0)$ is a right eigenvector of M with eigenvalue 1.*

2.7. Symplectic matrices and variational equations of Hamiltonian systems.

Definition 2.18. A $2n$-by-$2n$ real-valued matrix S is said to be *symplectic* if

$$S^T JS = J,$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

is the standard symplectic matrix of dimension $2n$, as introduced in the section on Hamiltonian systems.

J enjoys some nice properties, such as $J^T = J^{-1} = -J$ and $J^2 = -I_{2n}$. One consequence of these properties is that J itself is symplectic. It is also easy to see that the identity I_{2n} is symplectic. In fact, the set of all $2n$-by-$2n$ real-valued symplectic matrices, denoted $\mathrm{Sp}(2n, \mathbb{R})$, is a group under matrix multiplication. If $S \in \mathrm{Sp}(2n, \mathbb{R})$, then S^{-1} exists and is symplectic, and is given explicitly by

$$S^{-1} = J^{-1}S^T J. \tag{2.19}$$

Let us now consider a specific kind of linear differential equation:

$$\dot{z} = JB(t)z, \tag{2.20}$$

where $B(t)$ is a symmetric real-valued $2n$-by-$2n$ matrix.

Proposition 2.21. *The principal fundamental matrix $Z(t)$ of (2.20) is symplectic for all t.*

Proof. We have that $\dot{Z} = JB(t)Z$ and $Z(0) = I_{2n}$. Note that Z^T satisfies $\dot{Z}^T = Z^T B(t) J^T$ because $B(t)$ is symmetric. It follows that

$$\begin{aligned} \frac{d}{dt}(Z^T JZ) &= \dot{Z}^T JZ + Z^T J\dot{Z} \\ &= Z^T B(t) J^T JZ + Z^T JJB(t)Z \\ &= Z^T B(t)Z - Z^T B(t)Z \\ &= 0, \end{aligned}$$

so that $Z^T JZ$ is constant in t. At $t = 0$, we have $Z(0)^T JZ(0) = J$ because $Z(0) = I_{2n}$. Hence $Z(t)^T JZ(t) = J$ for all t. □

Why would we consider such a differential equation in the first place? The variational equation (2.15) of a Hamiltonian system is exactly of the form (2.20) because

$$DX_H(\varphi_H^t(w)) = D(J\nabla H)(\varphi_H^t(w)) = J\mathrm{Hess}(H)(\varphi_H^t(w)),$$

so $B(t) = \mathrm{Hess}(H)(\varphi_H^t(w))$ is real and symmetric. If $w_0 \in \mathbb{R}^{2n}$ is such that $\varphi_H^t(w_0)$ is T-periodic, then the variational equation (2.15) is T-periodic, and since the principal fundamental matrix $D\varphi_H^t(w_0)$ is symplectic for all t, the monodromy matrix $M = D\varphi_H^T(w_0)$ of (2.15) is symplectic. This shows:

Corollary 2.22. *Let $(\mathbb{R}^{2n}, H)$ be a Hamiltonian system with flow $\varphi_H^t(w)$. If $\varphi_H^t(w_0)$ is T-periodic, then the monodromy matrix $M = D\varphi_H^T(w_0)$ of (2.15) is symplectic.*

2.8. **Eigenvalues and eigenvectors of symplectic matrices.** Let $S \in \mathrm{Sp}(2n, \mathbb{R})$ be given. Suppose that v is a right eigenvector of S with eigenvalue λ, so that $Sv = \lambda v$. Since S is invertible, $\lambda \neq 0$. Observe that

$$\begin{aligned} S^T JSv &= Jv \\ \lambda S^T Jv &= Jv \\ (Jv)^T S &= \frac{1}{\lambda}(Jv)^T. \end{aligned}$$

Since J is invertible, $Jv \neq 0$, and hence $(Jv)^T$ is a left eigenvector of S with eigenvalue $1/\lambda$. This correspondence goes the other way as well: suppose that w is a left eigenvector of S with eigenvalue $\rho \neq 0$, so that $wS = \rho w$. It follows that $S^T w^T = \rho w^T$, and hence

$$(S^T)^{-1} w^T = \frac{1}{\rho} w^T.$$

Since S is symplectic, $S^T JS = J$, and thus $S = J^{-1}(S^T)^{-1}J = J^T(S^T)^{-1}J$. It follows that

$$\begin{aligned} S(wJ)^T &= J^T(S^T)^{-1}J(wJ)^T \\ &= J^T(S^T)^{-1}JJ^T w^T \\ &= J^T(S^T)^{-1}w^T \\ &= \frac{1}{\rho}J^T w^T \\ &= \frac{1}{\rho}(wJ)^T. \end{aligned}$$

Hence $(wJ)^T$ is a right eigenvector of S with eigenvalue $1/\rho$.

In particular, we have shown that λ is an eigenvalue of S if and only if $1/\lambda$ is an eigenvalue of S. Moreover, since we are working with real symplectic matrices, λ is an eigenvalue of S if and only if $\bar{\lambda}$ is an eigenvalue. Recall that the invserse of S is given by (2.19): $S^{-1} = J^{-1}S^T J$. This shows that S^{-1} is similar to S^T, and hence they have the same eigenvalues (always considered up to algebraic multiplicity). Moreover, S^T and S have the same eigenvalues, so S^{-1} and S have the same eigenvalues. This is another way of seeing that eigenvalues of a symplectic matrix come in reciprocal pairs, and this perspective shows that the algebraic multiplicity of λ is equal to the algebraic multiplicity of $1/\lambda$.

A reformulation of this result is that the characteristic polynomial $p(\lambda)$ of S is reflexive:

$$p(\lambda) = \lambda^{2n} p(1/\lambda).$$

This follows immediately from (2.19) and the well-known fact that $\det S = 1$. For more details, see [1], Sections 41 C and 42 B.

Let m_1 denote the algebraic multiplicity of the eigenvalue 1 of S, and let m_{-1} denote the algebraic multiplicity of the eigenvalue -1. If 1 is not an eigenvalue of S, we interpret this as $m_1 = 0$, and similarly for -1. Note that 1 and -1 are the only solutions over the complex numbers of the equation $z = 1/z$. Hence for all other complex numbers, z and $1/z$ are distinct. Recall that the product of all eigenvalues of S, counted with algebraic multiplicity, equals $\det S$. Since the eigenvalues of S come in reciprocal pairs of equal algebraic multiplicity and $\det S = 1$, it follows that $(-1)^{m_{-1}} = 1$, and hence m_{-1} is even. Also, since S has an even number of eigenvalues, and all eigenvalues other than ± 1 come in reciprocal pairs, we see that $m_1 + m_{-1}$ must be even. Since m_{-1} is even, so is m_1.

We summarize this discussion in the following theorem:

Theorem 2.23. *Let $S \in Sp(2n, \mathbb{R})$ be given.*

1. *If λ is an eigenvalue of S, then so are $1/\lambda, \bar{\lambda}$, and $1/\bar{\lambda}$. Moreover, the eigenvalues $\lambda, 1/\lambda, \bar{\lambda}, 1/\bar{\lambda}$ all have the same algebraic multiplicity.*
2. *The eigenvalues 1 and -1 of S must have even algebraic multiplicity.*
3. *A vector v is a right eigenvector of S with eigenvalue λ if and only if $(Jv)^T$ is a left eigenvector of S with eigenvalue $1/\lambda$.*

Recall the previous sections on left and right eigenvectors coming from first integrals. For a Hamiltonian system, the fact that a first integral gives rise to both a left and a right eigenvector of the monodromy matrix, both with eigenvalue 1, is no coincidence. Our computations showed that along a periodic orbit $\varphi_H^t(w_0)$, a first integral F gave us the left eigenvector $dF(w_0)$ and the right eigenvector $X_F(w_0)$ of the monodromy matrix M. This is an illustration of the correspondence between left and right eigenvectors of a symplectic matrix, because $-dF(w_0) = (JX_F(w_0))^T$.

Example 2.24. With this result in mind, we revisit Figure 1, which plots the eigenvalues of the monodromy matrix M as a function of m_1 and m_2 for the planar 3-body problem with masses $m_1+m_2+m_3 = 1$. For all values of m_1, m_2, m_3, M has four eigenvalues of 1, which correspond to the first integrals of energy and angular momentum (see the section on Poincaré's theorem). When none of the masses are too small (such as when $m_1 = m_2 = m_3 = 1$), the other four eigenvalues of M are $\rho, \rho, 1/\rho, 1/\rho$, where ρ is real and $\rho > 1$. This can be seen computationally.

As two of the masses get smaller, ρ approaches 1, so in the complex plane we can think of the eigenvalue pair $\rho, 1/\rho$ (the red points in Figure 2) as moving towards the point $z = 1$ (shown in blue) along the real axis. Once two of the masses get small enough, the pair $\rho, 1/\rho$ leaves the real axis at $z = 1$ and starts moving around the unit circle $S^1 = \{z : |z| = 1\}$ (shown in green).

2.9. **Euclidean and skew-scalar products.** We will use $\langle \cdot, \cdot \rangle$ to denote the standard Euclidean inner product on any $\mathbb{R}^k$: if $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathbb{R}^k$, then

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i y_i.$$

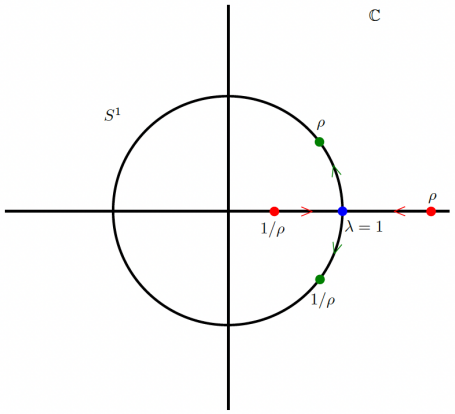

FIGURE 2. Eigenvalues of M in the complex plane.

If $x, y \in \mathbb{R}^{2n}$ are thought of as column vectors in an even-dimensional space, then observe that

$$\langle Jx, Jy\rangle = (Jx)^T Jy = x^T J^T Jy = x^T y = \langle x, y\rangle. \tag{2.25}$$

Definition 2.26. On even-dimensional spaces $\mathbb{R}^{2n}$, we define the *skew-scalar product* or *symplectic inner product* $\omega(\cdot,\cdot)$ by

$$\omega(x, y) = \langle x, Jy\rangle.$$

In coordinates, if $x = (q_1, \dots, q_n, p_1, \dots, p_n)$ and $y = (q_1', \dots, q_n', p_1', \dots, p_n')$, then

$$\omega(x, y) = \sum_{i=1}^{n} (q_i p_i' - p_i q_i').$$

From this formula, it is clear that $\omega(x, x) = 0$ for all $x \in \mathbb{R}^{2n}$. Equivalently, $\langle x, Jx\rangle = 0$, which means x and Jx are always orthogonal. In particular, the correspondence $v \leftrightarrow (Jv)^T$ between right and left eigenvectors of a symplectic matrix has the nice property that

$$(Jv)^T v = \langle Jv, v\rangle = \langle v, Jv\rangle = \omega(v, v) = 0.$$

The Poisson bracket (2.4) can be thought of as a skew-scalar product: if $F, G \in C^\infty(\mathbb{R}^{2n})$, then the definition (2.4) is equivalent to

$$\{F, G\} = \omega(\nabla F, \nabla G) = \langle \nabla F, J\nabla G\rangle = dF \cdot X_G. \tag{2.27}$$

In this language, F is a first integral of the Hamiltonian H if and only if their gradients are *skew-orthogonal*.

2.10. **Poincaré's Theorem.**

Theorem 2.28. *Let $(\mathbb{R}^{2n}, H)$ be a Hamiltonian system with flow $\varphi_H^t(w)$. Suppose there is a $w_0 \in \mathbb{R}^{2n}$ such that the orbit $\varphi_H^t(w_0)$ is T-periodic, $T > 0$. Suppose further that there is a set of k first integrals $F_1, \dots, F_k$ ($k \le n$) with $F_1 = H$, all in involution with each other, and that $F_1, \dots, F_k$ are functionally independent at w_0. Then M has at least $2k$ eigenvalues of 1.*

Here, *in involution* means that $\{F_i, F_j\} = 0$ for all i, j, and recall that *functionally independent* means the vectors $\nabla F_1(w_0), \ldots, \nabla F_k(w_0)$ are linearly independent. This theorem was proved by Poincaré in Volume I of [4], Sections 69-71.

We aim to give a detailed proof of this theorem in more modern language.

Proof. For each $i = 1, \ldots, k$, put $X_i = X_{F_i}(w_0)$. By hypothesis, the vectors $\nabla F_i(w_0)$, $i = 1, \ldots, k$ are linearly independent, and since J is invertible, the vectors $X_i = J\nabla F_i(w_0)$, $i = 1, \ldots, k$ are also linearly independent. From the section on first integrals of Hamiltonian systems and right eigenvectors, we know that each X_i is a right eigenvector of M with eigenvalue 1. Hence we have already found k eigenvalues of 1. How do we get k more?

Inspired by the left-right eigenvector correspondence of a symplectic matrix, we put $\tilde{X}_i = JX_i$ for each $i = 1, \ldots, k$. Other ways of writing $\tilde{X}_i$ are

$$\tilde{X}_i = JX_i = -dF(w_0)^T = -\nabla F_i(w_0).$$

From the section on eigenvalues and eigenvectors of symplectic matrices, or from the section on first integrals and left eigenvectors, we see that $\tilde{X}_i^T$ is a left eigenvector of M with eigenvalue 1 for each $i = 1, \ldots, k$. Equivalently, $\tilde{X}_i$ is a right eigenvector of M^T with eigenvalue 1. Since the F_i, $i = 1, \ldots, k$ are functionally independent at w_0, the $\tilde{X}_i = -\nabla F_i(w_0)$, $i = 1, \ldots, k$ are linearly independent.

In general, if you take a set of k linearly independent vectors and combine it with another set of k linearly independent vectors, the big set of $2k$ vectors is not guaranteed to be linearly independent. In our situation, we have more structure that avoids this issue. Since $F_1, \ldots, F_k$ are in involution, we have that $\{F_i, F_j\} = 0$ for each i, j, and thus $\langle X_i, \tilde{X}_j \rangle = 0$ for each i, j, by the previous section. That is, the two sets $X_1, \ldots, X_k$ and $\tilde{X}_1, \ldots, X_k$ are mutually orthogonal, which is enough to ensure linear independence of the combined set $X_1, \ldots, X_k, \tilde{X}_1, \ldots, \tilde{X}_k$. For more details, see the proposition in next section.

Extend $X_1, \ldots, X_k, \tilde{X}_1, \ldots, \tilde{X}_k$ to a basis $X_1, \ldots, X_k, \tilde{X}_1, \ldots, \tilde{X}_k, z_1, \ldots, z_r$ of $\mathbb{R}^{2n}$, such that $\langle \tilde{X}_i, z_j \rangle = 0$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, r$. Here, $r = 2(n-k)$. This is always possible because of the Gram-Schmidt process. Re-order this basis and call it $\mathcal{B}$:

$$\mathcal{B} = \{X_1, \ldots, X_k, z_1, \ldots, z_r, \tilde{X}_1, \ldots, \tilde{X}_k\}.$$

As we shall prove in the next section, M takes a particularly nice form in this basis:

$$[M]_{\mathcal{B}} = \begin{pmatrix} I_k & * & * \\ \mathbf{0}_{r\times k} & * & * \\ \mathbf{0}_{k\times k} & \mathbf{0}_{k\times r} & I_k \end{pmatrix}.$$

In this form, it is clear that M has at least $2k$ eigenvalues of 1. □

2.11. **Linear algebra details.**

Lemma 2.29. *If $v_1, \ldots, v_l \in \mathbb{R}^m$ are linearly independent, then the l-by-l Gram matrix G with entries $G_{ij} = \langle v_i, v_j \rangle$ is invertible.*

Proof. We prove that G is injective. Suppose not, so there is a nonzero $\xi \in \mathbb{R}^l$ such that $G\xi = 0$. Then

$$0 = \xi^T G \xi = \sum_{i=1}^{l}\sum_{j=1}^{l} \xi_i G_{ij} \xi_j = \sum_{i=1}^{l}\sum_{j=1}^{l} \langle \xi_i v_i \xi_j v_j \rangle = \left\langle \sum_{i=1}^{l} \xi_i v_i, \sum_{j=1}^{l} \xi_j v_j \right\rangle,$$

and hence $\sum_{i=1}^{l} \xi_i v_i = 0$. This is a contradiction because $v_1, \ldots, v_l$ are linearly independent, as desired. In fact, the same argument shows that G is positive-definite. □

Proposition 2.30. *Let $v_1, \ldots, v_l \in \mathbb{R}^m$ be linearly independent, and let $z_1, \ldots, z_l \in \mathbb{R}^m$ be linearly independent. If $\langle v_i, z_j \rangle = 0$ for each i, j, then $v_1, \ldots, v_l, z_1, \ldots, z_l$ are linearly independent.*

Proof. Suppose there are $a_1, \ldots, a_l, b_1, \ldots, b_l \in \mathbb{R}$ such that

$$a_1 v_1 + \cdots + a_l v_l + b_1 z_1 + \cdots + b_l z_l = 0. \tag{2.31}$$

For each $j = 1, \ldots, l$, taking the inner product of (2.31) with v_j gives

$$a_1 \langle v_1, v_j \rangle + \cdots + a_l \langle v_l, v_j \rangle = 0.$$

This gives a linear system of l equations in l variables $a_1, \ldots, a_l$, which has matrix form $GA = 0$. Here, G is the l-by-l Gram matrix with entries $G_{ij} = \langle v_i, v_j \rangle$ and A is the column vector with entries $a_1, \ldots, a_l$. Since G is invertible, the unique solution of $GA = 0$ is $A = 0$. Taking the inner product of (2.31) with each z_j and using the same argument shows that $b_1 = \cdots = b_l = 0$, as desired. □

Theorem 2.32. *Let $S \in Sp(2n, \mathbb{R})$. Suppose that $v_1, \ldots, v_k \in \mathbb{R}^{2n}$ are linearly independent right eigenvectors of S, each with eigenvalue 1. For each $i = 1, \ldots, k$, put $\tilde{v}_i = Jv_i$, so that $\tilde{v}_i$ is a right eigenvector of S^T with eigenvalue 1 (equivalently, $\tilde{v}_i^T$ is a left eigenvector of S with eigenvalue 1). Suppose that $\langle v_i, \tilde{v}_j \rangle = 0$ for each i, j. Then S has $2k$ eigenvalues of 1.*

Proof. Since J is invertible and $v_1, \ldots, v_k$ are linearly independent, we have that $\tilde{v}_1, \ldots, \tilde{v}_k$ are linearly independent. Since these two sets of vectors are mutually orthogonal, the previous proposition implies that $v_1, \ldots, v_k, \tilde{v}_1, \ldots, \tilde{v}_k$ are linearly independent. Extend this to a basis $v_1, \ldots, v_k, \tilde{v}_1, \ldots, \tilde{v}_k, z_1, \ldots, z_r$ of $\mathbb{R}^{2n}$ such that $\langle \tilde{v}_i, z_j \rangle = 0$ for each $i = 1, \ldots, k$ and $j = 1, \ldots, r$. This is always possible because of the Gram-Schmidt process. Re-order these vectors and call this basis $\mathcal{B}$:

$$\mathcal{B} = \{v_1, \ldots, v_k, z_1, \ldots, z_r, \tilde{v}_1, \ldots, \tilde{v}_k\}.$$

Since each v_i is a right eigenvector of S, we have $Sv_i = v_i$. Now let $l \in \{1, \ldots, r\}$ be given. Write

$$Sz_l = a_1 v_1 + \cdots + a_k v_k + b_1 z_1 + \cdots + b_r z_r + \tilde{a}_1 \tilde{v}_1 + \cdots + \tilde{a}_k \tilde{v}_k. \tag{2.33}$$

For each $j = 1, \ldots, k$, take the inner product of (2.33) with $\tilde{v}_j$ to get

$$\begin{aligned} \langle Sz_l, \tilde{v}_j \rangle &= \tilde{a}_1 \langle \tilde{v}_1, \tilde{v}_j \rangle + \cdots + \tilde{a}_k \langle \tilde{v}_k, \tilde{v}_j \rangle \\ &= \tilde{a}_1 \langle v_1, v_j \rangle + \cdots + \tilde{a}_k \langle v_k, v_j \rangle. \end{aligned} \tag{2.34}$$

Here, we use (2.25) to get $\langle \tilde{v}_i, \tilde{v}_j \rangle = \langle Jv_i, Jv_j \rangle = \langle v_i, v_j \rangle$. Also note that

$$\langle Sz_l, \tilde{v}_j \rangle = \langle z_l, S^T \tilde{v}_j \rangle = \langle z_l, \tilde{v}_j \rangle = 0.$$

Hence (2.34) becomes

$$\tilde{a}_1 \langle v_1, v_j \rangle + \cdots + \tilde{a}_k \langle v_k, v_j \rangle = 0,$$

valid for each $j = 1, \ldots, k$. As in the proof of the previous proposition, this system of linear equations can be written in matrix form $GA = 0$, and since the Gram matrix $G = (\langle v_i, v_j \rangle)_{ij}$ is invertible, the unique solution is $\tilde{a}_1 = \cdots = \tilde{a}_k = 0$. Thus

in the basis $\mathcal{B}$, none of the vectors $Sz_1, \dots, Sz_r$ have components along the vectors $\tilde{v}_1, \dots, \tilde{v}_k$.

Lastly, let $i \in \{1, \dots, k\}$ be given. Write

$$S\tilde{v}_i = \alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 z_1 + \cdots + \beta_r z_r + \tilde{\alpha}_1 \tilde{v}_1 + \cdots + \tilde{\alpha}_k \tilde{v}_k. \tag{2.35}$$

For each $j = 1, \dots, k$, take the inner product of (2.35) with $\tilde{v}_j$ to get

$$\begin{aligned} \langle S\tilde{v}_i, \tilde{v}_j \rangle &= \tilde{\alpha}_1 \langle \tilde{v}_1, \tilde{v}_j \rangle + \cdots + \tilde{\alpha}_k \langle \tilde{v}_k, \tilde{v}_j \rangle \\ &= \tilde{\alpha}_1 \langle v_1, v_j \rangle + \cdots + \tilde{\alpha}_k \langle v_k, v_j \rangle. \end{aligned} \tag{2.36}$$

This time the left side is no longer zero, but rather

$$\langle S\tilde{v}_i, \tilde{v}_j \rangle = \langle v_i, S^T \tilde{v}_j \rangle = \langle \tilde{v}_i, \tilde{v}_j \rangle = \langle v_i, v_j \rangle.$$

Hence (2.35) becomes

$$\tilde{\alpha}_1 \langle v_1, v_j \rangle + \cdots + \tilde{\alpha}_k \langle v_k, v_j \rangle = \langle v_l, v_j \rangle.$$

Note that the values $\tilde{\alpha}_j = \delta_{ij}$ satisfy this system of equations. By invertibility of the Gram matrix, $\tilde{\alpha}_j = \delta_{ij}$ is the unique solution. Hence in the basis $\mathcal{B}$, the only nonzero component of the vector $S\tilde{v}_i$ along the $\tilde{v}_1, \dots, \tilde{v}_k$ is in the direction of $\tilde{v}_i$, and that component is 1.

It follows immediately from these observations that

$$[S]_{\mathcal{B}} = \begin{pmatrix} I_k & * & * \\ \mathbf{0}_{r \times k} & * & * \\ \mathbf{0}_{k \times k} & \mathbf{0}_{k \times r} & I_k \end{pmatrix},$$

and thus S has $2k$ eigenvalues of 1, as desired. □

2.12. **Partial result on the non-integrability of the planar 3-body problem.**

2.13. **Extending the result.**

3. Part II: Quasi-periodic Floquet Theory

3.1. **Motivation.** insert some historical motivation

3.2. **Equivariant Floquet Theory.** There is a large amount of information surrounding linear differential equations

$$\dot{x} = A(t)x. \tag{3.1}$$

When $A(t)$ is periodic, the main description of these systems comes from Floquet theory. We describe an analagous theory for equivariant systems. Namely, systems of the form (3.1) where, rather than $A(t)$ being periodic,

$$A(t) = D(\frac{d}{dt}(\varphi_H^t(x_0)))$$

for some x_0 in phase space and the following assumptions hold:

(1) There exists a flow φ_F^s such that $\varphi_F^s \varphi_H^T = \varphi_H^T \varphi_F^s$
(2) For some fixed s and T, $\varphi_F^s(x_0) = \varphi_H^T(x_0)$
(3) $D\varphi_F^s$ depends only on the parameter s

Note that the definition for $A(t)$ arises naturally from the linearization of the system. One may wonder why (3) is a reasonable assumption to make. Due to Example 2.7, we see that if L corresponds to angular momentum,

$$D\varphi_L^s = \begin{pmatrix} \mathcal{R}_s & & \\ & \ddots & \\ & & \mathcal{R}_s \end{pmatrix}, \quad \mathcal{R}_s = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}$$

Then $D\varphi_L^s$ depends only on s, which is exactly assumption (3). Now one can ask what these assumptions imply about the system. Even though $A(t)$ is not necessarily periodic, there is a similar relation between $A(t)$ and $A(t+T)$.

Lemma 3.2. *For systems of the form (3.1), satisfying (1)-(3), $A(t)$ satisfies the following equation:*

$$A(t+T) = D\varphi_F^s A(t)(D\varphi_F^s)^{-1}$$

Proof. By the chain rule and assumption (3),

$$\frac{d}{dt}\{\varphi_F^s\varphi_H^t(x_0)\} = D\varphi_F^s(\frac{d}{dt}\varphi_H^t(x_0))$$

Differentiating both sides with respect to $\varphi_H^t(x_0)$, and using assumptions (1) and (2), we see that

$$D\frac{d}{dt}\{\varphi_F^s\varphi_H^t(x_0)\}D\varphi_F^s = D\varphi_F^s D(\frac{d}{dt}\varphi_H^t(x_0))$$

$$A(t+T)D\varphi_F^s = D\varphi_F^s A(t)$$

Since $D\varphi_F^s$ is the derivative of a flow, it is invertible, which directly leads to the result. □

To simplify notation, let $R_s = D\varphi_F^s$. We can now see how the fundamental matrix solution evolves with each period.

Lemma 3.3. *Let $X(t)$ be a fundamental matrix solution to (3.1), where assumptions (1)-(3) hold. Then there is a constant matrix M such that*

$$X(t+T) = R_s X(t) M \tag{3.4}$$

Proof. Since $X(t)$ is a fundamental matrix solution to (3.1),

$$\begin{aligned} \dot{X}(t+T) &= A(t+T)X(t+T) \\ &= R_s A(t) R_s^{-1} X(t+T) \end{aligned}$$

which implies that $R_s^{-1}X(t+T)$ is a solution to (3.1). Since the columns of $X(t)$ are linearly independent solutions to (3.1), we see that the columns of $R_s^{-1}X(t+T)$ must be linear combinations of the columns of $X(t)$. Thus

$$R_s^{-1}X(t+T) = X(t)M$$

for some constant matrix M. This yields the desired result. □

The constant matrix M in this lemma is called a *monodromy matrix*. Now assume that $Y(t)$ is another fundamental matrix solution to (3.1), so that $Y(t) = X(t)B$

for some invertible constant matrix B. By the lemma, there is a constant matrix $\tilde{M}$ such that satisfying $Y(t+T) = R_s Y(t)\tilde{M}$. Then

$$\begin{aligned} Y(t+T) &= X(t+T)B \\ R_s Y(t)\tilde{M} &= R_s X(t)MB \\ X(t)B\tilde{M} &= X(t)MB \\ \tilde{M} &= B^{-1}MB \end{aligned}$$

which implies that the eigenvalues of $\tilde{M}$ are the same as those of M. Thus the eigenvalues of a monodromy matrix are a property intrinsic to the system, and do not depend on any given fundamental matrix solution.

Now recall Liouville's theorem ([6], Lemma 3.11):

$$\det(X(t)) = \det(X(t_0)) \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))ds\right)$$

We provide an analagous version for the equivariant case.

Theorem 3.5. *Let $X(t)$ be the principal fundamental matrix to (3.1), under the assumptions (1)-(3). Then*

$$\det(M) = \frac{1}{\det(R_s)} \exp\left(\int_0^T tr(A(s))ds\right) \tag{3.6}$$

Proof.

$$\begin{aligned} \det(X(T)) &= \det(X(0)) \exp\left(\int_0^T \operatorname{tr}(A(s))ds\right) \\ &= \exp\left(\int_0^T \operatorname{tr}(A(s))ds\right) \\ \det(R_s)\det(M) &= \exp\left(\int_0^T \operatorname{tr}(A(s))ds\right) \\ \det(M) &= \frac{1}{\det(R_s)} \exp\left(\int_0^T \operatorname{tr}(A(s))ds\right) \end{aligned}$$

□

In addition to Liouville's theorem, there is an analagous version of Floquet's theorem for the equivariant system. (Compare this with [6], Theorem 3.15)

Theorem 3.7. *If $X(t)$ is the principal fundamental matrix solution to (3.1) where assumptions (1)-(3) hold, then there is a T-periodic matrix $P(t)$ and constant matrices Q_1 and Q_2,*

$$X(t) = \exp(tQ_1)P(t)\exp(tQ_2) \tag{3.8}$$

Proof. Let $X(t)$ be a principal fundamental matrix solution. Note that since R_s is invertible, $\det(R_s) \neq 0$. By Theorem 6.5, it is clear that $\det(M) \neq 0$. Thus by [6],

Lemma 3.34, we see that $R_s = \exp(TQ_1)$ and $M = \exp(TQ_2)$ for some constant matrices Q_1 and Q_2. Put

$$P(t) = \exp(-tQ_1)X(t)\exp(-tQ_2)$$

Observe that

$$\begin{aligned} P(t+T) &= \exp(-(t+T)Q_1)X(t+T)\exp(-(t+T)Q_2) \\ &= \exp(-tQ_1)\exp(-TQ_1)X(t+T)\exp(-TQ_2)\exp(-tQ_2) \\ &= \exp(-tQ_1)X(t)\exp(-tQ_2) \\ &= P(t). \end{aligned}$$

Thus we see that $P(t)$ is T-periodic. □

3.3. **Stability.** Analyzing this result closer, we see that the system has the same stability requirements as the normal Floquet case, with the following exception: The requirements on the Floquet multipliers also apply to the eigenvalues of the matrix R_s. We provide a slightly modified version of Corollary 3.17 from Teschl [6] below.

Theorem 3.9. *The system (3.1) where assumptions (1)-(3) hold remains bounded as $t \to \infty$ if the modulus of all the eigenvalues of R_s and of M is less than or equal to 1 and the algabraic and geometric multiplicity of the eigenvalues 1 are equal. If all the eigenvalues have modulus less than 1, the system converges to 0 as $t \to \infty$.*

Proof. By Lemma 3.3 in Teschl [6], the eigenvalues λ_i of R_s are related to the eigenvalues γ_i of Q_1 in the following way:

$$\lambda_i = exp(T\gamma_i)$$

so the exact same analysis in the proof of Corollary 3.17 in Teschl [6] applies here. □

To make this more precise, suppose v is an eigenvector of M with eigenvalue ρ. Then let $x(t) = X(t)v$. Observe then that $\dot{x}(t) = A(t)x(t)$ and

$$\begin{aligned} x(t+T) &= X(t+T)v \\ &= R_sX(t)Mv \\ &= \rho R_sX(t)v \\ &= \rho R_sx(t) \end{aligned}$$

We see that applying k iterates of T tells us about the asymptotic behavior of $x(t)$

$$x(t+Tk) = \rho^k R_s^k x(t). \tag{3.10}$$

Hence the asymptotic behavior of $x(t)$ is determined by $|\rho|$ and R_s.

3.4. **The Hamiltonian Case.** Suppose that (3.1) arose from the linearization of a Hamiltonian system. Then $A(t) = J\text{Hess}(H)(\varphi_H^t(x_0))$, where H is the Hamiltonian, and (3.1) takes the form

$$\dot{x} = J\text{Hess}(H)(\varphi_H^t(x_0))x \tag{3.11}$$

By Proposition 5.17, the principal fundamental matrix $X(t)$ of (3.11) is symplectic. If, in addition, we assume conditions (1)-(3), then

$$M = X(t)^{-1}R_s^{-1}X(t+T).$$

Evaluating this at $t = 0$, we find that $M = R_s^{-1}X(T)$, so that the monodromy matrix of (3.11) corresponding to the principal fundamental matrix is the product of R_s^{-1} and the symplectic matrix $X(T)$.

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