Dynamics of Aut(F_n): Ergodicity and Compact, Connected, Semi-simple Groups

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Abstract

In [Gel08], T. Gelander proved that the orbit of almost any point in $\mathrm{Hom}(F_n, G)$ under the natural action of $\mathrm{Aut}(F_n)$ is dense, and furthermore that this action is is ergodic. In this paper, we generalize this result to the Torelli subgroup, an important normal subgroup of $\mathrm{Aut}(F_n)$. That is, we prove that the orbit of almost any point in $\mathrm{Hom}(F_n, G)$ under action of $\mathrm{Tr}(n)$ is dense and that the action is ergodic.

1 Introduction

For this paper, let F_n be the free group of rank n, with standard generators $\{x_1, ..., x_n\}$. A word in the letters $\{x_i\}$ and their inverses is denoted $w\{x_i\}$. The automorphism group of the free group is denoted $\mathrm{Aut}(F_n)$. Note that an automorphism $\phi \in \mathrm{Aut}(F_n)$ is determined by its image when applied to the tuple of standard generators $(x_1, \ldots, x_n)$. The Nielsen moves, also referred to as product replacement moves, are the automorphisms that replace an element of the generating set with a product with another element:

replace the i-th coordinate with a product with the j-th or its inverse on the left

$$\begin{aligned}
L_{ij}^{\pm} &: (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, x_j^{\pm} x_i, \ldots, x_j, \ldots, x_n) \\
R_{ij}^{\pm} &: (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, x_i x_j^{\pm}, \ldots, x_j, \ldots, x_n) \\
P_{ij} &: (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)
\end{aligned}$$

It is a well-known fact from Nielsen's Theorem [MKS04] that $\mathrm{Aut}(F_n)$ is generated by the set of all Nielsen moves. For a group G, the set $\mathrm{Hom}(F_n, G)$ can be identified with G^n by fixing a set of free generators for F_n. Each $\phi \in \mathrm{Hom}(F_n, G)$ maps the free generating set to an n-tuple in G^n,

$$(x_1, \ldots, x_n) \stackrel{\phi}{\mapsto} (g_1, \ldots, g_n)$$

and $\mathrm{Aut}(F_n)$ acts on G^n by precompositions with ϕ. Under this identification $\mathrm{Hom}(F_n, G) \cong G^n$, Nielsen moves can be considered as acting on n-tuples $(g_1, \ldots, g_n)$.

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Let G be a compact, connected, and semi-simple Lie group. We note that G is an algebraic group over $\mathbb{R}$ so it is equipped with the Zariski topology. T. Gelander proved in [Gel08] that for $n \geq 3$, and compact, connected, semi-simple Lie groups G, almost every point in G^n has a dense orbit under action by $\mathrm{Aut}(F_n)$, and the action is ergodic. In this paper, we review the definition of a well-studied normal subgroup of $\mathrm{Aut}(F_n)$ called the Torelli group and provide a generating set to prove that under the same conditions, almost every point in G^n has a dense orbit under action by $\mathrm{Tr}(n)$. This allows us to prove that the action is ergodic.

2 The Torelli Subgroup

Lemma 2.1. *Let G be a group. Then, the commutator subgroup $[G, G]$ is characteristic; that is, for any $\phi \in \mathrm{Aut}(G)$, ϕ preserves the commutator subgroup:*

$$\phi([G, G]) = [G, G]$$

Since group automorphisms preserve the commutator subgroup, there is a natural map from an automorphism ϕ of F_n to an automorphism $\overline{\phi}$ of the abelianization $F_n/[F_n, F_n] \cong \mathbb{Z}^n$.

Definition 2.2. Let $\psi : \mathrm{Aut}(F_n) \to \mathrm{Aut}(\mathbb{Z}^n)$ be the natural mapping outlined above, defined by $\psi(\phi) = \overline{\phi}$. The map ψ is a homomorphism. For $n \geq 3$, define the *Torelli subgroup* as the kernel of this map:

$$\mathrm{Tr}(n) = \ker(\psi)$$

In particular, $\mathrm{Tr}(n)$ is a normal subgroup of $\mathrm{Aut}(F_n)$.

T. Gelander proved in [Gel08] that the action of $\mathrm{Aut}(F_n)$ on G^n is ergodic. We thus ask the question of whether subgroups of $\mathrm{Aut}(F_n)$ also act the same. In this paper, we will prove that $\mathrm{Tr}(n)$ acts ergodically on G^n.

Lemma 2.3. *Automorphisms of the form*

$$\phi : (x_1, x_2, \ldots, x_i, \ldots, x_n) \longmapsto (x_1, x_2, \ldots, x_i[w_1\{x_j, x_k\}, w_2\{x_j, x_k\}], \ldots, x_n) \tag{1}$$

are elements of $\mathrm{Tr}(n)$ *for arbitrary words $w_1\{x_j, x_k\}, w_2\{x_j, x_k\}$ and $i \neq j \neq k$.*

Proof. It suffices to show $\psi(\phi) = 1_{\mathrm{Aut}(F_n/[F_n,F_n])}$.

$$(x_1, x_2, \ldots, x_i, \ldots, x_n) \overset{\phi}{\mapsto} (x_1, x_2, \ldots, x_i \, [w_1\{x_j, x_k\}, w_2\{x_j, x_k\}] \, , \ldots, x_n)$$
$$\overset{\psi}{\mapsto} (x_1, x_2, \ldots, x_i, \ldots, x_n)$$

Since $\psi(\phi)$ is in $\mathrm{Aut}(\mathbb{Z}^n)$, this vanishes. □

where ψ maps into $\mathrm{Aut}(\mathbb{Z}^n)$, so the commutator $[w_1\{x_j, x_k\}, w_2\{x_j, x_k\}]$ vanishes (since $\mathbb{Z}^n$ is abelian), making the action trivial. Thus $\psi(\phi) = 1_{\mathbb{Z}^n}$ and indeed, ϕ is an element of $\mathrm{Tr}(n)$. Automorphisms of this form generate a subgroup $\langle\{\phi\}\rangle$ of $\mathrm{Tr}(n)$. In order to show that the action of $\mathrm{Tr}(n)$ on G^n is ergodic, it suffices to show that the action of this subgroup $\langle\{\phi\}\rangle$ on G^n is ergodic. To do this, we will use the following lemmas.

Lemma 2.4. *Let G be a compact, connected, semi-simple Lie group. Let $\Gamma \leq G$ be a dense subgroup of G. Then $[\Gamma, \Gamma]$, the commutator subgroup of Γ, is also dense in G.*

Proof. M. Goto proved in [Got49] that all compact, connected, semi-simple lie groups G are equal to their commutator subgroups; $G = [G, G]$. Every $g \in G$ can be written as a commutator $g = x^{-1}y^{-1}xy$. Since $\Gamma \leq G$ is dense in G, there are sequences x_n, y_n in Γ that converge to x, y respectively. Since taking inverses is a continuous operation, $x_n^{-1} \to x^{-1}$ and $y_n^{-1} \to y^{-1}$. Similarly, since multiplication is continuous,

$$x_n^{-1}y_n^{-1}x_n y_n \to x^{-1}y^{-1}xy = g$$

where $x_n^{-1}y_n^{-1}x_n y_n = [x_n, y_n]$ is a sequence in $[\Gamma, \Gamma]$. Thus $[\Gamma, \Gamma]$ is dense in G. □

Lemma 2.5. *Let $\Gamma \leq G$ be a dense subgroup. Then the right (resp. left) translate Γg (resp. $g\Gamma$) is dense.*

Proof. Let $U \subseteq G$ be an open set. It suffices to show $\Gamma g \cap U$ is nonempty for all $g \in G$. Since multiplication is continuous on G, Ug^{-1} is open. Then $\Gamma \cap Ug^{-1}$ is nonempty. Therefore, $\Gamma g \cap U$ is nonempty for all $g \in G$. The proof for the left translate is similar. □

Remark. *It is a fact that compact, connected Lie groups, G are endowed with a unique natural measure called the Haar measure. In this paper, we naturally consider such groups as measure spaces with this measure.*

Lemma 2.6. *Let G be a compact, connected Lie group. Let $\Gamma \leq G$ be a dense subgroup of G.*

Then, Γ acts ergodically on G by right (resp. left) translation.

Proof. Suppose that $\mathcal{A} \subseteq G$ is invariant under right translation by Γ. Then, the function $f = 1_{\mathcal{A}}$ is also invariant under Γ. Notice that for any function $g \in L^1(G)$, $f*g$ is also invariant under Γ. Indeed, $f(k^{-1}h)g(h)$ is invariant for $k \in \Gamma$, by assumption, whence $(f*g)(k) = \int f(k^{-1}h)g(h)dh$ is invariant for $k \in \Gamma$, as desired. Then, consider a sequence of approximants to the identity $g_1, g_2, \ldots$. For each i, $f * g_i \to f$ is continuous and Γ-invariant. Thus, since Γ is dense, $f * g_i$ is G-invariant for each i. Since the action of G on itself by right translation is obviously ergodic, this implies that $f * g_i$ is constant a.e. for each i; since $f * g_i \xrightarrow{L^1} f$, this implies that f is constant a.e. whence $\mathcal{A}$ is either null or conull, as desired. □

3 The Action of $\mathrm{Tr}(n)$ on G^n

Definition 3.1. Let $(g_1, g_2, \ldots, g_n) \in G^n$ and H be a group that acts on G^n. Define $\mathcal{O}_H(g_1, g_2, \ldots, g_n)$ to be the orbit of $(g_1, g_2, \ldots, g_n)$ under action by H. For each $i \in \{1, \ldots, n\}$, define $\mathcal{O}^i_H(g_1, g_2, \ldots, g_n)$ to be the image of $\mathcal{O}_H(g_1, g_2, \ldots, g_n)$ under the projection to the ith coordinate

$$\mathcal{O}^i_H(g_1, g_2, \ldots, g_n) \xrightarrow{\mathrm{Proj}_i} G.$$

We proceed to prove in Proposition 3.3 that for almost any point in G^n, $\mathcal{O}^k_{\mathrm{Tr}(n)}(g_1, g_2, \ldots, g_n)$ is dense in G. This allows us to prove in Proposition 3.5 that the orbit of almost any point is dense in G^n.

3.1 Dense Orbits

Definition 3.2 (Pairs that Generate Dense Subgroups of G)**.** T. Gelander proved in Lemma 1.4, in [Gel08] that, for compact, connected, semi-simple groups G, the set

$$U = \{(x_1, x_2) \in G \times G : \langle x_1, x_2 \rangle \text{ is dense in } G\}$$

Unless otherwise noted, when we write the set U, we refer to the set described above.

This will be used to prove the following:

Proposition 3.3. *Let G be a compact, connected semi-simple Lie group and $n \geq 3$. For $k, i \in \{1, \ldots, n\}$, define $\Delta_k = \left\langle L^{\pm}_{ki} \right\rangle$ to be the subgroup of $\mathrm{Aut}(F_n)$ generated by all left Nielsen moves on the k^{th} coordinate. Then*

$$\mathcal{O}^k_{\Delta_k \cap \mathrm{Tr}(n)}(g_1, g_2, \ldots, g_n) \text{ is dense in } G \text{ for a.e. } (g_1, g_2, \ldots, g_n) \in G^n \qquad (*)$$

Proof. Without loss of generality, we will prove the claim for $k = n$, the projection to the last coordinate of the n-tuple. Let U be the same set as in Definition 3.2 which is open, dense, and of full measure. Since G is compact, connected, and semi-simple, $U \subseteq G \times G$ and a.e. $(g_1, g_2, \ldots, g_n) \in G^n$ is such that $(g_1, g_2) \in U$. Fix one such tuple.

Now, we consider the orbit of the action of $\Delta_n \cap \mathrm{Tr}(n)$ on $(g_1, g_2, \ldots, g_n)$. Let H be the subgroup of G generated by $(g_1, g_2, \ldots, g_{n-1})$. H is dense in G. Let w_1, w_2 be arbitrary words in $g_1, \ldots, g_{n-1}$ and their inverses; we have $w_1, w_2 \in H$.

$$\begin{aligned}(g_1, g_2, ..., g_n) &\xmapsto{\Delta_n \cap \mathrm{Tr}(n)} (g_1, g_2, ..., [w_1, w_2]g_n) \\ &\xmapsto{\Delta_n \cap \mathrm{Tr}(n)} (g_1, g_2, ..., W g_n) \text{ where } W \in [H, H]\end{aligned}$$

$\mathcal{O}^n_{\Delta_n \cap \mathrm{Tr}(n)}(g_1, g_2, \ldots, g_n)$ is exactly the right translate $[H, H]g_n$. Since $\langle g_1, g_2 \rangle$ is a subgroup of H, H is dense in G. By Lemma 2.4, $[H, H]$ is dense in G. By Lemma 2.5 $[H, H]g_n$ is dense in G. The proof is similar for any $k \in \{1, \ldots, n\}$. □

We now extend the density of $\mathcal{O}^n_{\Delta_n \cap \mathrm{Tr}(n)}(g_1, g_2, \ldots, g_n)$ to prove that a.e. $\mathcal{O}_{\mathrm{Tr}(n)}(g_1, g_2, \ldots, g_n)$ is dense in all of G^n. To do this, we construct a set $X \subseteq G$ that suits a similar purpose as $U \subseteq G \times G$ from Definition 3.2.

Lemma 3.4. *We construct the set X:*

$$X = \{g \in G : \text{ the set } \{x \in G : \langle g, x \rangle \text{ is dense in } G\} \text{ has full measure in } G\}$$

Then, X has full measure.

Proof. It suffices to show that X^c is null. Let U be the same as in Definition 3.2. The complement of U

$$U^c = \{(x_1, x_2) \in G \times G : \langle x_1, x_2 \rangle \text{ is not dense in } G\}$$

must have zero measure. Thus, by Fubini's theorem,

$$X^c = \{g \in G : \text{ the set } \{x \in G : \langle g, x \rangle \text{ is not dense in } G\} \text{ has nonzero measure in } G\}$$

must have zero measure. □

Remark. *Since G is a compact Lie group, it is endowed with a bi-invariant metric $\|\cdot\|$. The bi-invariant metric on G may be extended to a bi-invariant metric on G^n: the taxicab metric.*

Proposition 3.5. *Let G be a compact, connected, semi-simple Lie group. Then $\mathcal{O}_{\mathrm{Tr}(n)}(g_1, \ldots, g_n)$ is dense in G^n for a.e. $(g_1, \ldots, g_n) \in G^n$.*

Proof. Let $(h_1, \ldots, h_n) \in G^n$ be an arbitrary n-tuple and let $\epsilon > 0$. It suffices to show there is $(g'_1, \ldots, g'_n) \in \mathcal{O}_{\mathrm{Tr}(n)}(g_1, \ldots, g_n)$ within an open ϵ-ball of $(h_1, \ldots, h_n)$, denoted $B_\epsilon(h_1, \ldots, h_n)$ for a.e. $(g_1, \ldots, g_n)$. For $i \in \{1, \ldots, n\}$, we fix the following sets:

$$X = \{g \in G : \text{ the set } \{x \in G : \langle g, x\rangle \text{ is dense in } G\} \text{ has full measure in } G\}$$
$$Y_i = \{x \in G : \langle g_i, x\rangle \text{ is dense in } G\}$$

X has full measure by Lemma 3.4

a.e. $g_i \in X$ so with probability 1, Y_i has full measure

Since X has full measure, almost any $(g_1, \ldots, g_n)$ is such that $g_1 \in X$. For such a tuple, the set Y_1 is full-measure; Y_1 is also open by Lemma 1.2 of [Gel08].

The set U from Definition 3.2 is dense, open, and has full measure, so $U \cap B_{\frac{\epsilon}{n}}(h_{n-1}, h_n)$ is open and non-empty. It contains an element $(x_{n-1}, x_n) \in U \cap B_{\frac{\epsilon}{n}}(h_{n-1}, h_n)$ and its open r-ball for some small $r > 0$; $B_r(x_{n-1}, x_n) \subseteq U \cap B_{\frac{\epsilon}{n}}(h_{n-1}, h_n)$.

Since Y_1 is open and has full measure, $B_{\frac{r}{2}}(x_n) \cap Y_1$ is open and non-empty. Since $\mathcal{O}^n_{\Delta \cap \mathrm{Tr}(n)}(g_1, \ldots, g_n)$ is dense in G, there exists $(g_1, \ldots, g'_n)$ in its orbit such that $g'_n \in B_{\frac{r}{2}}(x_n) \cap Y_1$. Then, $\langle g_1, g'_n\rangle$ is dense by virtue of g'_n being an element Y_1 and by projecting to the $(n-1)$-th coordinate, we have $\mathcal{O}^{n-1}_{\Delta_{n-1} \cap \mathrm{Tr}(n)}(g_1, g_2, \ldots, g'_n)$ is dense in G where $\mathcal{O}^{n-1}_{\Delta_{n-1} \cap \mathrm{Tr}(n)}(g_1, g_2, \ldots, g'_n)$ is contained in $\mathcal{O}_{\mathrm{Tr}(n)}(g_1, g_2, \ldots, g_n)$.

In other words, there exists an element in the orbit of the starting tuple, $(g_1, \ldots, g'_{n-1}, g'_n) \in \mathcal{O}_{\mathrm{Tr}(n)}(g_1, g_2, \ldots, g_n)$ such that $g'_{n-1} \in B_{\frac{r}{2}}(x_{n-1})$.

Then, since $\|g'_n - x_n\| < r/2$ and $\left\|g'_{n-1} - x_{n-1}\right\| < r/2$, we have that $(g'_{n-1}, g'_n) \in B_r(x_{n-1}, x_n) \subseteq U \cap B_{\frac{\epsilon}{n}}(h_{n-1}, h_n)$.

Thus, (g'_{n-1}, g'_n) generates a dense subgroup and $\left\|g'_{n-1} - h_{n-1}\right\|, \|g'_n - h_n\| < \frac{\epsilon}{n}$. We get that for $i \in \{1, \ldots, n\}$, $\mathcal{O}^i_{\Delta_i \cap \mathrm{Tr}(n)}(g_1, \ldots, g'_{n-1}, g'_n)$ is dense and thus there exists $(g'_1, \ldots, g'_n) \in \mathcal{O}_{\mathrm{Tr}(n)}(g_1, \ldots, g_n)$ such that $\|g'_i - h_i\| < \frac{\epsilon}{n}$ for all $i \in \{1, \ldots, n\}$. Thus, $\|(g'_1, \ldots, g'_n) - (h_1, \ldots, h_n)\| < \epsilon$. □

For $n \geq 4$, the argument above can be used without the construction of the sets X, Y_i. However, it is difficult to prove the statement in the case of $n = 3$ by appealing only to the set U from Definition 3.2. The construction of the set X allows the process used in the proof to be generalized for all $n \geq 3$.

4 Ergodicity of $\mathrm{Tr}(n) \curvearrowright G^n$

T. Gelander proved in [Gel08] that the action of $\mathrm{Aut}(F_n)$ on G^n is ergodic using the fact that $\mathcal{O}^k_{\langle L_{ki}, L_{kj}\rangle}(g_1, \ldots, g_n)$ coincides with the orbit under a dense subgroup $\langle g_i, g_j\rangle$ acting by left translation on G, $\mathcal{O}^k_{\langle g_i, g_j\rangle} g_k$, where $\langle g_i, g_j\rangle$ is dense for a.e. pair $(g_i, g_j) \in G \times G$. A similar argument can be made for $\mathrm{Tr}(n)$.

Namely, $\mathcal{O}^k_{\mathrm{Tr}(n)}(g_1, \ldots, g_n)$ coincides with $\mathcal{O}^k_{[H,H]} g_k$. Thus, the proof of Theorem 4.1 is an adaptation of T. Gelander's proof of Theorem 1.6 in [Gel08].

Theorem 4.1. *Let G be a compact, connected, semi-simple group, and $n \geq 3$. Then the action of $\mathrm{Tr}(n)$ on G^n is ergodic.*

Proof. Suppose for the sake of contradiction that the action is not ergodic. Then, there is some $\mathrm{Tr}(n)$-invariant set $\mathcal{A} \subseteq G^n$ that is neither null nor conull. Since the action of G itself by right translation is ergodic, $\mathcal{A}$ is not invariant under right translation by G on one of the coordinates, say the last factor. By Fubini's theorem, for a set of positive measure of $(g_1, \ldots, g_{n-1}) \in G^{n-1}$, the set $\{g \in G : (g_1, \ldots, g_{n-1}, g) \in \mathcal{A}\}$ is neither null nor conull.

Note that a.e. $(g_1, \ldots, g_{n-1})$ is such that $\langle g_1, g_2\rangle$ is dense in G (see Definition 3.2). Fix such a tuple $(g_1, \ldots, g_{n-1})$ and let $H = \langle g_1, \ldots, g_{n-1}\rangle$; obviously, H is dense in G. The orbits of the action of $\mathrm{Tr}(n)$ on $\{(g_1, \ldots, g_{n-1}, g) : g \in G\}$ projected to the last coordinate coincide with the orbits of the action of $[H,H]$ on G. Now, consider $\mathcal{A}_1 = \{g \in G : (g_1, \ldots, g_{n-1}, g) \in \mathcal{A}\}$ which is neither null nor conull by assumption. Since $\mathcal{A}$ is $\mathrm{Tr}(n)$-invariant, $\mathcal{O}_{\mathrm{Tr}(n)}(g_1, \ldots, g_{n-1}, g) \subseteq \mathcal{A}$ for all $g \in \mathcal{A}_1$. Thus, $\mathcal{A}_1$ is $[H,H]$-invariant. Since $[H,H]$ is dense in G, it acts ergodically on G and this is a contradiction. Therefore $\mathrm{Tr}(n)$ acts ergodically on G^n. □

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