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Discovering Auctions: Contributions of Paul Milgrom and Robert Wilson*

Alexander Teytelboym ⓡ

University of Oxford, Oxford, OX1 3UQ, UK
alexander.teytelboym@economics.ox.ac.uk

Shengwu Li ⓡ

Harvard University, Cambridge, MA 02138, USA
shengwu_li@fas.harvard.edu

Scott Duke Kominers ⓡ

Harvard University, Cambridge, MA 02138, USA
kominers@fas.harvard.edu

Mohammad Akbarpour ⓡ

Stanford University, Stanford, CA 94305, USA
mohamwad@stanford.edu

Piotr Dworczak ⓡ

Northwestern University, Evanston, IL 60208, USA
piotr.dworczak@northwestern.edu

Abstract

The 2020 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel was awarded to Paul R. Milgrom and Robert B. Wilson for "improvements to auction theory and inventions of new auction formats". In this survey article, we review the contributions of the laureates, emphasizing the subtle interplay between deep theoretical questions and practical design challenges that resulted in one of the most successful fields of economics.

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I. Introduction

The 2020 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel was awarded to Paul R. Milgrom and Robert B. Wilson for "improvements to auction theory and inventions of new auction formats". Auction design is one of the great success stories in economics – beautiful ideas deeply embedded in modern economic theory, which have led to real-life applications with far-reaching welfare gains. But auction design did not trace a simple, linear path. Instead, insights about auction design emerged from a continuous conversation between theory and practice.[1,2]

Prices play a crucial role in markets, coordinating production and consumption, and equilibrating supply and demand. To accomplish this task, they must use "knowledge which is not given to anyone in its totality" (Hayek, 1945), reflecting the private information of many individuals.

Wilson and Milgrom started with a fundamental question: how do prices come about? Competitive rational expectations models often predict that prices are fully revealing – that is, prices reflect all the information that traders hold. Under such pricing, each trader should ignore their own private information about underlying asset values when choosing what to buy or sell. But how can information come to be reflected in prices if no trader acts on it?

Auction theory opens the black box of price formation, studying how particular market institutions generate prices in the presence of strategic traders with private knowledge. In his pioneering work from the late 1970s, later extended by Milgrom, Wilson showed how auctions can be thought of as a foundation for the competitive price-formation process.

Auctions are an appealing foundation for prices because they are ubiquitous in real-world economic interactions. Auctions have been used to sell a wide variety of goods, such as art, fish, real estate, Treasury Bills, and, in 193 AD, the entire Roman Empire (Gibbon, 1776).

In early analyses of auctions (see, e.g., Vickrey, 1961), it was assumed that the bidders' values for a good are private and independently drawn. In such cases, under standard assumptions, the details of the auction are irrelevant – all common auction formats yield the same revenue and the same expected payoffs for bidders. In the course of exploring the

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[1] This Nobel Prize is one of several awarded for innovations in market and mechanism design, following those to: James Mirrlees and William Vickrey in 1996; Leonid Hurwicz, Eric S. Maskin, and Roger B. Myerson in 2007; Alvin E. Roth and Lloyd S. Shapley in 2012; Jean Tirole in 2014; and Oliver Hart and Bengt Holmström in 2016, surveyed in this journal by Dixit and Besley (1997), Drèze (1997), Mookherjee (2008), Jackson (2013), Serrano (2013), Fudenberg (2015), and Schmidt (2017).

[2] Milgrom (2021) and Wilson (2021) have reflected on their work in their respective Nobel Lecture articles. For other surveys, see Arozamena *et al.* (2021) and Biró and Magyarkuti (2021).

possibility that prices reflect the fundamental value of assets, Wilson and Milgrom developed theories of auctions for environments in which bidders' information is correlated. Under these richer informational assumptions, they then demonstrated that the choice of auction format can, in fact, affect revenue and bidders' payoffs. Milgrom and Weber (1982) developed the canonical framework for interdependent values, and proved that the standard auction formats can be unambiguously ranked in terms of their expected revenue. In this way, the literature on auctions as a price-formation process planted the seeds for a literature on practical auction design.

The link between auctions and price discovery – initially motivated by theoretical inquiry – proved useful when "putting auction theory to work". Wilson and Milgrom, together with various collaborators, realized that the desirable features of competitive-equilibrium outcomes could be achieved by designing auctions to mimic the properties of competitive equilibria, while accounting for strategic behavior under asymmetric information. For example, the classical tâtonnement processes from general equilibrium theory did not incorporate trader incentives. However, more structured versions of the tâtonnement process, such as the one introduced by Kelso and Crawford (1982), could serve as algorithms to discover prices in fairly complex allocation problems with privately informed and strategic agents. Somewhat paradoxically then, while Wilson and Milgrom had initially proposed auctions as an alternative to the abstract tâtonnement process in providing an explanation for emergence of competitive prices, they ended up embracing tâtonnement-like auction mechanisms in practical applications.

These insights, combined with novel design features such as the activity rule, culminated in the development of the simultaneous multi-round auction (SMRA), which was first used in the 1994 United States Federal Communications Commission (FCC) auctions for allocating wireless spectrum rights. This successful auction design, beyond its immediate welfare impacts, encouraged regulators around the world to recruit economists to help design markets. At the same time, the SMRA inspired a research program that strengthened fundamental links between auctions, matching markets, and the existence of equilibrium in markets with indivisibilities. Milgrom's work, in particular, has been instrumental to our understanding of how these connections relate to whether bidders view goods as substitutes or as complements. The resulting theoretical insights spurred the creation of numerous auction formats that can accommodate rich bidder preferences and that are now used in many applications.

Wilson and Milgrom led economic theorists to a deeper understanding of auction design under information and incentive constraints. From there, real-world applications of auction theory have revealed other important constraints, having to do with computation and communication complexity. In the past two decades, economists, computer scientists, and operations researchers have jointly tackled crucial questions of fast computation of allocation and payment rules, as well as the design of bidding languages that are effective at eliciting bidder preferences. One leading application of this work was the FCC's 2017 Incentive Auction, which Milgrom co-designed; and new challenges are emerging in many markets including electricity and online advertising. Future auction theorists may therefore continue asking the same types of questions as Wilson and Milgrom did – and come up with exciting new answers.

II. Auctions as a Strategic Foundation for Competitive Equilibria

One foundational idea in economics, with origins as far back as *The Wealth of Nations* (Smith, 1776), is that competitive markets lead to efficient allocation of resources and aggregation of economic information in equilibrium prices. By the late 1970s – when our investigation into the contributions of Wilson and Milgrom begins – the theoretical pillars of this idea had been developed and formalized. Building on the insights of Walras (1874), general equilibrium theory took center stage in economics following the influential work of Arrow and Debreu (1954) and McKenzie (1954). Walrasian equilibria are allocatively efficient, by the First Welfare Theorem. Muth (1961) paved the way for incorporating uncertainty and information into the competitive-equilibrium framework by proposing that agents have "rational expectations" and understand the link between equilibrium prices and the underlying unknown states of the world. This conceptual breakthrough allowed subsequent research to demonstrate formally that competitive prices could reveal all of the information in the economy to traders (Grossman, 1976; Radner, 1979; Allen, 1981). Related and equally prominent was the "efficient market hypothesis" laid out by the work of Samuelson (1965) and Fama (1965, 1970). In its strongest form, the efficient market hypothesis predicts that equilibrium prices reflect all payoff-relevant information, so that no individual trader can make excess profits on their own information; weaker forms of the hypothesis postulate that prices should, at the very least, incorporate all publicly available information.[3]

While highly influential, these theories remained controversial. Perhaps their main weakness was the black-box approach to equilibrium prices: both in general equilibrium theory and under the efficient market hypothesis, no explanation is provided as to how prices are formed. The "Walrasian auctioneer" is only a metaphor; and the tâtonnement process assumes

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[3] Early traces of this idea can be found in the pioneering, albeit initially underappreciated, work of the French mathematician Bachelier (1900), who put forth the idea that speculation is not possible when asset prices follow a random walk.

that participants naïvely declare their favorite bundles with no strategic manipulation. To make matters worse, it gradually became apparent that the two theories are logically inconsistent with one another. Kreps (1977) observed that when prices communicate payoff-relevant information to traders, a competitive equilibrium can fail to exist even in well-behaved cases. Grossman and Stiglitz (1980) argued that if prices aggregate all relevant information, then the information market cannot be in equilibrium, as then individual traders would have no incentives to acquire information – and if no one acquires any information, then how can prices aggregate it? Even if information were freely available, traders would lack reason to act on it, as they could do weakly better by conditioning their actions on the fully revealing equilibrium price. Additionally, the no-trade theorem of Milgrom and Stokey (1982) implies that prices are unlikely to be found as a consequence of different traders "betting" on the values of assets based on their private information, as such information-based trading is inconsistent with common knowledge of rationality.

The auction theory contributions of Wilson and Milgrom eventually helped clarify and explain some of these deep theoretical mysteries.[4] Wilson's early work on auctions, however, was inspired by real-life bidding scenarios: Wilson (1967) offered a pioneering game-theoretic analysis of an auction in which the object for sale has the same value to both bidders (the common-value model), but one of them has superior information about it;[5] Wilson (1969) studied a symmetric model in which both bidders have equally precise but different estimates of the common value. In a groundbreaking insight, Wilson realized that his common-value auction models could provide the missing foundation for competitive prices and their desirable informational properties. As a game theorist, Wilson was dissatisfied with the fact that classical theories were not strategic. The analysis of practical selling mechanisms made it salient that the descriptions of tâtonnement or the Walrasian auctioneer were too abstract to explain how information becomes incorporated into equilibrium prices.[6]

Auctions, by contrast, fit the bill perfectly; they describe a strategic interaction between players endowed with private information that results in a final price and allocation via a well-defined set of rules. Moreover, auctions have been widely used to buy and sell for millennia. The

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[4] Of course, we do not claim that the "mystery" of competitive prices has been completely solved. For example, the exact scope and validity of the efficient market hypothesis is still being debated; see Malkiel (2003) and Sewell (2011) for recent overviews of key contributions.

[5] Wilson credits Woods (1965) for identifying "an interesting real instance of competitive bidding under uncertainty with asymmetrical information" in the context of two major oil companies bidding via sealed tender for rights to an offshore parcel.

[6] From private communication with Robert B. Wilson.

common-value settings that Wilson introduced gave a precise meaning to the question of whether prices could reflect the "true value" of the good. In two leading contributions, Wilson (1977, 1985) used auction models to provide foundations for the information aggregation property and allocative efficiency of Walrasian equilibria, while Milgrom (1981b) explained how common-value auctions could resolve the Grossman–Stiglitz paradox.

Wilson (1977) investigated the claim that competitive prices can successfully uncover the fundamental value of an asset by aggregating dispersed information. To that end, he constructed a simple model of a sealed-bid first-price auction in a common-value environment. The key idea was that bidders privately observe informative signals about the unknown (but common to all) value of the asset.[7] It was clear that auctions would allow for market power when the number of bidders is small and/or when a bidder controls a signal with unique informational content. Thus, Wilson studied the limit of the symmetric case as the number of bidders goes to infinity – approaching perfect competition, while still retaining the strategic interaction and dispersed information aspects of an incomplete-information game.

The main difficulty in the analysis was dealing with the famous "winner's curse". The winner's curse arises when the fact of winning the auction is itself an informative signal of the asset's value to the winner. Wilson's earlier papers (Wilson, 1967, 1969), as well as the highly influential (although sometimes overlooked) PhD dissertation of Ortega-Reichert (1968), provided the necessary game-theoretic tools to deal with this conceptual and technical challenge. In a monotone equilibrium of a symmetric common-value auction, winning is equivalent to learning that the maximum of all other agents' estimates of the value are lower than that of the winner. As a result, a rational bidder will optimally bid by conditioning on the event that she observed the most optimistic signal realization, hence significantly revising downwards the estimate of the value compared with a naïve inference based solely on her private signal.

Wilson (1977) incorporated that reasoning into the equilibrium computation of optimal bids, while deriving the bidding function in closed form under some regularity conditions. The most significant such condition required that signal realizations can be ordered, and that higher signal realizations correspond to conditional distributions of the value that are higher in the stochastic order.[8] Additionally, Wilson assumed a one-to-one

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[7] In fact, Wilson's formulation allowed for bidders' utility to depend on their private signals even conditional on the true value of the asset. However, here we focus on the pure common-value case in which the asset is worth the same amount to all bidders.

[8] The ideas surrounding the relevant notions of stochastic orders were later clarified and expanded by Milgrom.

mapping between the value and the highest possible signal realization that a bidder could observe.

Under these assumptions, Wilson (1977) proved that the winning bid – the price – converges almost surely to the true value of the asset as the number of bidders grows. Therefore, market prices do aggregate all relevant information, in the sense that conditional on the equilibrium price, all remaining signals become redundant as sources of information about the value of the asset. This result was the first formal example demonstrating that a real-life market institution – an auction – can aggregate all the relevant information in a strategic environment.

These ideas were substantially generalized by Milgrom (1979), who continued to focus on a one-item first-price auction but studied general valuations of the bidders, allowing for risk aversion, non-quasi-linearity in monetary transfers, differences in utilities from holding the asset, and (moderate) disagreements in prior beliefs about the distribution of the common-value component. Milgrom defined the value of the asset to be the maximal amount that any bidder would be willing to pay if she could directly observe the common-value component. Finally, Milgrom assumed that bidders observe private signals that are distributed symmetrically and independently conditional on the common-value component. In this general environment, Milgrom (1979) proved that the winning bid converges in probability to the value of the asset if and only if the signal distribution satisfies a key condition, which states that for any possible value v of the asset, there exists a signal realization under which the posterior probability that the asset is worth v is arbitrarily more likely than the event that the asset is worth less than v. This condition was substantially weaker than Wilson's assumptions that effectively required that there exists a signal realization under which v is possible while any lower value has probability 0. From an economic perspective, Milgrom's condition clarified that it is possible for the equilibrium price to converge to the value of the asset even though no single bidder observes that value exactly. At the same time, Milgrom's result showed that the existence of arbitrarily informative signals is necessary for the information aggregation property.[9]

The full economic force of these insights was demonstrated by Milgrom (1981b), who considered a symmetric model in which bidders' utility functions are the same, but their values for the asset may depend not only on the common-value component but also on private signals. Milgrom (1981b) assumed that signals have the "monotone likelihood ratio property" (MLRP), a concept from statistics that he introduced to

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[9] For the case when the hazard rate of the signal is bounded, Di Tillio *et al.* (2021) quantified the amount of information revealed by the winning bid, and characterized conditions under which it is increasing or decreasing in the number of bidders.

economics (Milgrom, 1981a). The MLRP requires that signal realizations can be ordered in such a way that, regardless of the prior distribution of the common-value component, higher signal realizations lead to posterior distributions of the common-value component that are higher in the sense of first-order stochastic dominance.

Importantly, Milgrom (1981b) studied a uniform-price auction with k units of the asset, an extension of the second-price auction of Vickrey (1961). The benefit of this auction format for studying information aggregation is that it effectively turns bidders into price-takers – the price for any bidder is the kth highest competing bid. In contrast to the rational-expectations model, however, that price is not observed by an agent when deciding how much to bid. Milgrom showed that in a monotone equilibrium, each bidder bids her expected value for the asset conditional on her signal and the event that she "ties" for the kth unit of the asset, which is informationally equivalent to learning that the kth highest signal among competitors was equal to hers. Thus, a rational player bases her action on her own information, while incorporating the anticipated learning from the equilibrium price into her strategy. Milgrom (1981b) could then prove that this strategy remains optimal ex post, and hence would be unchanged if the bidder could actually observe her price in advance. At the same time, a bidder with no information would not make any profit. Thus, Milgrom showed that the game-theoretic analysis of the uniform-price auction can resolve the Grossman–Stiglitz paradox.[10] The price, at best, could reveal the information of others but not the information of the bidder herself; hence, there is no contradiction in assuming that observing the price leads to full learning and that agents act on (and pay for) their private information.

The adoption of a uniform-price auction in the work of Milgrom (1981b) seems to have been motivated by an attempt to explain informational paradoxes in strategic models. However, Milgrom (1981b) made it salient that when bidders' information is correlated (e.g., because of the existence of a common-value component), the auction format matters for the results of the auction; thus, it contained the seeds for later work on auction design.

While our review focuses on auctions, it is worth emphasizing that in the context of markets aggregating information, Milgrom also made a substantial contribution to the finance literature. Glosten and Milgrom (1985) constructed a model in which an initially uninformed market-maker gradually learns from the order flow by posting a bid–ask spread. Similarly to the bidders in the setting of Milgrom (1981b), the market-maker in the setting of Glosten and Milgrom (1985) chooses the bid–ask spread anticipating the informational content of each transaction.

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[10] As defined by Vives (2011) and Rostek and Weretka (2012), prices can be "privately revealing".

As the number of transactions grows, the bid–ask spread narrows and reflects all information available in the market. Together with Kyle (1985), the work of Glosten and Milgrom (1985) became a cornerstone of the market microstructure literature.[11] Similarly to auction theory, market microstructure analyzes explicit trading mechanisms by which the behavior of strategic and asymmetrically informed agents determines equilibrium prices.[12]

The work discussed thus far was primarily concerned with information aggregation, and studied a one-sided market. Wilson (1985) instead focused on the allocative efficiency property of Walrasian equilibria. The challenge was to incorporate dispersed information, strategic behavior, and realistic market institutions – all of which were absent in the original formulation – into a two-sided market. Wilson proposed a model with (potential) buyers and sellers differing in their private valuations for the asset. The trading mechanism was a uniform-price double auction. Under such a design, buyers and sellers submit their bids, and a price is found to maximize the number of units traded. Buyers with bids above the price and sellers with offers below the price all trade at that same price. Bidders are strategic, and realize that their bids can influence the equilibrium price. Unlike in the one-sided case, closed-form expressions for equilibrium strategies are not generally available.

In his analysis, Wilson (1985) relied on the notion of incentive efficiency developed by Holmström and Myerson (1983). Incentive efficiency of a trading mechanism requires that there exists no other trading rule that would improve some agent's expected gains from equilibrium trade without reducing others' expected gains. For that, it suffices that the trading mechanism maximizes a sum of agents' virtual valuations, properly weighted by type-dependent welfare weights, subject to feasibility. Similarly to the revenue-maximization case of Myerson (1981), virtual valuations in this context capture the idea that a change in behavior of one type of an agent influences the equilibrium rents of other types of that agent. Overall, to prove incentive efficiency of the double auction, it is enough to find a set of non-negative welfare weights under which the equilibrium outcome maximizes the corresponding welfare function.

Under the assumption of a regular distribution of buyers' and sellers' valuations, Wilson (1985) proved that the welfare weights converge to 1 as

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[11] While Wilson's work had less immediate repercussions for the study of market microstructure, Wilson (1979) provided perhaps the earliest traces of the quadratic-Gaussian framework that permeates the microstructure today. Other important contributions include those of Grossman (1981), Kyle (1989), and Klemperer and Meyer (1989); see Rostek and Yoon (2021) for a survey.

[12] Price discovery and information aggregation remain central research questions in market microstructure; for a review of classical contributions, see O'Hara (1995).

the number of traders goes to infinity. As a result, in the limit, a double auction maximizes the sum of all agents' utilities. The key intuition behind this result is that when the market becomes large, each trader's marginal contribution to the determination of the equilibrium price becomes small. Thus, even though bidders are strategic, they effectively become "price-takers" in a large market.[13]

This result showed that allocative efficiency can be achieved in a setting with private information and strategic agents by a real-life mechanism – a double auction. This focus on practical mechanisms was central to Wilson's work. Some earlier mechanism-design approaches to efficiency relied largely on more abstract mechanisms whose designer must carefully set up payments that depend on the detailed structure of the participant's values and their distributions (see, e.g., d'Aspremont and Gérard-Varet, 1979). In contrast, Wilson emphasized that a double auction can serve the role of a "market institution" because it is detail-free, in the sense that its rules do not vary with the details of the economic environment. The equilibrium of the double auction did require the agents themselves to "cope with the complexity of the common knowledge features". However, Wilson's conviction that theory should be grounded in economic practice soon led to him to formulate an even stronger prescription for theoretical work: "I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems". This quotation from Wilson (1987) became known as the "Wilson doctrine" and continues to be one of key principles guiding research progress in market and mechanism design.[14]

III. The Theory of Auction Design

By opening the black box of price-formation, auction theory enabled economists to study how prices and allocations depend on the fine details of market institutions. Rather than limiting attention to mechanisms tailored to particular assumptions about information, auction theorists studied real-world auction formats that had seen use across a variety of contexts.

Of the standard auction formats, four are especially ubiquitous (Cassady, 1967). In a "first-price auction", all bidders simultaneously submit bids, and the highest bidder wins the object at a price equal to the highest bid. In a

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[13] The idea that it is approximately optimal for agents to behave as price-takers in large markets was also explored by Roberts and Postlewaite (1976).

[14] The recent research on "robust" mechanism design, for example, seeks to systematically reduce auction theory's reliance on strong common-knowledge assumptions about bidders' information (Bergemann and Morris, 2005; Chung and Ely, 2007; Bergemann *et al.*, 2017, 2019; Brooks and Du, 2021).

"Dutch auction", the auctioneer starts by calling out a high price and then gradually reducing it until one bidder claims the object at the standing price (and thus ends the auction). Hence, each bidder must decide on the price at which she will claim the object, provided it is still available at that price. The winner is the bidder with the highest claim-price, and she pays her claim-price – so the claim-prices are isomorphic to bids in a first-price auction; that is, Dutch auctions and first-price auctions are strategically equivalent (Vickrey, 1961).

In an "English auction", meanwhile, the auctioneer starts by calling out a low price and then gradually raises it. When all but one bidder has withdrawn, the final bidder wins at the standing price. We assume that each bidder chooses, at each moment, whether to stay in the auction or to quit irrevocably; moreover, each bidder observes when other bidders quit.[15] In a "second-price auction", bidders simultaneously submit bids, and the highest bidder wins at a price equal to the second-highest bid.

The rules of the auction alone are not enough to conduct a game-theoretic analysis. To complete the description of the game, we must specify what each bidder knows, and how their preferences depend on what is known. Much early research in auction theory relied on either the independent private-values model of Vickrey (1961) or common-value models such as those discussed in Section II.[16] These models assume that bidders are risk-neutral, and that the value of winning the object is additively separable from the utility for money; however, they make divergent assumptions about information and valuations.

Recall that in the common-value framework, the object for sale is worth the same amount V to every bidder, with V drawn from a known distribution. Each bidder receives a signal about V, and these signals are independent conditional on V.

In the independent private-values model, by contrast, each bidder i has their own value v_i for the object, drawn independently from a continuous distribution with full support on a bounded interval. This model plausibly describes persons bidding for a good that will be consumed, without the possibility of resale, with the variation in values due to idiosyncratic tastes. A useful benchmark is the symmetric independent private-values model, in which every bidder's value is drawn from the same distribution.

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[15] Some variants of the English auction do not satisfy these assumptions (Cassady, 1967; Ashenfelter, 1989). In one variant, bidders call out successively higher bids until no bidder is willing to call out a yet higher bid. In another variant, bidders make bids using hidden gestures that preserve their anonymity. The variant studied by Milgrom and Weber (1982) is the "clock" version of the English auction, sometimes called a "Japanese auction".

[16] Common-value models were also studied by Rothkopf (1969), Reece (1978), and Maskin and Riley (1980).

Under private values, each bidder in an English auction has a dominant strategy – namely, to bid until the price reaches her value, and then to quit. Quitting thresholds in the English auction are isomorphic to bids in a second-price auction, so in the second-price auction it is a dominant strategy to bid one's value. However, the situation is different in the common-value model: in an English auction, seeing other bidders quit yields information about the common value, so the optimal bidding strategy is not straightforward, and the equivalence to second-price auctions does not hold.

The starkest prediction of the symmetric independent private-values model is that the details of the auction format do not matter. All four standard formats yield the same expected revenue, and even the same interim expected payoffs (see Vickrey, 1962; Ortega-Reichert, 1968; Myerson, 1981; Riley and Samuelson, 1981). Hence, while the independent private-values model is a tractable benchmark, it does not yield strong guidance for auction design.

Milgrom and Weber (1982) made a breakthrough by proposing a model of "interdependent values". This model allows for richer preferences and information structures, and nests both the common-value model and the symmetric independent private-values model as special cases. Working with the interdependent-values model yielded tractable characterizations of bidding strategies in second-price and ascending auctions, which had previously not been well understood except under private values. Milgrom and Weber (1982) showed that under interdependent values, the standard auction formats could be ranked in terms of expected revenue, often strictly. Revenue equivalence under independent private values thus turned out to be the extreme case of a more general model.

The interdependent-values model has two key ingredients. The first ingredient is to allow values to be partly private and partly common. Each bidder i observes a real-valued signal s_i, and her value for the object $v_i(s)$ is a function of all the signals, non-decreasing in each argument, and invariant when we permute the signals of the other bidders. The private-values model is a special case, setting $v_i(s) = s_i$. The common-value model is another special case, setting $v_i(s) = \mathbb{E}(V \mid s)$, where V is the common value of the object.

The second ingredient of the interdependent-values model is to assume that the bidders' signals are "affiliated" random variables. This means that if we take any pair of signals, and condition on all the other signals, then that pair has the weak MLRP (Jewitt, 1991, p. 177). Hence, raising one signal is unambiguously good news about every other signal (Milgrom, 1981a), which enables clean comparative statics. Affiliation is stronger than non-negative correlation, and allows for independent signals and also for conditionally independent signals of a common value.

Having unified two benchmarks in a general model, Milgrom and Weber (1982) examined the relationship between information, auction formats, and expected revenues. Under interdependent values, the standard formats can be ranked in terms of revenue, often strictly. English auctions yield the most expected revenue, followed by second-price auctions, followed by first-price and Dutch auctions. The key insight is that in a first-price auction or a Dutch auction, the price paid by the winning bidder depends only on her own signal. By contrast, in an English auction, when one bidder quits, the remaining bidders can infer that bidder's signal and adapt their behavior accordingly. Consequently, the price paid by the winning bidder is increasing in the signals of all the losing bidders. The winning bidder's signal is affiliated with the other signals, so when her own signal rises, her expected payment rises faster in the English auction than in the first-price auction. (The second-price auction is an intermediate case, in which the winner's payment is increasing in the highest signal from the losing bidders.) This insight, known today as the "linkage principle", connects practical auction design to its roots in price-formation: the English auction raises more revenue than the first-price auction precisely because it enables the expected price to reflect every bidder's private information.

The interdependent-values model further implies that the auctioneer should be completely transparent about her own private information. That is, suppose we extend the model so that the auctioneer observes some signals, also affiliated, with each bidder's value function v_i non-decreasing in the auctioneer's signals. These could represent, for instance, knowledge of the provenance of a painting, or an independent geologist's assessment of the mineral resources being sold. Milgrom and Weber (1982) proved that in all four standard auction formats, the auctioneer maximizes her expected revenue by committing in advance to fully disclose her own signals. This suggests that auctions can aggregate not only the buyers' information, but also the information of a self-interested seller.[17]

IV. Auction Theory at Work

With auction theory in hand, we could, in principle, give specific advice to real-world auctioneers about auction formats. But why not go further? By employing mechanism design, at least in theory, it is possible to maximize the auctioneer's goals under given assumptions about preferences and information. It turns out, however, that in settings with correlated information, simple design questions can lead to paradoxical answers.

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[17] As it turns out, the linkage principle also implies that the seller can further increase her revenue by linking the winner's payment to future signals of the object's value, for example, through royalty rates (Riley, 1988; DeMarzo *et al.*, 2005; Board, 2007).

Myerson (1981) observed that even mild correlations between bidders' signals could allow unusual mechanisms to extract all of the surplus from the bidders, and Crémer and McLean (1988) found a general method for constructing such mechanisms. But these sorts of mechanisms crucially rely on unrealistically precise knowledge about the joint distribution of private information, which auctioneers would not have access to in practice.

Wilson and Milgrom instead embraced a form of "non-ideal" auction theory that takes seriously the hidden reasons for real-world institutions.[18] Wilson (1987) observed that trading institutions do not seem to exploit fine details about the distribution of private information: "[the] rules of these markets are not changed daily as the environment changes"; he furthermore argued that "the task of theory is to explain how practitioners are (usually) right". Similarly, Milgrom and Weber (1982) held that auction theory should account for the popularity of common auction formats, and seek to understand "which form will (or should) be used in any particular circumstance". This approach served as the foundation for many practical advances in auction design.

An additional challenge for applying the theory in practice was that many real-world auctions involve the sale of multiple units of a single good (emission permits, natural resources, financial instruments) or multiple distinct objects (spectrum licenses). To this day, it is an ongoing research effort to understand which results from the single-unit case extend to the multi-unit case;[19] for the multi-object case, meanwhile, it quickly became apparent that general strategic analysis can be exceptionally complicated. Thus, in early applications of auction theory, no single model provided a comprehensive recommendation for the ultimate design; rather, the designer had to rely on intuitions and insights built from the analysis of simpler tractable cases, carefully extrapolating them to the actual design problem.[20] A further consequence was that the exposure to real-life design problems taught theorists what really matters in practice.[21] In this section, we review the application of multi-unit auctions to the sale of government debt and electricity, and then examine the use of multi-product auctions in the sale of

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[18] These "hidden reasons" were often made explicit by follow-up research. For example, Lopomo (2001) clarified the circumstances under which the English auction is optimal among a large class of mechanisms by studying the class of all posterior-implementable trading procedures; Akbarpour and Li (2020) introduced a notion of "credibility" of mechanisms that helps explain the popularity of first-price and ascending auctions.

[19] For example, the revenue-equivalence theorem carries over under much stronger assumptions (Engelbrecht-Wiggans, 1988); however, the linkage principle need not hold in general (Perry and Reny, 1999).

[20] See Gilboa *et al.* (2014) for a formalization of this idea.

[21] See, for example, Bulow and Klemperer (1996), as well as Milgrom (2000) and Klemperer (2002) for some discussions of the insights generated by practical auction design.

spectrum licenses. In the following section, we review some of the follow-on theoretical work that emerged from the exposure of auction theory (and auction theorists) to practical problems.

Government Debt and Electricity Auctions

The two most prominent applications of multi-unit auctions are the sale of government debt and electricity. In multi-unit auctions, bidders submit demand functions (or supply functions in the context of multi-unit procurement auctions). In practice, auction designers typically settle on one of two standard formats: a pay-as-bid auction, in which winning bidders pay their bids, or a uniform-price auction, in which winning bidders pay the market-clearing price for each unit won.[22] The US government, for example, used a pay-as-bid auction to sell Treasury Bills starting in 1929 (Garbade, 2008), but switched to a uniform-price auction in 1998. Moreover, dozens of countries and states (e.g., the United Kingdom and California) have liberalized their electricity markets in the past 30 years, using either pay-as-bid or uniform-price auctions to clear their wholesale markets (Fabra *et al.*, 2006).[23] (The uniform-price auction is becoming a more popular format for government debt, but pay-as-bid auctions are still commonly used in electricity markets.)

In a leading paper, Wilson (1979) observed that bidding on quantities of the same good creates new complications and challenges for both theoretical analysis and practical design. He considered a symmetric common-value setting in which bidders specify the share of the good they would request at each possible price – and made three insights. First, uniform-price divisible goods auctions can have a multiplicity of bidding equilibria. Intuitively, when bidder i bids with a demand curve, only the quantity requested by i at the market-clearing price matters for i's payoff; however, the quantities requested away from the market-clearing price affect the incentives of other bidders. Therefore, it is possible to support multiple equilibria that differ in the shapes of the bidders' demand curves. Second, in uniform-price auctions, bidders might be able to coordinate on equilibria that generate low revenue for the auctioneer, irrespective of the number of bidders in the auction. In these equilibria, bidders bid high on the first few units and drop their demand sharply close to their market share, yielding a low market-clearing price. Third, Wilson (1979) argued that pay-as-bid auctions do not necessarily resolve the low-revenue problem of uniform-price auctions

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[22] There is also a multi-unit version of the Vickrey auction in which the winning bidders pay the opportunity cost of the units they win.

[23] In electricity markets, bidders are suppliers, so the auctioneer is interested in selecting the lowest bids.

because bidders will respond to the choice of format by shading their bids, so as to pay the same as in some equilibria of the uniform-price auction.

Wilson's insights were prescient. In particular, bid manipulation in uniform-price auctions has been observed in many practical and experimental settings.[24] For example, in their early analysis of the liberalized British electricity market, Green and Newbery (1992) and Wolfram (1998) pointed out noncompetitive behavior by electricity suppliers in uniform-price auctions. The strategies adopted by the bidders – a flat supply curve followed by a sharp increase, known as "hockey-stick" bids – strongly echoed the warnings of Wilson (1979).[25] The UK wholesale electricity market eventually switched to a pay-as-bid format in 2001.[26]

Meanwhile, policymakers have also been experimenting with auctions beyond the standard uniform-price and pay-as-bid formats. For example, the Bank of Spain introduced a hybrid auction format in which all bids above the (quantity-weighted) average winning price are paid at the average winning price, and all bids below are paid in full (Alvarez and Mazón, 2019). The idea was to capture the best of both the pay-as-bid and the uniform-price auction by reducing incentives for bid manipulation (as bids close to the clearing price are paid in full) while giving a decent incentive to bid truthfully (as higher bids are never paid in full).

The FCC Spectrum Auctions

In 1993–1994, the US FCC sought to use an auction to allocate electromagnetic radio spectrum bands for use in personal communications services (PCS) such as mobile phones, pagers, and wireless networks. Milgrom and Wilson, in collaboration with R. Preston McAfee, John McMillan, and the FCC's Evan Kwerel (McAfee and McMillan, 1996), led the design and organization of this auction, which was the first of its kind and, according to McMillan (1994), "one of the biggest and most complicated [auctions] in history"; for a survey, see McAfee *et al.* (2010).

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[24] For an experimental context, see, for example, List and Lucking-Reiley (2000).

[25] Another way of looking at what happened was that bidders were simply exercising their market power in an auction setting. As Klemperer (2002) pointed out, "the most important issues in auction design are the traditional concerns of competition policy—preventing collusive, predatory, and entry-deterring behavior".

[26] Wilson made a vast contribution to electricity market design (Wilson, 2002; Chao and Oren, 2021) by combining insights from economics, engineering, and operations research (see, e.g., Chao *et al.*, 2000; Chao and Wilson, 2002; Wilson, 2008). Wilson's engagement with electricity markets led to his sweeping theory of nonlinear pricing (i.e., pricing that is not proportional to the quantity purchased; Wilson, 1993). Nonlinear pricing (e.g., two-part tariff pricing) is relevant in retail electricity markets because power generators have high fixed costs and low or zero marginal costs, making linear pricing infeasible. Wilson's analysis has also inspired a huge literature on nonlinear pricing and mechanism design; see Armstrong (2016) for a survey.

The idea of allocating spectrum by auction had been proposed decades earlier by Herzel (1951) and Coase (1959) in the context of selling television broadcasting licenses, but the question of spectrum allocation in the PCS context was far more complex: unlike television licenses, individual spectrum licenses varied significantly, both in the region sizes and geographies they covered, as well as the amounts of bandwidth they offered. At the same time, some prospective bidders were small players who mainly sought to purchase local licenses, while others were large national telecoms for whom buying licenses would only be valuable if they were able to acquire national networks.

This setting necessitated a completely new type of auction format that could balance the demands of the different types of bidders. Simply selling the licenses sequentially would make price discovery impossible, as for many bidders the value of a given license depended heavily on which other licenses that bidder could obtain. A single-round, sealed-bid process with package bidding, meanwhile, risked disenfranchising smaller bidders, who would lack the information to bid effectively against the national players (Milgrom, 2020).

Bringing together their earlier work in auction theory with ideas of McAfee and McMillan (1987), Milgrom and Wilson developed the SMRA, which introduced two key innovations: ascending bids, with a discrete bid increment, and an activity rule restricting bidders' ability to bid later in the auction if they do not bid early on.

Under the SMRA mechanism, all the spectrum licenses are auctioned simultaneously in a series of rounds. In each round, bidders enter sealed bids for licenses; these bids are then posted and circulated to all the bidders, in particular identifying the "standing high bid". Bidding increases by a discrete increment in each round – to outbid a previous round's high bid, one must bid higher by at least a predetermined minimum increment. Additionally, bidders' eligibility to bid in a given round is constrained by their activity in previous rounds – the quantity of licenses a bidder bids for in a later round cannot significantly exceed the quantity that bidder bids for in earlier rounds.[27]

Echoing an insight of Kelso and Crawford (1982), Milgrom (2000, Theorems 1–3) showed that under truthful bidding, the SMRA achieves a competitive equilibrium allocation (up to small errors driven by the discreteness of the bid increment) whenever bidders consider spectrum licenses to be (one-for-one) substitutes, in the sense that when the price of one license increases, bidders' demand for the other licenses does not

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[27] In practice, these activity rules are typically implemented with a small amount of slack to avoid bidders losing eligibility due to technical errors or other mistakes in the bidding process.

decrease. Hence, the SMRA closes the loop with the idea of auctions as a source of equilibrium price discovery discussed in Section II; in theory, at least under substitutability, the SMRA can achieve the tâtonnement outcome, even under strategic bidding.[28] Moreover, under the SMRA, those bids increase monotonically, unlike in tâtonnement, where prices are typically allowed to both increase and decrease.

That said, there is an important boundary to the result: one-for-one substitutability is in some sense necessary (see also Milgrom and Strulovici, 2009). Milgrom (2000, Theorem 4) showed that if even a single bidder considers some licenses to be complements, then it is possible that competitive equilibria might not exist. This result corresponds to a real-world pricing challenge in the context of complementarity: if a bidder only values a given license, A, in conjunction with a second license, B, then the bidder's willingness to pay for A depends on the price of B, and vice versa. In an auction, this can lead to an "exposure problem": a bidder could be stuck overpaying for a given license when she is outbid on a complementary one (see also Goeree and Lien, 2014).[29]

Moreover, as Ausubel and Cramton (1995) pointed out and Milgrom (2000) also noted, if bidders have multi-unit demand, then analysis under truthful bidding reflects at most partial equilibrium. Indeed, in such cases bidders almost always have some incentive to underreport their demand in order to reduce prices (Ausubel *et al.*, 2014). Nevertheless, there is still a sense in which the outcomes suggested by the theory really did translate into practice; the FCC spectrum auctions appear to have been quite efficient, with bidders having managed to "build their desired aggregations" (McAfee and McMillan, 1996), and with similar licenses selling for similar prices (Cramton, 1995, 1997; Cramton *et al.*, 1998).

This success in equilibrium price discovery has been credited in large part to the second innovation of Milgrom and Wilson in the design of the SMRA: the activity rule. As Milgrom (2000) recounted, the idea for an activity rule reflected two concerns that were, to some degree, in conflict

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[28] Milgrom (2000) showed this for "straightforward bidding"; see also Gul and Stacchetti (2000) and Ausubel (2006).

[29] Formally, if a bidder values licenses A and B at a and b, respectively, but values the pair together at $a + b + c$, then she risks an exposure problem whenever she bids, for example, more than a for A. Indeed, suppose she bids $a + \delta$ for A while holding the high bid for B, and is then outbid on B at a price of more than $b + c - \delta$; in this case, her value for the pair of licenses is not high enough to continue bidding on B, and she ends up left holding A at a price above her willingness to pay. One solution might be to bundle A and B into a single license C (Adams and Yellen, 1976). However, this can be challenging in practice because some bidders might regard A and B as complements while other bidders might regard them as substitutes. Even if all bidders regard A and B as complements, it might be unclear whether bundling them is indeed optimal from a revenue perspective (Levin, 1997).

with each other. The spectrum auctions needed to end within a reasonable period of time, but in a regular ascending auction, there is generally no specific need for a bidder to bid quickly, as all bidding paths lead to the same equilibrium outcome. Meanwhile, auctions with a fixed end time encourage "sniping", whereby bidders wait until the very last minute to place their bids, in hopes of winning a bargain.[30]

Under the Milgrom–Wilson design, the SMRA would continue running until there are no new bids on any license. While this, in principle, could cause the auction to run indefinitely, the activity rule encourages bidders to bid early on because their ability to bid in later rounds is tied to their bidding activity in earlier rounds. This speeds up the auction process; even more importantly, it serves to increase bidders' information early in the auction, which improves price discovery (Milgrom, 2000).

The SMRA was first used in the FCC's July 1994 paging licenses sale, which raised $617 million. A broader PCS auction using the SMRA ran from the end of 1994 into 1995, raising over $7 billion. As already noted, these auctions were notable not just for their revenue, but also for their apparent efficiency; the auctions were "widely regarded as a success" by both the FCC and auction participants (Federal Communications Commission, 1997). Soon afterward, the SMRA was adopted in other spectrum auction contexts around the world. Subsequent innovations (see Porter and Smith, 2006; Bichler and Goeree, 2017) included changes to the rules regarding between-round information sharing, to reduce the potential for collusion (see, e.g., Cramton and Schwartz, 2002; Klemperer, 2003), as well as innovations in the activity rule (Ausubel and Baranov, 2020). In fact, in early 2021, the FCC used a clock version of the SMRA format to conclude one of the largest auctions ever held, raising over $81 billion (Federal Communications Commission, 2021).

V. From Practice Back to Theory

The exposure to real-life auction design problems inspired by the early work of Milgrom and Wilson generated and revived a number of theory research programs. This new wave of theoretical work tried both to explain the successes and failures of various practical designs, and to provide specific guidelines to policymakers in settings that often differed significantly from the idealized single-unit models discussed in Sections II and III. We survey these innovations here. First, we briefly mention some key contributions in

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[30] Milgrom (2000) first noticed this behavior in the "silent auctions" commonly held by charities (see also Milgrom, 2020); more recently, this behavior has been commonly observed on eBay and similar online auction platforms (Roth and Ockenfels, 2002; Bajari and Hortaçsu, 2004; Ariely *et al.*, 2005).

the theoretical literature on multi-unit auctions initiated by Wilson (1979) and influenced by design challenges in electricity markets and Treasury Bill auctions. Then, we turn to two vast programs on dynamic auctions inspired by the design of the SMRA.

Multi-Unit Auctions

Wilson (1979) showed that some equilibria in the uniform-price auction can exhibit low revenues. But are such equilibria likely? And can they be eliminated? Klemperer and Meyer (1989) suggested an ingenious solution: uncertainty of supply. They showed that under quadratic utilities, if there is enough uncertainty regarding available supply, then the bidding equilibrium is unique and symmetric. At least in a simple model, equilibrium multiplicity stops being a problem with enough uncertainty. More recent theoretical work has shown that by introducing reserve prices, it might be possible to ensure high revenue in all the equilibria (Burkett and Woodward, 2020).

Back and Zender (1993) used a version of the Wilson (1979) model to analyze the Treasury Bill auctions described in Section IV. They showed that in a uniform-price auction, bidders are able to submit very steep bidding functions, thereby reducing competition from other bidders and enforcing a collusive outcome at the reserve price.[31] Back and Zender (1993) also attempted to compare revenues under uniform-price and pay-as-bid auctions, but in their general model they were still stymied by the residual multiplicity of equilibria in both auction formats. Later, Ausubel *et al.* (2014) found conditions on structure of supply uncertainty for the existence of unique, symmetric, and linear equilibria in both uniform-price and pay-as-bid auctions with diminishing marginal values.[32] For this special setting, they were able to give a sharp revenue ranking in which pay-as-bid auctions dominate both the uniform-price and Vickrey auctions.[33]

In a different direction, Ausubel (2004) considered whether a dynamic auction for multiple units could outperform a static Vickrey auction in an interdependent-value setting, reminiscent of the results of Milgrom and Weber on the difference between an ascending auction and a second-price auction for a single item. Ausubel (2004) proposed a novel design of the “clinching auction” that implements the efficient allocation in an ex post equilibrium. In the symmetric, constant marginal values

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[31] Indeed, a version of such an extreme collusive outcome played out recently in Faroese fishing quota auctions (Marszalec *et al*., 2020).

[32] For uniform-price auctions, the structure of these equilibria follows from the results of Klemperer and Meyer (1989).

[33] Pycia and Woodward (2020) showed that that this revenue ranking holds under much more general conditions.

setting, the clinching auction outperforms the Vickrey auction in terms of efficiency and – whenever both auctions are efficient – in terms of revenue.

A large empirical literature has attempted to estimate bidders' valuations from observed bids in multi-unit auctions, especially in government debt and electricity contexts (see Athey and Haile, 2007; Hortaçsu and McAdams, 2018).[34] For example, Borenstein *et al.* (2002) found large deviations from marginal-cost bidding in uniform-price electricity auctions during the 2000 "California electricity crisis", consistent with bidders' exercising their market power. For auctions of government debt, meanwhile, the empirical evidence on the revenue (and efficiency) ranking of uniform-price versus pay-as-bid auctions turns out to be rather mixed (see, e.g., Hortaçsu and McAdams, 2010; Marszalec, 2017; Hortaçsu *et al.*, 2018).

Auctions as Tâtonnement: The Role of Substitutes

In their seminal paper, Kelso and Crawford (1982) established a crucial connection between dynamic auctions, matching theory, and tâtonnement from general equilibrium theory.[35] In the Kelso and Crawford (1982) setting, there are firms that wish to hire several workers, and workers who are interested in matching with firms. Because of heterogeneity and indivisibility of workers, both the core and competitive equilibria may fail to exist under general preferences. But Kelso and Crawford (1982) showed that if firms view workers as (gross) "substitutes", then a core allocation always exists and can be found with a "salary-adjustment process" analogous to both an ascending auction (in which workers are goods and firms are buyers) and the Gale and Shapley (1962) "deferred acceptance" algorithm. The substitutability condition says, roughly, that an increase in the price (salary) of one good (worker) weakly increases the buyer's (firm's) demand for all other goods (workers). If all firms view workers as substitutes, then an auction can start at low prices, proceed to raise prices for over-demanded goods, and eventually clear all markets without having to lower prices due to over-supply. In the deferred acceptance analogy of such an auction, firms make salary offers to their favorite workers, and if some worker rejects a salary offer, then the associated firm can only make

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[34] One reason why empirical analysis of auctions has proven fruitful is that real-world auctions are especially structured economic environments, governed by precise rules that are known to all the participants. As Wilson (1987) observed, "[game] theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge[.]" This makes it natural to use game-theoretic predictions to engage in structural estimation.

[35] These connections were also made in a unit demand setting by Crawford and Knoer (1981) and Demange *et al.* (1986).

offers to that worker at higher salaries. In other words, we can imagine a Walrasian auctioneer calling out salaries for each worker, eliciting firms' demand and monotonically adjusting prices towards equilibrium. However, unlike in the classical tâtonnement, in which all market participants are treated symmetrically and no participant is committed to an offer, the tâtonnement process based on the deferred acceptance algorithm treats the workers and the firms asymmetrically because the proposing side (e.g., firms) commits to its offers while the receiving side (e.g., workers) does not (Crawford and Knoer, 1981). As Milgrom (2000) pointed out, this connection between dynamic auctions and monotone tâtonnement is at the heart of the design and successful price discovery in the SMRA.

The success of the SMRA revived theoretical and practical interest in modeling markets for substitutable indivisible goods. Gul and Stacchetti (1999) and Milgrom and Strulovici (2009) laid out the theoretical foundations for the existence and structure of competitive equilibria in the presence of substitutes. On the auction design end, Ausubel and Milgrom (2006) pointed out that the (rarely used) Vickrey auction has exceptionally desirable theoretical properties when goods are substitutes. But even with substitutable valuations, the Vickrey auction can be complex for bidders. Gul and Stacchetti (2000), Parkes and Ungar (2000), Ausubel and Milgrom (2002), and Ausubel (2006) thus proposed formats for dynamic implementation of Vickrey auctions with heterogeneous substitutable goods, while Lahaie *et al.* (2008), Milgrom (2009), and Klemperer (2010) suggested simple and effective bidding languages for sealed-bid auctions of substitutes.

Hatfield and Milgrom (2005) brought the connections established by Kelso and Crawford (1982) into the heart of the modern theory of matching markets.[36] In particular, they defined an abstract notion of a "contract" between a worker and a firm, which can list terms of the match beyond a salary. For example, a contract could specify working hours or the length of parental leave. Hatfield and Milgrom (2005) showed that if firms regard contracts with workers as substitutes, then the deferred acceptance algorithm will find a "stable" outcome (i.e., a set of contracts robust to recontracting by workers and firms).[37]

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[36] Relatedly, see the work of Fleiner (2003), as well as that of Echenique (2012), who showed how to embed the Hatfield and Milgrom (2005) model into a version of the Kelso and Crawford (1982) framework with salaries (see also Schlegel, 2015).

[37] Both Crawford and Knoer (1981) and Kelso and Crawford (1982) pointed out that convergence to the stable outcomes in their models with salaries works even when their models are generalized to allow for contracts (which they called "endogenous job characteristics"); see also Roth (1984) and Fieiner (2003).

However, Hatfield and Milgrom (2005) pointed out that even under substitutability, the worker-proposing deferred acceptance algorithm is not strategy-proof for workers. In order to recover strategy-proofness, firms' preferences must satisfy the "law of aggregate demand"; that is, the condition that if a firm is offered more contracts, then it does not accept fewer contracts than before. There is, in fact, a close relationship between the "law of aggregate demand" and the "activity rule" in the SMRA, which meant that telecoms were not allowed to bid on few licenses when prices were low (i.e., when many contracts were offered) and then bid on many licenses when prices were high (i.e., when few contracts were offered).

Finally, Hatfield and Milgrom (2005) introduced a very general auction-like process called the "cumulative offer mechanism", which finds a stable outcome whenever it terminates in a feasible outcome. Subsequent work showed that substitutability is not in fact necessary for stability in many-to-one matching with contracts (Hatfield and Kojima, 2008); in such settings, the cumulative offer mechanism can be used to find stable outcomes under much weaker substitutability conditions (Hatfield and Kojima, 2010; Hatfield and Kominers, 2019; Hatfield *et al.,* 2021). These weakened versions of substitutability opened up a vast array of applications for matching with contracts, from cadet–branch matching (Sönmez, 2013; Sönmez and Switzer, 2013) and the design of the Israeli Psychology Masters Match (Hassidim *et al.*, 2017) to proposing richer priority structures for college admissions (Yenmez, 2018).[38]

New Auction Formats for Complements

What happens when the auctioneer is not selling substitutes? After all, the substitutability assumption can be rather strong and describes only a small set of valuations found in real-world markets. In spectrum auctions, for example, the need to assemble a portfolio of spectrum bands can lead to complementarities. Similarly, the power plants participating in electricity auctions often face start-up costs and therefore find it prohibitively expensive to supply small quantities of power. Theoretically, market-clearing prices might exist in the absence of substitutability although only in fairly restricted preference domains – see, for example, Bikhchandani and Mamer (1997), Danilov *et al.* (2001), Sun and Yang (2006), Baldwin and Klemperer (2019), and Rostek and Yoder (2020) – but even then, there is no guarantee that an ascending auction, such as the SMRA, would find equilibrium prices even if they exist.

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[38] The theory of matching with contracts also led to a more general theory of matching in multi-layered supply chains (Ostrovsky, 2008) and multilateral trading networks (Hatfield *et al.*, 2013; Fleiner *et al.*, 2019).

One solution might be to run a sealed-bid package auction; this could, in principle, avoid prices drifting away from equilibrium and allow bidders to bid on a package of goods, thereby avoiding the exposure problem in the SMRA. The natural candidate for a sealed-bid auction for complements is, of course, the Vickrey auction. However, as Ausubel and Milgrom (2006) pointed out, the Vickrey auction has undesirable properties in the presence of complementarity: payments might not be monotone in bids, prices might be outside the core (yielding low revenues), and it is easy for bidders to collude or enter shill bids. Worst still, sealed-bid package auctions for complements can end up being inefficient because bidders are typically unable to enter valuations over all possible bundles (Parkes, 2006; Milgrom, 2007).

Milgrom's theoretical work has been at the forefront of auction designs that can accommodate complements. One of the leading examples is developing the now ubiquitous "combinatorial clock auction" (CCA; Ausubel *et al.*, 2006). This auction combines a "clock phase", in which prices rise and bidders state their demand (similar to the clock auction used in energy auctions at the time), followed by the "supplementary bid round" in which bidders submit final sealed-bid package bids. The prices from the clock phase typically act as minimum bids for the allocation phase. If bidders bid truthfully in the clock phase, then they can all benefit from price discovery, and they only have to focus on the most desirable packages along the price trajectory of the clock phase. In fact, the clock phase might get close to an equilibrium allocation if the degree of complementarity in bidders' preferences is not too strong (Milgrom, 2017). Although theoretically the CCA leaves scope for complex bidding strategies (Janssen and Karamychev, 2016; Levin and Skrzypacz, 2016; Janssen and Kasberger, 2019), its popularity suggests that bidders are not always able to find simple opportunities for manipulation.

Many aspects of the CCA format have evolved and improved over time (Ausubel and Baranov, 2014). For example, the original formulation proposed pricing the allocation phase using the "proxy auction" of Ausubel and Milgrom (2002). However, Day and Raghavan (2007), Day and Milgrom (2008), and Day and Cramton (2012) suggested an ingenious nearest-Vickrey auction in which Vickrey prices are projected to the closest point of the minimum-revenue core. This idea aims to yield acceptable levels of revenue (by being in the core) while giving bidders a strong incentive to bid truthfully (by being as close as possible to Vickrey prices while remaining in the core). Although various modifications of the nearest-Vickrey rule have been suggested, for example, those of Erdil and Klemperer (2010) and Bünz *et al.* (2018), the nearest-Vickrey rule has been used in most CCAs for spectrum around the world.

VI. New Threads

The first 30 years of auction theory focused on design under informational and incentive constraints; that is, maximizing a given objective function subject to a need to incentivize participants to reveal their valuations/preferences. But modern technologies have enabled the emergence of economic systems that are far too complex for exact optimization. Thus, the past 20 years of research on auctions, pioneered by economists, computer scientists, and operations reseachers together, have embraced a new set of challenges: how to maximize allocative objectives subject to constraints on computation and communication.

For instance, it is computationally difficult to coordinate electricity generation and transmission (Lavaei and Low, 2011; Bienstock and Verma, 2019), assign radio spectrum broadcast rights subject to legally mandated interference constraints (Leyton-Brown *et al.*, 2017), and find value-maximizing allocations in combinatorial auctions (Lehmann *et al.*, 2006). Moreover, the widespread adoption of the Internet and smartphones has led to global transaction networks, online marketplaces, and social networks; these have created many new market design problems, which often have to be solved in milliseconds.[39] These challenges have encouraged a closer study of how market rules affect outcomes with computational limitations.

The 2017 FCC Incentive Auction, which Milgrom co-designed, is now a leading example of market design in the presence of computational challenges (Milgrom, 2017).[40] In this auction, the US government sought to buy back spectrum licenses from television broadcasters for use in wireless mobile applications. The challenge was that broadcasters held local licenses across a wide range of spectrum bands, which meant that those who stayed on the air would need to be moved to new channels in order to organize the spectrum sold in the auction into a national network (see Rosston, 2012). The question of which broadcasters could be feasibly repacked to different channels without creating broadcast interference was computationally intractable, which made it impossible to compute the optimal allocation, even abstracting from incentive issues. In response to these challenges, Milgrom and his collaborators proposed a heuristic allocation algorithm based on a descending clock auction, with a scoring rule under which a broadcaster's price for selling a license only decreased if an appropriate repacking could be found for use in the event that the broadcaster decided to remain on the air (Milgrom and Segal, 2020). Such

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[39] For example, it is computationally difficult to determine which advertisement to show for a given search keyword (Mehta *et al.*, 2007).

[40] See also Kominers and Teytelboym (2020) for a further discussion of complexity in market design, framed around Milgrom (2017) and the Incentive Auction.

a mechanism would be strategy-proof, at least for broadcasters with unit supply; and indeed, the use of a clock mechanism meant that broadcasters' decisions during the auction would be particularly straightforward.[41] But how can we know whether the underlying heuristic would provide a "good" allocation?

In fact, computer scientists have been developing theory for precisely this sort of problem for years. Broadly speaking, in settings where computing the optimal allocation is infeasible, computer scientists develop computationally tractable allocation algorithms and then prove formal guarantees on how close those algorithms are to optimal.

Following this approach, Milgrom (2017) showed that as long as TV stations were not "too complementary" (in a precise sense), an ascending auction would yield an outcome that is close to the efficient one. And indeed, the Incentive Auction seems to have led to significant efficiency gains in practice: it enabled the FCC to repurpose 70 MHz of high-value spectrum for mobile broadband, created 14 MHz of new unlicensed spectrum for wireless innovation, covered all the FCC's expenses, provided $10 billion in compensation to broadcast television licensees, and generated an additional $7 billion for the US Treasury (Leyton-Brown *et al.*, 2017).[42]

Algorithmic Mechanism Design

The design and analysis of the Incentive Auction reflected years of research in computer science, in a field widely known as "algorithmic mechanism design". To understand how algorithmic mechanism design works, we first examine the classic Vickrey–Clarke–Groves (VCG) mechanism as an example (Vickrey, 1961; Clarke, 1971; Groves, 1973). If our goal is to achieve a welfare-maximizing allocation, and if finding such an allocation (given market participants' valuations) is computationally tractable, then the VCG payment rule guarantees truthful reporting, which means that we can implement efficient outcomes.

However, at least two issues can arise in practice. First, it is possible that calculating an efficient allocation can be computationally difficult (this often happens, for example, when bidders have multidimensional preferences over different combinations of goods). If we can at best approximate solutions to

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[41] Inspired by the format of the Incentive Auction, Li (2017) proposed a formal sense ("obvious strategy-proofness") in which truthful bidding can be straightforward enough so as to be potentially intuitive to bidders.

[42] Besides these immediate welfare consequences, the innovative auction design spurred many new research programs. For example, Dütting *et al.* (2017) investigated the "deferred acceptance" heuristic used in the Incentive Auction, and showed that it guarantees a constant fraction of the optimal social welfare. Dworczak (2020), meanwhile, introduced a class of "cutoff mechanisms" that remain truthful even in the presence of signaling concerns due to an aftermarket.

the welfare-maximization problem, then, as Lehmann *et al.* (2002) pointed out, VCG payments no longer guarantee truthfulness. Indeed, because VCG payments are defined as the difference between the values of two optimization problems, replacing exact solutions with even highly accurate approximations can yield arbitrarily inaccurate payment calculations (e.g., the computed prices could be negative in an otherwise standard auction). Second, using VCG mechanisms in practice entails communication challenges: a direct implementation of the VCG mechanism requires access to bidders' valuations, which can be prohibitively difficult to communicate to the auctioneer when bidders have combinatorial valuations (Parkes and Ungar, 2000; Lehmann *et al.*, 2002; Dughmi and Vondrák, 2015).

These difficulties inherent in implementing the VCG mechanism in practice have motivated research to seek alternatives that have less demanding computation and/or communication requirements. In certain well-behaved environments, Archer and Tardos (2001) and Lehmann *et al.* (2002) showed that truthful revelation can be incentivized by way of alternative payment rules, so long as the outcome of the approximation algorithm is monotone in an appropriate sense (resembling the monotonicity of the allocation studied by Myerson, 1981). This implementation result gives another way to understand the strategy-proofness of the Incentive Auction: the heuristic algorithm used there was indeed monotone (Milgrom and Segal, 2020), which meant that broadcasters were incentivized to bid truthfully even though the computational difficulties in finding feasible repackings meant that the efficient allocation could at best be approximated.

Meanwhile, if we relax to Bayesian implementation, then in fairly general settings there exists a "black-box" reduction from the problem of designing allocation mechanisms (that must satisfy incentive-compatibility constraints) to the problem of designing allocation algorithms (Hartline and Lucier, 2010; Hartline *et al.*, 2011; Dughmi *et al.*, 2017).[43] The computational and communication complexity of implementing approximately efficient outcomes can also become less severe if one considers indirect mechanisms (Daskalakis and Syrgkanis, 2016).

Computation ...

Similar tools have enabled algorithmic mechanism designers to address some of the oldest open questions of economics, such as allocation under multiple-good monopoly. In that setting, the seller has a limited supply

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[43] Here, the term "black-box reduction" refers to the idea that the mechanism is only assumed to have input–output access to an allocation algorithm, without knowing precisely what that algorithm is. As a result, the final mechanism does not need to fine-tune the given allocation algorithm in order to obtain (almost) the same welfare guarantees; it just accesses the algorithm as a "black box" in a computationally efficient fashion.

of heterogeneous items for sale. There are many interested buyers, and the seller seeks to design an auction to maximize her revenue. This problem is known to be fairly complex (Hart and Reny, 2015) and indeed there is no known closed-form solution. Yet Cai *et al.* (2012) provided a computationally efficient solution by reducing the revenue maximization question to welfare maximization. Their approach gave a broad generalization of the classic result of Myerson (1981), in whose setting the revenue-optimal auction is the welfare-optimal auction with bids transformed into virtual values. Cai *et al.* (2012) showed that the same reduction applies in a multi-object setting, but instead of finding the virtual transformation and pricing rule in closed form, they provided a computationally efficient algorithm to determine them in any given case.

Algorithmic mechanism designers have also made substantial progress in characterizing when particularly simple auction mechanisms are approximately optimal. For example, consider a single-item auction with valuations that are independently but not identically distributed. We know that the revenue-maximizing mechanism with asymmetric bidders can be complex (Myerson, 1981). Yet Hartline and Roughgarden (2009) proved that simple posted-price mechanisms achieve at least a quarter of the optimal revenue, independent of both the number of the bidders and the distribution of values.

In a more recent paper, Daskalakis *et al.* (2017) provided characterization results for the optimality of multi-dimensional mechanisms in terms of the type distribution. These results can be applied, for example, to provide necessary and sufficient conditions for the optimality of simple mechanisms that are used in practice, such as selling the "grand bundle", or selling items one-by-one. Another line of work examines multi-item auction environments and shows that, even in settings in which the optimal multi-item mechanism is known to have a very complex structure, there are simple mechanisms that guarantee a constant fraction of the optimal revenue (Chawla *et al.*, 2007; Cai *et al.*, 2019; Babaioff *et al.*, 2020; Daskalakis *et al.*, 2020).

A more recent direction in algorithmic mechanism design has challenged the information structure assumptions of classic auction theory, bringing us closer to the spirit of the Wilson (1987) doctrine. For instance, what if the auctioneer does not know the full distribution of values, and instead can only observe a few samples from that distribution when designing her mechanism? Again, the classic single-item auction setting serves as a useful benchmark. Suppose the auctioneer observes m samples from the valuations of the bidders. We can ask how large m has to be, as a function of $\epsilon > 0$, so that a $(1-\epsilon)$-approximation of the optimal revenue is achievable. Cole and Roughgarden (2014) proved that having a polynomial (in $1/\epsilon$) number of samples is necessary and sufficient, effectively showing that the only way to achieve a sufficiently strong constant approximation of the optimal revenue is through a detailed understanding of bidders' valuation distributions (see also Hartline and Roughgarden, 2009).[44]

. . . and Communication

Meanwhile, studying communication complexity has led us to revisit fundamental results in economics. Consider the First Welfare Theorem – it says that announcing supporting prices is sufficient to confirm the (Pareto) efficiency of an allocation, but not that prices are necessary. Indeed, the First Welfare Theorem is silent as to whether there might be efficient non-price mechanisms in an economy.

However, in two classic papers, Hurwicz (1960) and Reiter (1974) showed that price mechanisms are in some sense special: in convex economies, the Walrasian price mechanism verifies efficient allocations with the minimal amount of communication. Recently, Nisan and Segal (2006) substantially generalized that finding by showing that prices play an indispensable role in any social choice problem with privately known preferences, even if the problem is non-convex.[45] Thus, if we wish to reach an efficient allocation, then in some sense we must find a way of discovering prices.

Yet finding market-clearing prices requires eliciting players' preferences, and this can be extremely complex. For example, to fully specify her preferences in a combinatorial auction with m items, a bidder must report her value for each of the $2^m - 1$ possible packages. When m is large, this is prohibitively difficult. Hence, many real-world mechanisms (such as the SMRA and CCA) work by quoting price lists for different potential allocations and asking bidders to report demand given those prices across a series of rounds. Yet, even here, communication complexity bounds our ability to achieve efficient outcomes: Nisan and Segal (2005) showed that no demand-query mechanism can produce an efficient allocation (or even a near-efficient one) without exponentially many communication rounds (see also Segal, 2007).

This has motivated further research on bidding languages that reduce the complexity of communicating preferences to the market-clearing mechanism, while still making it possible to find (nearly) efficient prices, in contexts such as combinatorial auctions (see, e.g., Boutilier and Hoos, 2001; Cavallo *et al.*, 2005; Nisan, 2006; Bichler *et al.*, 2011) and

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[44] This result generalizes to multi-item settings (Gonczarowski and Weinberg, 2021).

[45] Parkes (2002) also showed the necessity of revealing supporting prices in order to verify efficiency, albeit in a more restricted communication language domain.

combinatorial assignment problems (such as course allocation; see, e.g., Budish *et al.*, 2017; Budish and Kessler, 2021). Meanwhile, we have started developing practical market-clearing mechanisms that can approximate efficient outcomes without prohibitively large amounts of communication (see, e.g., Blumrosen *et al.*, 2007; Kos, 2012; Mookherjee and Tsumagari, 2014; Ashlagi *et al.*, 2020).

Thus, we find ourselves once again back at the questions Wilson and Milgrom started with: using theory to reason about when and how we can find efficient prices – and then designing mechanisms to reach the associated allocations, with close attention to constraints imposed by the need to make our mechanisms work in practice.

VII. Conclusion

The beautiful conceptual insights of Milgrom and Wilson helped place auctions and auction design squarely within the core of economic thinking. Their pioneering analyses helped solidify game theory as a leading framework for studying markets and market institutions. At the same time, their applied work brought auctions to the forefront of allocation processes all around the world.

This work paved the way for new applications of auctions and other price discovery mechanisms in a range of market design contexts: everything from selling online advertising (Edelman *et al.*, 2007; Lahaie *et al.*, 2007; Varian, 2007, 2009; Agarwal *et al.*, 2009; Athey and Ellison, 2011; Varian and Harris, 2014; Arnosti *et al.*, 2016; Ostrovsky and Schwarz, 2016) and trading financial securities (Budish *et al.*, 2015; Du and Zhu, 2017; Duffie and Zhu, 2017; Kyle and Lee, 2017) to allocating food to food banks (Prendergast, 2017, 2020). More broadly, the work of Milgrom and Wilson has fed into a growing understanding – also pushed by Roth (2002) and others – of the role of the economist as an "engineer", working to improve real-world markets through constant feedback between theory and practice.[46]

On this, Milgrom (2000) wrote the following.

> "[T]heoretical analyses have clearly proved their worth in the practical business of auction design. Drawing on both traditional and new elements of auction theory, theorists have been able to analyze proposed designs, detect biases, predict shortcomings, identify trade-offs, and recommend solutions.

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[46] Roth and Wilson share a background in operations research; Milgrom holds a master's degree in statistics. Cherrier and Saïdi (2019) have described the key role that interdisciplinary interactions played in shaping the modern field of market design.

> It is equally clear that designing real auctions raises important practical questions for which current theory offers no answers. [. . .] Because of such limits to our knowledge, auction design is a kind of engineering activity. It entails practical judgments, guided by theory and all available evidence, but it also uses ad hoc methods to resolve issues about which theory is silent. As with other engineering activities, the practical difficulties of designing effective, real auctions themselves inspire new theoretical analyses, which appears to be leading to new, more efficient and more robust designs".

Wilson, meanwhile, made the following remark.[47]

> "The ongoing computerization of marketplaces will continue to make market design a multidisciplinary endeavor, which already occupies computer scientists as well as economists. And economic engineering more broadly—'design economics'—will likely continue to grow in its ability to help structure contracts, firms, and organizations and collaborations of all sorts. [. . .]
>
> We've learned that maximizing gains from trade is more about participants' information and incentives than intersecting demand and supply curves. So concepts from game theory have been useful guides in efforts to improve the performance of trading platforms. But scholarly theorizing is minor compared to hands-on engineering using knowledge of an industry's technology and practices, and familiarity with participants' concerns is necessary if one is to help them obtain better outcomes overall. Deep involvement discovers key features unanticipated by abstract views of markets. I foresee more economists improving the allocation of scarce resources rather than (just) studying it".

The conversation between theory and practice continues. Auction design – and market design more broadly – invites us to use economic theory and analysis to improve real-world market institutions. Much remains to be done.

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[47] Quoted from Roth and Wilson (2019).

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