PROBLEM SETS
NORTHWESTERN RTG SUMMER SCHOOL 2023

1. Invariant Random Subgroup

1.1. **Raz Slutsky.**

(1) Let G be a locally compact second countable group. Prove that $\mathrm{Sub}(G)$ is a compact space [1].

(2) Let Γ be a discrete countable group. Consider the space 2^Γ with the product topology. Show that the subset of subgroups inside 2^Γ is closed, and that it is homeomorphic to $\mathrm{Sub}(\Gamma)$.

(3) Let $\{\Gamma_n\}$ be a sequence of subgroups, and suppose that there exists $U \subset G$, an identity neighbourhood, such that $\Gamma_n \cap U = \{id\}$ for every Γ_n. Suppose that $\Gamma_n \to H$. Show that H is discrete. Show that if all the Γ_n are torsion-free, then so is H.

(4) Let G be a finitely generated group. Prove that every finite-index subgroup of G is an isolated point in $\mathrm{Sub}(G)$.

(5) Show that if G surjects on S^1, then G is not an isolated point in $\mathrm{Sub}(G)$.

(6) Show that if G acts on (X, μ) ergodically, then the associated invariant random subgroup is also ergodic.

(7) Fix a left-invariant metric d on G/K, a symmetric space. Show that the following sets form a basis of open sets around $\{e\}$ in $\mathrm{Sub}(G)$.

$$U_R = \{H \in \mathrm{Sub}(G) \mid \nexists h \in H \backslash \{e\} \; ; \; d([e], h[e]) \leqslant R\}$$

where $[e]$ is the projection of $e \in G$ to G/K. What does this say about a sequence of IRS's μ_n which converge to $\delta_{\{e\}}$? What does it say about a sequence of lattice IRS's, μ_{Γ_n} which converge to $\delta_{\{e\}}$?

1.2. **Kurt Vinhage.**

(1) Show that $\mathrm{Sub}(\mathbb{R}^n)$ is path connected.

1.3. **Mikolaj Fraczyk.**

(1) Let G be a simple Lie group. If Γ is a lattice and Γ' is a normal subgroup of Γ, then Γ' is confined.

2. Superrigidity and Arithmeticity

2.1. **Homin Lee.** Let Γ be a lattice in G. (Here, G will be always a simple non-compact "Lie group".) Reference is [Mor15].

(1) Prove that $\mathrm{SL}_2(\mathbb{Z})$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. (Hint: Use the hyperbolic plane.)

(2) (Unimodular group) Let μ be a left Haar measure on G. Define a measure $\widetilde{\mu}$ as $\widetilde{\mu}(A) = \mu(A^{-1})$ for any measurable set A. Show that $\widetilde{\mu}$ is a right Haar measure. Also, show that there is a homomorphism $\delta : G \to \mathbb{R}^+$ such that

$$\mu(gAg^{-1}) = \delta(g)\mu(A)$$

for all measurable set A in G. Conclude that G is unimodular, that is the left Haar measure is also the right Haar measure.

[1]Hint: First show that the space of closed subsets of a topological space is compact (using the Alexander sub-basis theorem). Then, show that the space of closed subgroups is a closed subset of it.

(3) (A property of higher rank)Let $G = \mathrm{SL}(3,\mathbb{R})$. Find subgroups $L_1, \ldots, L_r$ in G such that
  (a) $G = L_r \cdot \cdots \cdot L_1$,
  (b) $L_i \cap A$ is non-compact for all $i = 1, \ldots, r$, and
  (c) $\{a \in A : al = la, \quad \forall l \in L_i\}$ is non-compact for all $i = 1, \ldots, r$.

  Convince yourself that it is not true for rank 1 groups.

(4) Let Γ be a subgroup in $\mathrm{SL}(n,\mathbb{Q})$. Assume that

$$\Gamma \subset \{A = (a_{i,j})_{i,j} \in \mathrm{Mat}_{n\times n}(\mathbb{Q}) : \text{for all } i,j, \text{the denominator of } a_{i,j} < N\}$$

  for some N. Show that there is a finite index subgroup Γ' in Γ such that $\Gamma' \subset \mathrm{SL}(n,\mathbb{Z})$.

(5) (Compact factor) Let L, H be Lie groups. Let Λ be a lattice in the Lie group L. Let $\varphi : L \to H$ be a continuous surjective homomorphism with a compact kernel. Show that $\varphi(\Lambda)$ is a lattice in H.

(6) (Commensurator) Show that $\mathrm{SL}_3(\mathbb{Q}) \subset \mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\mathrm{SL}_3(\mathbb{Z}))$.

(7) (Restriction of Scalar) Let $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ and $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We will show that Γ can be realized as an arithmetic group ("integer points") so that it is a lattice in G by Borel and Harish–Chandra's theorem. Let $k = \mathbb{Q}(\sqrt{2})$, $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ and $\Delta : k \to \mathbb{R}^2$ be $\Delta(x) = (x, \sigma(x))$ where σ is the Galois conjugation, $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$ for $a, b \in \mathbb{Q}$. Prove the followings.
  (a) $\Delta(\mathcal{O})$ is discrete in $\mathbb{R}^2$. (This already suggests that Γ can be realized as a discrete subgroup in G using the embedding $\Sigma : \Gamma \to G$, $\Sigma(\gamma) = (\gamma, \sigma(\gamma))$.)
  (b) Show that $\{(1,1), (\sqrt{2}, -\sqrt{2})\}$ is a $\mathbb{Q}$-basis of $\Delta(k)$. Find a $\mathbb{Q}$ basis of $\Delta(k^2) = \{(v, \sigma(v)) : v \in k^2\}$. Let $\Delta(k^2) = V_{\mathbb{Q}}$. Note that $V_{\mathbb{Q}}$ is a $\mathbb{Q}$-form on $\mathbb{R}^4$, that is, $V_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space and $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^4$. Also $\Delta(\mathcal{O}^2)$ can be thought as an "integer lattice" in $V_{\mathbb{Q}}$.
  (c) Using the $\mathbb{Q}$-basis of $\Delta(k^2)$ you found in (c), show that $\Sigma(\Gamma)$ can be written as a subgroup of $\mathrm{SL}(4,\mathbb{Z})$.
  (d) Let $Mat_{4\times 4}(\mathbb{R})_{\mathbb{Q}} = \{A \in Mat_{4\times 4}(\mathbb{R}) : A(V_{\mathbb{Q}}) = V_{\mathbb{Q}}\}$. Show that

$$Mat_{4\times 4}(\mathbb{R})_{\mathbb{Q}} = \left\{ \begin{bmatrix} A & B \\ \sigma(A) & \sigma(B) \end{bmatrix} : A, B \in Mat_{2\times 2}(k) \right\}.$$

  (e) As $W = Mat_{4\times 4}(\mathbb{R})$ is a 16 dimensional vector space, we can think about a polynomial $f : W \to W$. We call that f is defined over $\mathbb{Q}$ with respect to a $\mathbb{Q}$-form $W_{\mathbb{Q}}$ if $f(W_{\mathbb{Q}}) \subset W_{\mathbb{Q}}$ where $W_{\mathbb{Q}} = Mat_{4\times 4}(\mathbb{R})_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of W induced by $V_{\mathbb{Q}}$. Find a set of polynomials Q such that 1) for all $f \in Q$ is defined over $\mathbb{Q}$ with respect to $W_{\mathbb{Q}}$ and 2) the common zero set G' of for all $f \in Q$ is isomorphic to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_4(\mathbb{R})$.
  (f) Similarly, define $Mat_{4\times 4}(\mathbb{R})_{\mathbb{Z}} = \{A \in Mat_{4\times 4}(\mathbb{R})_{\mathbb{Q}} : A(\Delta(\mathcal{O}^2)) \subset \Delta(\mathcal{O}^2)\}$. Show that $Mat_{4\times 4}(\mathbb{R})_{\mathbb{Z}} \cap G'$ is $\Sigma(\Gamma)$. Therefore, Γ is arithmetic, especially, Γ is a lattice in G.

2.2. **Kurt Vinhage.**

(1) Let $\Lambda < \mathbb{R}^n$ be a discrete subgroup. Prove that Λ is Zariski dense if and only if Λ is a lattice.

(2) Let $A \in \mathrm{GL}(n,\mathbb{R})$. We have a homomorphism $\varphi : \mathbb{Z} \to \mathrm{GL}(n,\mathbb{R})$ that is defined by $\varphi(1) = A$. Show that the $\mathbb{R}$ action on

$$\{(v,t) : v \in \mathbb{R}^n, 0 \leqslant t \leqslant 1\}/(v,0) \sim (Av,1)$$

  with $r.[(v,t)] = [(v, r+t)]$ is equivalent to the $\mathbb{R}$ action on

$$(\mathbb{R} \times \mathbb{R}^n)/\mathbb{Z}$$

  with $r.[(t,v)] = [(r+t, v)]$.

2.3. **Nick Miller.**

(1) Show that the inclusion $\mathrm{SL}_3(\mathbb{Q}) \subset \mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\mathrm{SL}_3(\mathbb{Z}))$ is proper, i.e., $\mathrm{SL}_3(\mathbb{Q}) \neq \mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\mathrm{SL}_3(\mathbb{Z}))$. Does $\mathrm{PGL}_3(\mathbb{Q})$ coincide with $\mathrm{Comm}_{\mathrm{PGL}_3(\mathbb{R})}(\mathrm{PGL}_3(\mathbb{Z}))$?

(2) Suppose you have semi simple $\mathbb{Q}$-group H (you can think about it as a subgroup of $\mathrm{SL}_n(\mathbb{R})$ such that there are finitely many polynomials (variables are matrix entries) with rational coefficients so that the intersection of zero sets is H) and a lattice $\Gamma < H(\mathbb{Q}) < H$ for which the image under the natural maps to $H(\mathbb{Q}_p)$ is bounded. Show that Γ is commensurable with $H(\mathbb{Z})$.

(3) Let $G = \mathrm{SL}(3, \mathbb{R})$ (more generally, simple Lie group). Let Γ be a lattice in G. Show that Γ has an infinite index in $\mathrm{Comm}_G(\Gamma)$ if and only if $\mathrm{Comm}_G(\Gamma)$ is dense in G.

(4) Let H be a group and L be a subgroup of H. Show that $\mathrm{Aut}_H(H/L)$ is isomorphic to $N_H(L)/L$.

2.4. **Amir Mohammadi.**

(1) Construct irreducible cocompact lattice in $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. Also, construct a reducible cocompact lattice in $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. (Hint: $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \simeq \mathrm{SO}(2, 2)^{\circ}$)

3. Kakutani equivalence

3.1. **Kurt Vinhage.**

(1) Let $T : (X, \mu) \to (X, \mu)$ be a probability measure preserving transformation, and $r : X \to \mathbb{R}_+$ be an L^1-function. Show that the suspension space can be constructed by quotienting $X \times \mathbb{R}$ by the action of the transformation $\tilde{T}(x, t) = (T(x), t - r(x))$.
   (a) Show that both the vertical flow and $\tilde{T}$ preserve the measure $\mu \times \mathrm{Leb}$ on $X \times \mathbb{R}$.
   (b) Show that $\mu \times \mathrm{Leb}$ induces a finite measure on the suspension space which is invariant under the vertical flow.
   (c) Find a formula for the total mass of the suspension space with respect to the invariant measure.

(2) Let $T : (X, \mu) \to (X, \mu)$ be a probability measure preserving transformation, and $r_1, r_2 : X \to \mathbb{R}_+$ be two L^1-functions. Show that the suspensions with roofs r_1 and r_2 are Kakutani equivalent by finding an explicit equivalence. [*Hint*: The time change functions $\tau(x, t)$ will be piecewise linear!]

(3) Let T, X, μ r_1 and r_2 be as in the previous problem. If there exists an L^1 function $f : X \to \mathbb{R}$ such that $r_2(x) = r_1(x) + f(T(x)) - f(x)$ for all $x \in X$, show the the suspesion flows with roofs r_1 and r_2 are measurably conjugate by finding an explicit conjugacy. [*Hint*: Let the segment above x "borrow" a segment of length $f(T(x))$ from the segment above $T(x)$ and "lend" a segment of length $f(x)$ to $T^{-1}(x)$]

(4) Let $\varphi_t : (X, \mu) \to (X, \mu)$ and $\psi_t : (Y, \nu) \to (Y, \nu)$ be probability measure preserving flows, and assume that $H : X \to Y$ is a Kakutani equivalence such that $H_*\mu = \nu$. Show that H is a measurable isomorphism. [*Hint*: First show that if $\tau : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism preserving Lebesgue and fixing 0, then $\tau = \mathrm{Id}_{\mathbb{R}}$]

3.2. **Homin Lee.**

(1) Let G_1 and G_2 be groups. Let X_1 and X_2 be G_1 and G_2 spaces, respectively. Let $T : G_1 \times X_1 \to X_1$ and $S : G_2 \times X_2 \to X_2$ be action maps. Assume that there is an orbit equivalence $H : X_1 \to X_2$ with respect to G_1 and G_2 actions. Define a map $\alpha : G_1 \times X_1 \to G_2$ as

$$\alpha(g, x) = h \iff H(T(g)(x)) = [S(h)](H(x)).$$

Show that α satisfies a cocycle equation, that is

$$\alpha(pq, x) = \alpha(p, T(q)(x))\alpha(q, x).$$

(2) Let G be a discrete group and X be a topological spce. Let $\rho_1 : G \times X \to X$ and $\rho_2 : G \times X \to X$ be 2 continuous actions on the space X by the group G_1 and G_2 respectively. Assume that ρ_1 and ρ_2 are orbit equivalent to each other. As above, we can define a cocycle $\alpha : G \times X \to G$ from the orbit equivalence. Show that ρ_1 and ρ_2 are topologically conjugate if and only if α is cohomologous with the identity map, that is there is a map $\varphi : X \to G$ such that $\alpha(g, x) = \varphi(\rho_1(g)(x))^{-1} g\varphi(x)$.

3.3. **Daren Wei.**

(1) Let $p(t) = \sum_{k=0}^{d} a_k t^k$ be a polynomial of degree d. For every $\epsilon > 0$, show that there exists $C = C(d) \geqslant 1$ such that if $|p(t)| \leqslant \epsilon$ for all $t \in [0, T]$, then $|a_k| < CT^{-k}\epsilon$ for all $k = 0, \ldots, d$. Conversely, show that if $|a_k| \leqslant C^{-1}T^{-k}\epsilon$ for all k, then $|p(t)| < \epsilon$ for all $t \in [0, T]$.

(2) Let G a connected group. Show that if H is a discrete normal subgroup of G, then H is contained in the center of G.

(3) Let's define $\bar{f}$ as follows.

Definition 3.1. If $w, w' \in \{1, 2, \ldots, m\}^n$, then

$$\bar{f}_n(w, w') = 1 - k/n$$

where k is the maximal integer for which we can find subsequences $i_1 < i_2 < \ldots < i_k$, $j_1 < j_2 < \ldots < j_k$ with $w(i_r) = w'(j_r)$ for $1 \leqslant r \leqslant k$.

(a) Show $\bar{f}_n$ defines a metric on $\{1, 2, \ldots, m\}^n$.
(b) if $w = 1212\ldots12$, i.e. n copies of 12, $w' = 2121\ldots21$, i.e. n copies of 21, what is $\bar{f}_n(w, w')$?
(c) if w is k copies of $123\ldots m$ and w' is in the form k copies of 1, k copies of 2, …, k copies of m, what is $\bar{f}_{mk}(w, w')$?

(4) Suppose that T is an ergodic measure preserving transformation acting on (X, μ) and $A \subset X$ is a measurable subset. For every $x \in X$, let $r_A(x) = \min\{k \geqslant 1 : T^k x \in A\}$ and define T_A as

$$T_A x = T^{r_A(x)} x.$$

(a) Show that $\int_A r_A(x) d\mu = 1$.
(b) Show that T_A is ergodic with respect to measure $\mu_A(\cdot) = \frac{\mu(\cdot)}{\mu(A)}$.
(c) If $A_1 \subset A_2$, show that $(T_{A_2})_{A_1} = T_{A_1}$.

(5) Suppose that S is an ergodic measure preserving transformation on (Y, ν) and $h : Y \to \mathbb{Z}^+$ has finite integral. Let $Y^h = \{(x, i) : x \in Y, 1 \leqslant i \leqslant h(x)\}$ and define S^h as

$$S^h(x, i) = \begin{cases} (x, i+1), & \text{if } i + 1 \leqslant h(x); \\ (Sx, 1), & \text{if } i + 1 > h(x). \end{cases}$$

Moreover, let ν^h be the product measure of ν and Lebesgue measure on $\mathbb{R}$ that normalized by $\int_Y h d\nu$.
(a) Show that S^h is ergodic with respect to measure ν^h.
(b) Suppose that $A \subset Y$ is a measurable subset, show that $(S_A)^{r_A}$ is measurable isomorphic to S.

4. Homogeneous dynamics

4.1. **Osama Khalil.**

(1) Let $G = \mathrm{SL}(2,\mathbb{R})$, Show that

$$\pi : N^{-1} \times A \times N^{+} \to G, \quad \pi(a,b,c) = abc$$

is a diffeomorphism near (id,id,id) onto its image where

$$N^{-} = \left\{ \begin{bmatrix} 1 & 0 \\ * & 0 \end{bmatrix} \right\}, \quad A = \left\{ \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \right\}, \quad N^{+} = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}.$$

4.2. **Amir Mohammadi.**

(1) Fix $0 \leqslant \delta < 1$. Show that there is $c(\delta) > 0$ such that for any $t > 0$ and $v \in \mathbb{R}^2 \backslash \{(0,0)\}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{||a_t r_\theta v||^{1+\delta}} \leqslant \frac{c(\delta) e^{-t(1-\delta)}}{||v||^{1+\delta}}$$

where $a_t = \mathrm{diag}(e^{2t}, e^{-2t})$ and r_θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(2) Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. For every $0 < \alpha < 1$ there is $\beta = \min\{\alpha/2, (1-\alpha)/2\}$ such that

$$\int_0^1 \frac{dr}{||a_t u_r w||} < \frac{C e^{-\beta t}}{||w||^{\alpha}}$$

for all $\omega \in \mathfrak{g} \backslash \{0\}$.

5. (General) Dynamics

5.1. **Osama Khalil.**

(1) (Weyl's Theorem) Let $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ and denote by $R_\alpha : S^1 \to S^1$ the rotation by α. Prove that for every $x \in S^1$, the orbit of x under α is equidistributed, i.e., for every open set $E \subseteq S^1$,

$$\lim_{N\to\infty} \frac{\#\left\{0 \leqslant n \leqslant N-1 : R_\alpha^n(x) \in E\right\}}{N} = \mathrm{Leb}(E).$$

This can be done in the following steps:

(a) Prove that it suffices to show that for every continuous function $f \in C(S^1)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(R_\alpha^n(x)) = \int_{S^1} f \, d\mathrm{Leb}. \tag{5.1}$$

(b) Use density of trig polynomials in $C(S^1)$ to show that it is enough to prove (5.1) for the functions $f(y) = e^{2\pi i k y}, k \in \mathbb{Z}$.

(c) Prove (5.1) for $f_k(y)$ [Hint: geometric sums].

(2) Show that $h_{top}(R_\alpha) = 0$ for any $\alpha \in \mathbb{R}$.

(3) Let $f : S^1 \to S^1$ be an expanding map and μ be an invariant probability measure under f. Let $\varepsilon := \inf_x |f'(x)|/2$ and let Ξ be a finite measurable partition of S^1 such that each element of Ξ has diameter at most ε. Prove that Ξ realizes the metric entropy of μ, i.e. prove that

$$h_\mu(f) = \lim_{n\to\infty} \frac{\log H\left(\bigvee_{k=0}^{n-1} f^{-k}(\Xi)\right)}{n}.$$

Recall that for any partition $\mathcal{P}$, $H(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P)$.

(4) (van der Corput's Trick) Prove that there is a constant $C \geqslant 1$ so that for every $N, H \geqslant 1$ and every sequence $(a_n)_n$ of complex numbers of magnitude at most 1, we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} a_n\right| \leqslant \left(\frac{1}{H}\sum_{h=0}^{H-1}\frac{1}{N}\sum_{n=1}^{N} a_n a_{n+h}\right)^{1/2} + \frac{CH}{N}.$$

[Hint: Cauchy-Schwarz.]

(5) Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Prove that the sequence $(n^2\alpha)_{n\geqslant 1}$ is equidistributed $\mod 1$, i.e., for every open set $E \subseteq [0,1)$,

$$\lim_{N\to\infty}\frac{\#\left\{0 \leqslant n \leqslant N-1 : n^2\alpha \mod 1 \in E\right\}}{N} = \mathrm{Leb}(E).$$

[Hint: Apply van der Corput's trick with $a_n = \exp(2\pi i k n^2\alpha)$ for arbitrary fixed k then use Weyl's criterion (Weyl's Theorem).]

(6) Show that $h_{top}(R_\alpha) = 0$ for any $\alpha \in \mathbb{R}$.

(7) Let $f : S^1 \to S^1$ be an expanding map and μ be an invariant probability measure under f. Let $\varepsilon := \inf_x |f'(x)|/2$ and let Ξ be a finite measurable partition of S^1 such that each element of Ξ has diameter at most ε. Prove that Ξ realizes the metric entropy of μ, i.e. prove that

$$h_\mu(f) = \lim_{n\to\infty}\frac{H\left(\bigvee_{k=0}^{n-1} f^{-k}(\Xi)\right)}{n}.$$

Recall that for any partition $\mathcal{P}$, $H(\mathcal{P}) = -\sum_{P\in\mathcal{P}}\mu(P)\log\mu(P)$.

5.2. **Solly Coles.**

(1) Prove irrational rotations are minimal.
(2) Find a symbolic coding for the expanding map $E_m : \mathbb{S}^1 \to \mathbb{S}^1$, $E_m(x) = mx(\mathrm{mod}1)$. Here $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.
(3) Describe the periodic orbits of $E_m(x)$ (notice that the growth rate as period goes to infinity is the topological entropy).
(4) Show topological entropy is a (topological) conjugacy invariant.
(5) Show that the definitions of entropy by separated sets and spanning sets are equal.
(6) Calculate topological entropy of the cat map.
(7) Show the cat map is transitive (or harder - mixing).
(8) Show Lebesgue is ergodic for E_m.
(9) Show $\deg(f)$ is C^0 locally constant for f a smooth expanding map of the circle.
(10) Show that the Lyapunov exponent $\lambda(f,x)$ is independent of base point x for expanding map f (up to measure 0).

5.3. **Kurt Vinhage.**

(1) Find an expanding map f such that the function $\lambda(x) = \limsup_{n\to\infty}(1/n)\log\left[(f^n)'(x)\right]$ is nowhere continuous. (Hint: it is constant almost everywhere by Birkhoff ergodic theorem. Get it to be different at every periodic point.
(2) Let f be an expanding map of the circle and μ be an f-invariant probability measure. Recall the metric d_n given by the maximal distance between the points along the orbit of two points up to time $n-1$. Given $0 < \epsilon < 1$ and n, let $N(\epsilon, n)$ denote the minimal number of ϵ-balls in the metric d_n needed to cover a Borel set of measure at least 1/2. Prove that the following special case of a result of Anatoly Katok which says that the metric entropy $h_\mu(f)$ is equal to

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}(1/n)\log N(\epsilon, n)$$

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References

[Mor15] D. W. Morris, *Introduction to arithmetic groups*, Deductive Press, 2015. MR3307755 ↑1