Online Appendix of "Mechanism Design with Aftermarkets: Cutoff Mechanisms"

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Abstract

This Online Appendix consists of two parts. The first part contains the missing proofs extending the results of the paper to continuous distributions of types. The second part presents additional results: (i) results on robust implementation of cutoff rules, (ii) results on indirect implementation of cutoff rules, and (ii) an extension of the single-agent model to a case when the agent participates in an aftermarket regardless of the outcome in the mechanism (even after losing).

OA.1 Extension to continuous distributions of types

In this appendix, I show how to extend the results from Sections 3 and 4 to the model with continuous type spaces introduced in Section 6. I also formally prove all results stated in Section 6.

I assume that each Θ_i is a closed interval, and f_i is a density with respect to the Lebesgue measure on Θ_i. Because I do not distinguish between two mechanisms that induce the same distribution of posterior beliefs for every prior, it is without loss of generality to assume that the cardinality of the message space is at most a continuum – I will thus assume throughout that, for all $i \in \mathcal{N}$, $\mathcal{S}_i \subset \mathbb{R}^+$ and that $\mathcal{S}_i$ is endowed with a Borel σ-field.

The definition of DS implementability remains the same, except that sums are replaced by integrals: The payoff to agent i from reporting $\hat{\theta}_i$ to a direct mechanism $(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{t})$, when her true type is θ_i and other agents report truthfully is

$$\int_{\mathcal{S}_i} u_i(\theta_i;\, f_i^s) d\pi(s|\, \hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) x_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) - t_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}).$$

I first prove that cutoff rules are implementable for any prior distribution of types and any monotone aftermarket. The proof uses the monotonicity condition

$$\pi_i(S|\theta_i,\, \boldsymbol{\theta}_{-i}) x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) \text{ is non-decreasing in } \theta_i \text{ for all measurable } S \in \mathcal{S}_i \text{ and } \boldsymbol{\theta}_{-i} \in \Theta_{-i}. \quad \text{(M)}$$

Proof of Theorem 1. If $(\boldsymbol{x}, \boldsymbol{\pi})$ is a cutoff rule, then condition (M) follows directly from Definition 9 of cutoff rules (see Section 6). I will show that condition (M) implies implementability for any prior distribution and any (monotone) aftermarket. Fix i and $\boldsymbol{\theta}_{-i}$. Using the definition of cutoff mechanisms, for any $\tau \in \Theta_i$,

$$\int_{\mathcal{S}_i} u_i(\tau;\, f_i^s) d\pi_i(s|\, \tau,\, \boldsymbol{\theta}_{-i}) x_i(\tau,\, \boldsymbol{\theta}_{-i}) = \int_0^{\tau} \int_{\mathcal{S}_i} u_i(\tau;\, f_i^s) d\gamma_i(s|\, c,\, \boldsymbol{\theta}_{-i}) dx_i(c,\, \boldsymbol{\theta}_{-i}).$$

For any $\theta_i \geq \hat{\theta}_i$, we have

$$\int_{\hat{\theta}_i}^{\theta_i} \int_{\mathcal{S}_i} \left[u_i(\theta_i;\, f_i^s) - u_i(\hat{\theta}_i;\, f_i^s)\right] d\gamma_i(s|\, c,\, \boldsymbol{\theta}_{-i}) dx_i(c,\, \boldsymbol{\theta}_{-i}) \geq 0,$$

where the inequality follows from monotonicity of the aftermarket. Therefore,

$$\int_{\mathcal{S}_i} \left[u_i(\theta_i;\, f_i^s) - u_i(\hat{\theta}_i;\, f_i^s)\right] \left[d\pi_i(s|\, \theta_i,\, \boldsymbol{\theta}_{-i}) x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) - d\pi_i(s|\, \hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) x_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i})\right] \geq 0. \quad \text{(OA.1.1)}$$

To show that condition (OA.1.1) implies implementability, I can use the same argument as in the case of discrete types (see Appendix A.1): When the type space Θ_i is continuous, I use the result from the appendix in Dworczak and Zhang (2017) which states that it is enough to check the matching-efficiency condition for all finite subsets of the type space. Thus, the proof goes through without any modifications for a continuous type space. $\square$

Next, I prove Proposition 1 showing that condition (M) characterizes cutoff rules.

Proof of Proposition 1. The proof is analogous to the proof for the discrete type case but the infinite type and signal spaces require additional care. I only have to prove the "if" direction. I fix i and $\boldsymbol{\theta}_{-i}$, and drop them from the notation to simplify exposition. Denote $\beta_S(\tau) \equiv \pi(S|\, \tau) x(\tau)$, for any measurable $S \subseteq \mathcal{S}$. Unlike in the discrete-type case, β_S corresponds to the probability that the signal lies in the set S to account for the fact that $\mathcal{S}$ can be an infinite space.[1] Because $\beta_S(\tau)$ is non-decreasing, it has one-sided limits everywhere and is continuous almost everywhere. According to the convention that I identify mechanisms that differ on a measure-zero set of types, it is without loss of generality to assume that $\beta_S(\tau)$ is right-continuous in τ. It follows that β_S induces a positive $\sigma-$additive measure μ_S on C defined by

$$\mu_S((a,\, b]) = \beta_S(b) - \beta_S(a).$$

Because a $\sigma-$additive measure on the Borel $\sigma-$field is uniquely defined by the values it takes on intervals, the above definition uniquely characterizes μ_S.

I will show that the measure μ_S, for any S, is absolutely continuous with respect to the cutoff distribution dx induced by the allocation rule x. For any $a,\, b \in C$, $a < b$, we have

$$\beta_S(b) - \beta_S(a) \leq \beta_{\mathcal{S}}(b) - \beta_{\mathcal{S}}(a) = x(b) - x(a).$$

[1] Each $\beta_S(\tau)$ is a measurable function of τ because both $x_i(\tau,\, \boldsymbol{\theta}_{-i})$ and $\pi_i(S|\, \tau,\, \boldsymbol{\theta}_{-i})$ were assumed to be measurable in τ.

It follows that if $x(b) = x(a)$, then $\beta_S(b) - \beta_S(a) = 0$. Because a and b were arbitrary, μ_S is absolutely continuous with respect to dx.

By the Radon-Nikodym Theorem, for any S, there exists a measurable positive function g_S supported on C that is a density of μ_S with respect to the measure dx. In particular,

$$\beta_S(\theta) = \pi(S|\theta)x(\theta) \equiv \mu_S([0,\, \theta]) = \int_0^\theta g_S(c)dx(c), \tag{OA.1.2}$$

for all θ and measurable $S \subseteq \mathcal{S}$.

With $S = [0,\, s]$, I define $G_c(s) \equiv g_{[0,\, s]}(c)$, for any $s \in \mathcal{S} = C$. My goal is to prove that $G_c(s)$ is a cdf when treated as a function of s (for dx-almost all c). Because the equality

$$\pi(S|\theta)x(\theta) = \int_0^\theta g_S(c)dx(c)$$

must hold for all θ, we can conclude that $G_c(0) = 0$, $G_c(1) = 1$, for dx-almost all $c \in C$. I will show that $G_c(s)$ is non-decreasing in s, for dx-almost all $c \in C$. Consider $s < s'$, and note that

$$\begin{aligned}\int_0^\theta g_{[0,\, s']}(c)dx(c) = \pi([0, s'])x(\theta) = \pi([0, s])x(\theta) + \pi((s, s'])x(\theta)\\ = \int_0^\theta g_{[0,\, s]}(c)dx(c) + \int_0^\theta g_{(s,\, s']}(c)dx(c),\end{aligned}$$

where $g_{(s,\, s']}(c)$ is obtained by taking $S = (s,\, s']$ and applying formula (OA.1.2). It follows that

$$\int_0^\theta \left[g_{[0,\, s']}(c) - g_{[0,\, s]}(c) - g_{(s,\, s']}(c)\right] dx(c) = 0,$$

for all $\theta \in \Theta$. Thus, $g_{[0,\, s']}(c) = g_{[0,\, s]}(c) + g_{(s,\, s']}(c)$ for dx-almost all c, and in particular, because $g_{(s,\, s']}(c)$ is non-negative, $g_{[0,\, s']}(c) \geq g_{[0,\, s]}(c)$, or $G_c(s') \geq G_c(s)$. Because $s < s'$ were arbitrary, $G_c(s)$ is non-decreasing in s. Finally, by monotonicity of $G_c(s)$ and equation (OA.1.2), $G_c(s)$ is right-continuous in s, for dx-almost all c.

Therefore, $G_c(s)$ is a cumulative distribution function, for dx-almost all c. We can thus define γ, for dx-almost all $c \in C$, by

$$\gamma([0,\, s]|\, c) = G_c(s),$$

for any $s \in [0,\, 1]$. (It is irrelevant how we define γ on the remaining dx-measure zero set of points c.) Because a σ-additive distribution γ is uniquely determined by the value it assigns to sets of the form $[0,\, s]$, for all $s \in [0,\, 1]$, by equation (OA.1.2) we get

$$\pi(S|\, \theta)x(\theta) = \int_0^\theta \gamma(S|\, c)dx(c),$$

for all measurable $S \subseteq \mathcal{S}$. Therefore, $(x,\, \pi)$ satisfies (6.1). Because i and $\boldsymbol{\theta}_{-i}$ were arbitrary, $(\boldsymbol{x},\, \boldsymbol{\pi})$ is a cutoff rule. $\square$

The results on optimal cutoff mechanisms with a single agent (Subsection 4.1) go through with virtually no change to the argument (of course, sum operators are replaced with integrals).[2] The Matthews-Border condition (M-B) has a direct analog for a continuous type space, so the only difficulty in extending the results to the multi-agent model lies in proving Lemma 2: Because the signal space is now potentially infinite, I cannot directly apply Lemma 3 from Gershkov *et al.* (2013) because Gershkov *et al.* only allow for a finite set of social alternatives. I circumvent this difficulty by proving an approximation result.

I say that a sequence of mechanism frames $\{(\boldsymbol{x}, \boldsymbol{\pi}^n)\}_{n=1}^{\infty}$ on the same signal space $\times_{i\in\mathcal{N}}\mathcal{S}_i$ converges to $(\boldsymbol{x}, \boldsymbol{\pi})$, if, for all i, $\pi_i^n(\cdot\,|\,\boldsymbol{\theta})x_i(\boldsymbol{\theta})$ converges to $\pi_i(\cdot\,|\,\boldsymbol{\theta})x_i(\boldsymbol{\theta})$ in the weak* topology of measures on $\mathcal{S}_i$, for almost all $\boldsymbol{\theta}$. Call a mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ $\mathcal{S}$-finite if there are finitely many signal realizations in the support of $\boldsymbol{\pi}$.

Lemma OA.1. *A mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ is a cutoff rule if and only if it is the limit of $\mathcal{S}$-finite cutoff rules with the same allocation rule $\boldsymbol{x}$.*

Proof of Lemma OA.1. First, suppose that a sequence of $\mathcal{S}$-finite cutoff rules $\{(\boldsymbol{x}, \boldsymbol{\pi}^n)\}_{n=1}^{\infty}$ converges to some mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$. I have to show that $(\boldsymbol{x}, \boldsymbol{\pi})$ is a cutoff rule.

Fix $\boldsymbol{\theta}$ and $i \in \mathcal{N}$. Convergence in the weak* topology means that for any continuous bounded function g on $\mathcal{S}_i$, we have

$$\lim_n \int_{\mathcal{S}_i} g(s)d\pi_i^n(s|\,\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_{\mathcal{S}_i} g(s)d\pi_i(s|\,\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}).$$

Because for each n, $(\boldsymbol{x}, \boldsymbol{\pi}^n)$ is a ($\mathcal{S}$-finite) cutoff rule, we have

$$\int_{\mathcal{S}_i} g(s)d\pi_i^n(s|\,\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i}\int_{\mathcal{S}_i} g(s)d\gamma_i^n(s|\,c, \boldsymbol{\theta}_{-i})dx_i(c, \boldsymbol{\theta}_{-i}),$$

for some probability measure $\gamma_i^n(\cdot\,|c, \boldsymbol{\theta}_{-i})$ on $\mathcal{S}_i$. By the Banach-Alaoglu theorem, the set of probability measures is compact in the weak* topology, so (after passing to a subsequence if necessary) we can assume that γ_i^n converges to some γ_i. Thus

$$\lim_n \int_{\mathcal{S}_i} g(s)d\gamma_i^n(s|\,c, \boldsymbol{\theta}_{-i}) = \int_{\mathcal{S}_i} g(s)d\gamma_i(s|\,c, \boldsymbol{\theta}_{-i}).$$

By the Lebesgue dominated convergence theorem,

$$\lim_n \int_0^{\theta_i}\int_{\mathcal{S}_i} g(s)d\gamma_i^n(s|\,c, \boldsymbol{\theta}_{-i})dx_i(c, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i}\left(\int_{\mathcal{S}_i} g(s)d\gamma_i(s|\,c, \boldsymbol{\theta}_{-i})\right)dx_i(c, \boldsymbol{\theta}_{-i}).$$

Combining the above equations,

$$\int_{\mathcal{S}_i} g(s)d\pi_i(s|\,\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i}\int_{\mathcal{S}_i} g(s)d\gamma_i(s|\,c, \boldsymbol{\theta}_{-i})dx_i(c, \boldsymbol{\theta}_{-i}).$$

[2] The results on concavification now follow from the Online Appendix of Kamenica and Gentzkow (2011) where they extend their methods to continuous state spaces.

Because the above equality is true for all continuous bounded functions g, the two measures must be equal, i.e.

$$\pi_i(S|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i(\theta_i,\,\boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \gamma_i(S|\,c,\,\boldsymbol{\theta}_{-i})dx_i(c,\,\boldsymbol{\theta}_{-i}),$$

for all measurable $S \subseteq \mathcal{S}_i$. Thus, $(\boldsymbol{x},\,\boldsymbol{\pi})$ is a cutoff rule.

Conversely, suppose that $(\boldsymbol{x},\,\boldsymbol{\pi})$ is a cutoff rule. I have to find a sequence $\{(\boldsymbol{x},\,\boldsymbol{\pi}^n)\}_{n=1}^{\infty}$ of $\mathcal{S}$-finite cutoff rules that converges to $(\boldsymbol{x},\,\boldsymbol{\pi})$.

Fix $\boldsymbol{\theta}$ and $i \in \mathcal{N}$, and consider the measure $\gamma_i(\cdot\,|\,\theta_i,\,\boldsymbol{\theta}_{-i})$ satisfying equation (6.1), defined on $\mathcal{S}_i$. Take an arbitrary discrete approximation of the probability measure $\gamma_i(\cdot\,|\,\theta_i,\,\boldsymbol{\theta}_{-i})$, i.e., a sequence $\{\gamma_i^n(\cdot\,|\,\theta_i,\,\boldsymbol{\theta}_{-i})\}_{n=1}^{\infty}$ of finite-support measures on $\mathcal{S}_i$ that converges in weak* topology to γ_i.[3] For each n, define a mechanism frame $(\boldsymbol{x},\,\boldsymbol{\pi}^n)$ by

$$\pi_i^n(S|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i(\theta_i,\,\boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \gamma_i^n(S|\,c,\,\boldsymbol{\theta}_{-i})dx_i(c,\,\boldsymbol{\theta}_{-i}),$$

for all $\boldsymbol{\theta}$, $i \in \mathcal{N}$, and measurable $S \subseteq \mathcal{S}_i$. Because γ_i^n has finite support, $(\boldsymbol{x},\,\boldsymbol{\pi}^n)$ is an $\mathcal{S}$-finite cutoff rule. By the same argument as in the first part of the proof, $(\boldsymbol{x},\,\boldsymbol{\pi})$ is a limit of $\{(\boldsymbol{x},\,\boldsymbol{\pi}^n)\}_{n=1}^{\infty}$. □

I am now ready to extend the proof of Lemma 2.

Proof of Lemma 2. The "only if" direction requires no changes in the continuous-type case (except that I now use the continuous-type version of Proposition 1 proven above). I focus on the "if" direction. First, because the results of Gershkov *et al.* (2013) allow for a continuous type space, the proof technique extends with no changes in the argument to the case of $\mathcal{S}$-finite mechanism frames. Consider a general mechanism frame $(\bar{\boldsymbol{x}},\,\bar{\boldsymbol{\pi}})$ (not necessarily $\mathcal{S}$-finite). By Lemma OA.1, $(\bar{\boldsymbol{x}},\,\bar{\boldsymbol{\pi}})$ can be represented as a limit of a sequence of $\mathcal{S}$-finite reduced-form cutoff rules $\{(\bar{\boldsymbol{x}},\,\bar{\boldsymbol{\pi}}_n)\}_{n=1}^{\infty}$.[4] By the result for $\mathcal{S}$-finite mechanism frames, we know that for each n there exists a ($\mathcal{S}$-finite) cutoff rule $(\boldsymbol{x},\,\boldsymbol{\pi}_n)$ such that $(\boldsymbol{x}^{\boldsymbol{f}},\,\boldsymbol{\pi}_n^{\boldsymbol{f}}) = (\bar{\boldsymbol{x}},\,\bar{\boldsymbol{\pi}}_n)$, where $(\boldsymbol{x}^{\boldsymbol{f}},\,\boldsymbol{\pi}_n^{\boldsymbol{f}})$ denotes the reduced-form of $(\boldsymbol{x},\,\boldsymbol{\pi}_n)$ under $\boldsymbol{f}$. Passing to a subsequence if necessary, we can assume that $(\boldsymbol{x},\,\boldsymbol{\pi}_n)$ converges to some $(\boldsymbol{x},\,\boldsymbol{\pi}_\star)$. Then, $(\boldsymbol{x},\,\boldsymbol{\pi}_\star)$ is also a cutoff rule. Moreover, $(\boldsymbol{x}^{\boldsymbol{f}},\,\boldsymbol{\pi}_\star^{\boldsymbol{f}}) = (\bar{\boldsymbol{x}},\,\bar{\boldsymbol{\pi}})$ (because this equality holds along the sequence). □

The remaining part of the proof of Theorem 3 is fully analogous to the discrete-type case.

Finally, I extend the results from Section 4.2.1. With continuous distributions, I say that the a distribution with density g *monotone-likelihood-ratio dominates* a distribution with full-support density f (denoted $g \succ^{MLR} f$) if $g(\theta)/f(\theta)$ is bounded and non-decreasing. The proof of Lemma 5 and Proposition 2 is then virtually identical with a continuous type space – in fact, except for using the sum operator instead of the integral operator, the proof in the main text did not make use of finiteness of the type space.

[3] Such an approximation can be constructed by discretizing the compact domain of γ_i.

[4] Lemma OA.1 was stated for cutoff rules but it also applies to reduced-form cutoff rules which, for any fixed i, are equivalent to one-agent cutoff rules.

OA.1.1 Missing proofs for Section 6

OA.1.1.1 Proof of Proposition 4

I will apply the results from Sections 3 and 4 for a continuous type space – the previous section showed that these results extend to this case. Because I have assumed that agents are symmetric, I drop all the subscripts.

Case 1. Suppose that W is concave and non-decreasing. Because W is concave, and $M(\bar{f})$ is linear in $\bar{f}$, the functional $\mathcal{W}$ is concave. Thus, for any interim allocation function $\bar{x}$, it is optimal to disclose no information, by Corollary 2. Using Proposition 2, we can write the problem as

$$\max_{\bar{x}} \left(\int_0^1 \bar{x}(\theta)f(\theta)d\theta\right) W(M(f^{\bar{x}})) \tag{OA.1.3}$$

subject to

$$\bar{x}(\theta) \text{ is non-decreasing in } \theta, \tag{OA.1.4}$$

$$\int_\tau^1 \bar{x}(\theta)f(\theta)d\theta \leq \frac{1-F^N(\tau)}{N}, \ \forall \tau \in [0, 1], \tag{OA.1.5}$$

where (OA.1.5) is a version of the Matthews-Border condition (4.6) for continuous type spaces, and $f^{\bar{x}}$, defined analogously as (4.1),

$$f^{\bar{x}}(\theta) = \frac{\bar{x}(\theta)f(\theta)}{\int_\Theta \bar{x}(\tau)f(\tau)d\tau}$$

is the belief over the winner type conditional on no disclosure. We can also write the objective function explicitly as

$$\left(\int_0^1 \bar{x}(\theta)f(\theta)d\theta\right) W\left(\frac{\int_0^1 \theta\bar{x}(\theta)f(\theta)d\theta}{\int_0^1 \bar{x}(\theta)f(\theta)d\theta}\right).$$

Consider an auxiliary problem in which we fix $\int_0^1 \bar{x}(\theta)f(\theta)d\theta = \beta$ for some $\beta \leq 1/N$. Since W is non-decreasing, the problem becomes

$$\max_{\bar{x}} \int_0^1 \theta\bar{x}(\theta)f(\theta)d\theta, \tag{OA.1.6}$$

subject to (OA.1.4), (OA.1.5), and

$$\int_0^1 \bar{x}(\theta)f(\theta)d\theta = \beta \tag{OA.1.7}$$

In the above problem, we can think of constraint (OA.1.7) as an equal mass condition. Intuitively, it is optimal to shift as much mass as possible to the right, subject to constraint (OA.1.5), which will thus hold with equality for large enough τ. Formally, I will show optimality of $\bar{x}(\theta) =$

$F^{N-1}(\theta)\mathbf{1}_{\{\theta\geq r\}}$, where r is chosen so that condition (OA.1.7) holds. Using integration by parts,

$$\int_0^1 \theta\bar{x}(\theta)f(\theta)d\theta = \int_0^1 \underbrace{\left(\int_\theta^1 \bar{x}(\tau)f(\tau)d\tau\right)}_{\Gamma(\theta)} d\theta$$

Ignoring constraint (OA.1.4) for now, the problem is to maximize the above expression over Γ subject to $\Gamma(0) = \beta$, Γ is non-increasing, and $\Gamma(\theta) \leq (1-F^N(\theta))/N$, for all θ. Clearly, this problem is solved by $\Gamma(\theta) = \min\{\beta,\, (1-F^N(\theta))/N\}$. But then $\Gamma(\theta) = \int_\theta^1 F^{N-1}(\tau)\mathbf{1}_{\{\theta\geq r\}}f(\tau)d\tau$, by the definition of r. Moreover, $F^{N-1}(\theta)\mathbf{1}_{\{\theta\geq r\}}$ satisfies constraint (OA.1.4), so it is a solution to problem (OA.1.6).

In the second step, I optimize over $\beta \in [0,\, 1/N]$ in condition (OA.1.7), which corresponds to optimizing over $r \in [0,\, 1]$ in the optimal solution to the auxiliary problem. By plugging in the optimal solution from the auxiliary problem to (OA.1.3), we obtain

$$\max_{r\in[0,\,1]} \left(\int_r^1 F^{N-1}(\theta)f(\theta)d\theta\right) W\left(\frac{\int_r^1 \theta F^{N-1}(\theta)f(\theta)d\theta}{\int_r^1 F^{N-1}(\theta)f(\theta)d\theta}\right).$$

This corresponds to equation (6.3) in Proposition 4, and thus the first case is proven.

Case 2. Consider the case when W is concave and decreasing. Following the same steps as previously, I consider the auxiliary problem with constraint (OA.1.7). Because W is decreasing, the objective is

$$\min_{\bar{x}} \int_0^1 \theta\bar{x}(\theta)f(\theta)d\theta,$$

subject to (OA.1.4), (OA.1.5), and (OA.1.7). This time, all the mass under $\bar{x}$ should be shifted to the left, subject to the monotonicity constraint (OA.1.4). Thus, the optimal $\bar{x}$ will be constant, equal to β. Because $\beta \leq 1/N$, such $\bar{x}$ satisfies the Matthews-Border condition (OA.1.5), and corresponds to allocating the object uniformly at random.

In the second step, because W was assumed non-negative, optimization over β yields $\beta = 1/N$, that is, β should be set to the maximal feasible level. Such a mechanism always allocates the good (to a randomly selected agent).

Case 3. Finally, assume that W is convex. Then, the functional $\mathcal{W}$ is convex, so it is optimal to fully disclose the cutoff representing the interim allocation rule $\bar{x}$, by Corollary 2. Full disclosure means that any posterior belief $\bar{f} \in M_f$ is decomposed into a distribution over truncations of the prior distribution f. Recall that $\bar{x}$ can be treated as a cdf of the cutoff. Therefore,

$$\mathrm{co}^{M_f}\mathcal{W}(\bar{f}) = \int_0^1 W(m(c))\,\frac{1-F(c)}{\int_0^1 \bar{x}(\theta)f(\theta)d\theta}d\bar{x}(c).$$

The additional term $(1-F(c))/(\int_0^1 \bar{x}(\theta)f(\theta)d\theta)$ appears because, by definition, the payoff W is a conditional expected payoff conditional on allocating the good. The distribution with cdf $\bar{x}$ is the ex-ante distribution of the cutoff for agent i. Conditional on agent i being the winner, the posterior distribution over the cutoff for agent i must be adjusted (intuitively, lower cutoffs are more likely). The ex-ante probability of cutoff c is transformed into a conditional probability by conditioning on the event $\tilde{\theta} \geq c$. The objective function (4.8) can be written as

$$\max_{\bar{x}} \int_0^1 W(m(c))\,(1 - F(c))d\bar{x}(c).$$

Using integration by parts (by assumption, W is differentiable) we obtain

$$\int_0^1 W(m(c))\,(1 - F(c))d\bar{x}(c) = -W(m(0))\bar{x}(0^-) - \int_0^1 \frac{d}{dc}\left[W(m(c))\,(1 - F(c))\right] x(c)dc.$$

Because $\bar{x}$ represents a cdf in the above equation, $\bar{x}(0^-)$, the left limit of $\bar{x}$ at 0, is equal to zero. By letting $w(c) \equiv W(m(c))$, the objective function can be written as

$$\max_{\bar{x}} \int_0^1 \frac{-\frac{d}{dc}\left[W(m(c))\,(1 - F(c))\right]}{f(c)}\bar{x}(c)f(c)dc = \max_{\bar{x}} \int_0^1 \underbrace{\left[w(c) - w'(c)\frac{1 - F(c)}{f(c)}\right]}_{J_w(c)}\bar{x}(c)f(c)dc.$$

The conclusion of Proposition 4 now follows from an argument analogous to the one used in previous cases. If $J_w(c)$ is non-positive for $c \leq \underline{r}$, and positive non-decreasing for $c \geq \underline{r}$, then it is optimal to set $\bar{x}(\theta) = 0$ for $\theta \in [0,\, \underline{r}]$, and push all the mass under $\bar{x}$ on $[\underline{r},\, 1]$ to the right, subject to constraint (OA.1.5). This gives us $\bar{x}(\theta) = F^{N-1}(\theta)\mathbf{1}_{\{\theta \geq \underline{r}\}}$. Under this $\bar{x}$, the distribution of the cutoff has a continuous part which is the distribution of a second highest type conditional on that type exceeding $\underline{r}$, and an atom at $\underline{r}$, with mass equal to the probability that the second highest type is below $\underline{r}$. (Notice that full disclosure of the cutoff leads to the same posterior beliefs over the winner's type as full disclosure of the second highest type. This is because, when the second highest type is below $\underline{r}$, the exact value of the second highest type does not influence the allocation for the highest type.)

To finish the proof of Proposition 4, I have to show that when $W(c)$ is increasing and log-concave, then there exists $\underline{r}$ such that $J_w(c)$ is non-positive for $c \leq \underline{r}$, and positive non-decreasing for $c \geq \underline{r}$. It is enough to prove that $J_w(c) \geq 0$ implies $J_w'(c) \geq 0$.

We have $m'(c) = (m(c) - c)f(c)/(1 - F(c))$. The inequality $J_w(c) \geq 0$ implies that $m(c) - c \leq W(m(c))/W'(m(c))$. Using the assumption that $W'' \geq 0$, and the above inequality,

$$J_w'(c) = W'(m(c)) - W''(m(c))(m(c) - c) \geq W'(m(c)) - W''(m(c))\frac{W(m(c))}{W'(m(c))}.$$

Using the fact that $W' \geq 0$, the above expression is greater than zero if and only if $(W')^2 \geq W''W$ which is equivalent to log-concavity of W.

OA.1.1.2 The optimal mechanism for Example 6

In this appendix, I formally derive the optimal mechanism from Example 6 case (a). That is, I find the optimal cutoff mechanism for the resale model in the setting of Example 2 assuming that the winner has a high (h) or low (l) ex-post value for the object (θ_i is the ex-ante probability of having a high value), and there is a single third party in the aftermarket with value $v > h$ who makes a take-it-or-leave-it offer to the winner; agents are ex-ante identical (it is then without loss of generality to look at symmetric mechanisms), and the distribution of types is continuous with cdf F and density f on $\Theta = [0, 1]$.

Proposition OA.1. *Suppose that $\mathbb{E}[\max_{i\in\mathcal{N}} \tilde{\theta}_i] < (h-l)/(v-l)$ and $\lambda \geq (h-l)/(v-l)$. One of the following two mechanisms maximizes total surplus among all cutoff mechanisms:*

(a) For some type $\theta^\star \in (0, 1)$[5]

$$x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \begin{cases} (1/N)\mathbf{1}_{\{\theta_{-i}^{(1)} < \theta^\star\}} & \theta_i < \theta^\star \\ \mathbf{1}_{\{\theta_i \geq \theta_{-i}^{(1)}\}} & \theta_i \geq \theta^\star \end{cases},$$

and $\mathcal{S}_i = \{s_L, s_H\}$ with

$$\pi_i(s_L | \theta_i, \boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}) = (1/N)\mathbf{1}_{\{\theta_{-i}^{(1)} \leq \theta^\star\}};$$

(b) $x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \mathbf{1}_{\{\theta_i \geq \max(\theta_{-i}^{(1)}, r^\star)\}}$ and $\mathcal{S}_i = \{s\}$, where $r^\star$ is defined by

$$\mathbb{E}[\max_{i\in\mathcal{N}} \tilde{\theta}_i | \max_{i\in\mathcal{N}} \tilde{\theta}_i \geq r^\star] = \frac{h-l}{v-l}.$$

Mechanism (a) is strictly better than mechanism (b) (and hence optimal) whenever F satisfies a regularity condition (see OA.1.39). The regularity condition is satisfied by all distributions $F(\theta) = \theta^\kappa$ for $\kappa > 0$.

I focus on the mechanism from point *(a)* in the discussion. I will argue that this cutoff rule can be implemented by the indirect mechanism described in Example 6 case (a). There is a bid space $B = [l, h]$, and each agent places a bid $b \in B$. When all bids are below some minimum bid $b^\star$, the object is allocated uniformly at random. Otherwise, the object is allocated to the highest bidder who pays either *(i)* the second highest bid when the second highest bid is above $b^\star$ or *(ii)* a discounted price $p^\star < b^\star$ when the second highest bid is below $b^\star$. In terms of the disclosure rule, the low signal is sent for sure if no one bid above $b^\star$, and with probability $1/N$ conditional on the event that only the winner bid above $b^\star$ (the high signal is sent otherwise).

The threshold type $\theta^\star$ is uniquely pinned down by the requirement that conditional on observing the high signal the third party is indifferent between offering a high and a low resale price (see OA.1.35). I define

$$b^\star = \lambda h + (1-\lambda)(\theta^\star h + (1-\theta^\star)l) \quad \text{and} \quad p^\star = ((N-1)/N)b^\star.$$

[5] $\theta^\star$ is defined by equation (OA.1.35).

Next, I argue that bidders have a dominant strategy in this auction that implements the optimal allocation rule: Agents with $\theta < \theta^\star$ bid $\theta h + (1-\theta)l < b^\star$, while agents with $\theta \geq \theta^\star$ bid $\lambda h + (1 - \lambda)(\theta h + (1-\theta)l) \geq b^\star$.

Consider a bidder with type $\theta < \theta^\star$ when the highest competing bid is b. If $b < b^\star$, the bidder is indifferent between all bids below $b^\star$. Consider a deviation to a bid above $b^\star$. Then, the bidder wins for sure and gets a payoff of

$$\lambda \left(\frac{1}{N}(\theta h + (1-\theta)l) + \frac{N-1}{N} h \right) + (1-\lambda)(\theta h + (1-\theta)l) - p^\star < \frac{1}{N}(\theta h + (1-\theta)l),$$

where the right hand side is her equilibrium payoff. Now consider a deviation when the highest competing bid is $b \geq b^\star$. Then, the bidder gets 0 from playing her equilibrium strategy. Suppose she deviated to bidding above b; she would receive

$$\lambda h + (1-\lambda)(\theta h + (1-\theta)l) - b \leq \lambda h + (1-\lambda)(\theta h + (1-\theta)l) - b^\star < 0.$$

Thus, all types below $b^\star$ have a dominant strategy. The proof for types above $b^\star$ is analogous and thus omitted.[6]

Assuming that the regularity condition holds, Proposition OA.1 provides an example of a setting where *all* optimal mechanisms reveal information when $N \geq 2$. Indeed, when a mechanism reveals no information, there are two cases to consider: Either the mechanism induces a low resale price, or a high resale price. If the mechanism induces a low resale price, then the setting becomes equivalent to an auction with no aftermarket – thus, it is optimal to allocate to the highest type with no reserve price. It follows from the proof of Proposition OA.1 that such a mechanism is strictly worse than the mechanism in point *(a)* whenever $\lambda > (h-l)/(v-l)$. More generally, this mechanism cannot be optimal for any $\lambda > 0$: It is enough to notice that a strict improvement can be achieved by disclosing the second highest bid whenever it is high enough to induce a high resale price. If the mechanism induces a high resale price, then the optimal allocation rule (among all allocation rules that induce the high resale price) is unique and given in point *(b)* of Proposition OA.1; thus, again the mechanism from point *(a)* is strictly better.

Proof of Proposition OA.1 Because agents are ex-ante identical, there exists an optimal mechanism that is symmetric – I can thus restrict attention to symmetric mechanisms. The action taken by the third party is binary, so it is without loss of generality to consider mechanisms that send two signals, $\mathcal{S}_i = \{s_L, s_H\}$, such that the third party offers a high resale price following the signal s_H.[7] By the results in Section 4, I can solve for the optimal mechanism in reduced form. Let $x(\theta)$ denote the (symmetric) interim expected allocation rule, and let $y(\theta) = \pi(s_L|\,\theta)x(\theta)$ denote the probability that the good is allocated and a low price is recommended, when an agent reports θ. In

[6] In the case $\lambda = 1$ the auction requires a modification because all types above $\theta^\star$ bid exactly h. Conditional on a tie at h, the object should be allocated to the highest type; therefore, in this case, bidders are asked to report their types – they are willing to report truthfully because they get a payoff of zero regardless of how the tie is broken.

[7] Recall that I assumed that the third party offers the high resale price whenever she is indifferent.

a cutoff mechanism, both $x(\theta)$ and $y(\theta)$ are non-decreasing in θ. I let $\phi(\theta) \equiv (v-h)-(v-l)(1-\theta)$. Below, I consider a relaxed problem by omitting the constraint that $x(\theta)-y(\theta)$ is non-decreasing, and that the third party finds it optimal to offer a low price following signal s_L (I later verify that these constraints are satisfied). The relaxed problem of maximizing surplus takes the form:

$$\max_{x,\,y}\ \lambda\left\{v\int_0^1 x(\theta)f(\theta)d\theta-(v-h)\int_0^1 \theta y(\theta)f(\theta)d\theta\right\} \\ +(1-\lambda)\left\{\int_0^1 (\theta h+(1-\theta)l)x(\theta)f(\theta)d\theta\right\} \quad \text{(OA.1.8)}$$

subject to

$$0\le y(\theta)\le x(\theta)\le 1,\ \forall\theta, \quad \text{(OA.1.9)}$$

$$x,\ y \text{ are non-decreasing}, \quad \text{(OA.1.10)}$$

$$\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \ge \int_0^1 y(\theta)\phi(\theta)f(\theta)d\theta, \quad \text{(OA.1.11)}$$

$$\int_\tau^1 x(\theta)f(\theta)d\theta \le \frac{1}{N}(1-F^N(\tau)),\ \forall\tau\in[0,\,1]. \quad \text{(OA.1.12)}$$

Constraint (OA.1.11) states that the third party finds it optimal to offer a resale price h after seeing the signal s_H, and constraint (OA.1.12) is the Matthews-Border condition that ensures that the interim expected allocation rule $x(\theta)$ is feasible. The objective function (OA.1.8) is equal to the per-agent total expected surplus.

I will solve the problem (OA.1.8)-(OA.1.12) in two steps. In the first step, I optimize over y treating x as given. In the second step, I optimize over x.

Step 1. Optimization over y for fixed x. For a fixed non-decreasing function x, the first-step problem can be expressed as (terms not depending on y are omitted):

$$\min_y \int_0^1 \theta y(\theta)f(\theta)d\theta \quad \text{(OA.1.13)}$$

subject to

$$0\le y(\theta)\le x(\theta),\ \forall\theta, \quad \text{(OA.1.14)}$$

$$y \text{ is non-decreasing}, \quad \text{(OA.1.15)}$$

$$\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \ge \int_0^1 y(\theta)\phi(\theta)f(\theta)d\theta. \quad \text{(OA.1.16)}$$

Consider two candidate solutions y_1 and y_2 to the above problem. I say that y_1 *dominates* y_2 if *(i)* $\int_0^1 y_1(\theta)f(\theta)d\theta=\int_0^1 y_2(\theta)f(\theta)d\theta$, and *(ii)* $\int_0^\alpha y_1(\theta)f(\theta)d\theta \le \int_0^\alpha y_2(\theta)f(\theta)d\theta$ for all $\alpha\in[0,\,1]$. Intuitively, y_1 induces a measure that first-order stochastically dominates the measure induced by y_2.

Consider a feasible y_1 and let $\alpha \equiv \int_0^1 y_1(\theta)f(\theta)d\theta$. Consider a function y_2 which satisfies (OA.1.14), (OA.1.15), $\alpha=\int_0^1 y_2(\theta)f(\theta)d\theta$, and is dominated by y_1. Then, $Y_1(\theta)=\int_0^\theta y_1(\tau)f(\tau)d\tau$

first-order stochastically dominates $Y_2(\theta) = \int_0^\theta y_2(\tau)f(\tau)d\tau$. It follows that

$$\int_0^1 \psi(\theta)y_2(\theta)f(\theta)d\theta \leq \int_0^1 \psi(\theta)y_1(\theta)f(\theta)d\theta,$$

for any non-decreasing function $\psi(\theta)$. Because $\phi(\theta)$ is non-decreasing in θ, the above inequality implies that any such y_2 satisfies (OA.1.16) and achieves a weakly lower value of the objective (OA.1.13) than y_1.

A solution to the problem (OA.1.13) - (OA.1.16) exists because the objective function is continuous on a compact domain. If $y^\star$ denotes an optimal solution, let $\alpha^\star = \int_0^1 y^\star(\theta)f(\theta)d\theta$. Then, we can impose the constraint

$$\int_0^1 y(\theta)f(\theta)d\theta = \alpha^\star, \tag{OA.1.17}$$

without changing the value of the problem (OA.1.13) - (OA.1.16).

It follows that to find an optimal solution, it is enough to find a function $y^\star$ that satisfies (OA.1.14), (OA.1.15), (OA.1.17), and is dominated by any other function y satisfying these constraints. Such a function exists and takes the form

$$y^\star(\theta) = \begin{cases} x(\theta) & \theta < \theta^\star \\ \bar{x} & \theta \geq \theta^\star \end{cases}, \tag{OA.1.18}$$

for some $\theta^\star \in [0, 1]$, and $\bar{x} \in [x^-(\theta^\star), x^+(\theta^\star)]$, where $x^-(\theta^\star)$ and $x^+(\theta^\star)$ denote the left and the right limit of x at $\theta^\star$, respectively. Indeed, $y^\star$ crosses any other y satisfying (OA.1.14), (OA.1.15), and (OA.1.17) once and from above, so it is dominated (in the sense defined above) by any such y.

Step 2. Optimization over x. Having solved for the optimal $y^\star$ given x, in step 2, I optimize over x.

I will consider two cases. In case *(1)*, I only look at allocation rules x that satisfy $\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq 0$. Then, it is immediate that $y^\star(\theta) = 0$; indeed, such $y^\star$ is then feasible for (OA.1.14) - (OA.1.16) and achieves the lower bound in (OA.1.13). In case *(2)*, $\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta < 0$, and the optimal $y^\star$ has a non-zero $\bar{x} \in [x^-(\theta^\star), x^+(\theta^\star)]$ pinned down by a binding constraint (OA.1.16):

$$\int_{\theta^\star}^1 (x(\theta) - \bar{x})\phi(\theta)f(\theta)d\theta = 0.$$ [8]

I proceed by finding the optimal x separately for cases *(1)* and *(2)* defined above. At the end, I compare the two constrained optima to find the globally optimal mechanism.

Case 1: $\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq 0$.

Because in this case a high price is always quoted in the aftermarket (whenever it happens), the

[8] If there are multiple $(\bar{x}, \theta^\star)$ satisfying these restrictions, then it must be that $x(\theta) = \bar{x}$ in some (possibly one-sided) neighborhood of $\theta^\star$, so $y^\star$ is defined uniquely.

problem (OA.1.8) - (OA.1.12) becomes

$$\max_x \ \lambda v \int_0^1 x(\theta) f(\theta) d\theta + (1-\lambda) \int_0^1 (\theta h + (1-\theta) l) x(\theta) f(\theta) d\theta) \tag{OA.1.19}$$

subject to

$$0 \leq x(\theta) \leq 1, \, \forall \theta, \tag{OA.1.20}$$

$$x \text{ is non-decreasing}, \tag{OA.1.21}$$

$$\int_0^1 x(\theta) \phi(\theta) f(\theta) d\theta \geq 0, \tag{OA.1.22}$$

$$\int_\tau^1 x(\theta) f(\theta) d\theta \leq \frac{1}{N}(1 - F^N(\tau)), \, \forall \tau \in [0, \, 1]. \tag{OA.1.23}$$

By an analogous argument as in the derivation of the optimal y, an optimal x should dominate any x' satisfying conditions (OA.1.20), (OA.1.21), and (OA.1.23). Informally, optimality requires that x "shifts mass as much as possible to the right," subject to constraints. Thus, an optimal x satisfies (OA.1.23) with equality for all $\tau \geq \beta$, and is zero on $[0, \, \beta]$, where $\beta \geq 0$ is the smallest number such that constraint (OA.1.22) holds. Either

$$\int_0^1 x(\theta) \phi(\theta) f(\theta) d\theta \geq 0, \tag{OA.1.24}$$

in which case $\beta = 0$, or $\beta > 0$ is defined by

$$\int_\beta^1 x(\theta) \phi(\theta) f(\theta) d\theta = 0. \tag{OA.1.25}$$

Since x satisfies the Matthews-Border condition (OA.1.23) with equality on $[\beta, \, 1]$, it is induced by a joint rule that gives the good to the agent with the highest type, conditional on at least one agent having a type above β. That is

$$x(\theta) = \begin{cases} 0 & \theta < \beta \\ F^{N-1}(\theta) & \theta \geq \beta \end{cases}.$$

If condition (OA.1.24) holds, $\beta = 0$. However, with the above x, (OA.1.24) is equivalent to $\mathbb{E}[\max_{i \in \mathcal{N}} \tilde{\theta}_i] \geq (h-l)/(v-l)$ which is ruled out by assumption. Thus, we mus have $\beta > 0$, and the optimal mechanism for case *(1)* is an efficient auction with a strictly positive reserve price and no information revelation (since the low signal is sent with probability zero). It is straightforward to verify that the mechanism frame described in point *(b)* of Proposition OA.1 is a cutoff mechanism that induces the above reduced form.

Case 2: $\int_0^1 x(\theta)\phi(\theta) f(\theta) d\theta < 0$.

In case *(2)*, problem (OA.1.8) - (OA.1.12) becomes, after incorporating the form of the optimal

y,

$$\max_{x,\theta^\star,\bar{x}} \lambda\left\{\int_0^{\theta^\star}[v-(v-h)\theta]x(\theta)f(\theta)d\theta + v\int_{\theta^\star}^1 x(\theta)f(\theta)d\theta - \bar{x}(v-h)\int_{\theta^\star}^1 \theta f(\theta)d\theta\right\}$$
$$+(1-\lambda)\left\{\int_0^1(\theta h+(1-\theta)l)x(\theta)f(\theta)d\theta\right\} \quad \text{(OA.1.26)}$$

subject to

$$0 \le x(\theta) \le \bar{x},\ \forall\theta \le \theta^\star, \quad \text{(OA.1.27)}$$

$$\bar{x} \le x(\theta) \le 1,\ \forall\theta \ge \theta^\star, \quad \text{(OA.1.28)}$$

$$x \text{ is non-decreasing,} \quad \text{(OA.1.29)}$$

$$\int_{\theta^\star}^1 (x(\theta)-\bar{x})\phi(\theta)f(\theta)d\theta \ge 0, \quad \text{(OA.1.30)}$$

$$\int_\tau^1 x(\theta)f(\theta)d\theta \le \frac{1}{N}(1-F^N(\tau)),\ \forall\tau\in[0,\,1]. \quad \text{(OA.1.31)}$$

Fix $\bar{x}$ and consider the optimal choice of $\theta^\star$ and x on $[\theta^\star,\,1]$. Because the objective function is point-wise increasing in x in this region, $\theta^\star$ and x should be chosen to satisfy the Matthews-Border condition (OA.1.31) with equality. Thus, $x(\theta) = F^{N-1}(\theta)$ for $\theta \in [\theta^\star,\,1]$.

Next, consider choosing x on $[0,\,\theta^\star]$, fixing $\theta^\star$ and $\bar{x}$. In the objective function, under the integral sign, $x(\theta)$ multiplies the function $\lambda[v-(v-h)\theta]+(1-\lambda)[\theta h+(1-\theta)l]$ that is positive non-increasing as long as $\lambda \ge (h-l)/(v-l)$, which is true by assumption. Because x cannot be decreasing (due to constraint OA.1.29), the optimal x must be therefore constant on $[0,\,\theta^\star]$, equal to some $\gamma \le \bar{x}$ such that condition (OA.1.31) is satisfied (formally, this is established by an argument analogous to the one used in the derivation of the optimal y).

With these two steps, the problem reduces to

$$\max_{x(0)\le\bar{x},\theta^\star} \lambda\left\{x(0)\int_0^{\theta^\star}[v-(v-h)\theta]f(\theta)d\theta + v\int_{\theta^\star}^1 F^{N-1}(\theta)f(\theta)d\theta - \bar{x}(v-h)\int_{\theta^\star}^1\theta f(\theta)d\theta\right\}$$
$$+(1-\lambda)\left\{x(0)\int_0^{\theta^\star}[\theta h+(1-\theta)l]f(\theta)d\theta + \int_{\theta^\star}^1[\theta h+(1-\theta)l]F^{N-1}(\theta)f(\theta)d\theta\right\} \quad \text{(OA.1.32)}$$

subject to

$$\int_{\theta^\star}^1 (F^{N-1}(\theta)-\bar{x})\phi(\theta)f(\theta)d\theta \ge 0, \quad \text{(OA.1.33)}$$

$$x(0)(F(\theta^\star)-\tau) \le \frac{1}{N}(F^N(\theta^\star)-\tau^N),\ \forall\tau\in[0,\,F(\theta^\star)]. \quad \text{(OA.1.34)}$$

Constraint (OA.1.34) can only be binding at the ends of the interval because the function on the left hand side is affine in τ, and the function on the right is concave in τ. Thus, (OA.1.34) becomes $x(0) \le (1/N)F^{N-1}(\theta^\star)$. Because the objective function is increasing in $x(0)$, it is optimal to set $x(0)$ to its upper bound: $x(0) = \max(\bar{x},\,(1/N)F^{N-1}(\theta^\star))$. The objective is also strictly increasing in

$\theta^\star$. This means that constraint (OA.1.33) must bind. Suppose that $\bar{x} > (1/N)F^{N-1}(\theta^\star)$. Then, by decreasing $\bar{x}$ slightly, we increase the objective function and preserve constraint (OA.1.33). Thus, $x(0) = \bar{x} = (1/N)F^{N-1}(\theta^\star)$ at the optimal solution. Because the objective function is increasing in $\theta^\star$, the solution is obtained by finding the highest $\theta^\star$ for which equation (OA.1.33) binds, that is,

$$\int_{\theta^\star}^{1} (F^{N-1}(\theta) - \frac{1}{N} F^{N-1}(\theta^\star))\phi(\theta) f(\theta) d\theta = 0. \tag{OA.1.35}$$

Because we are in case *(2)*, by assumption, (OA.1.33) is violated with $\theta^\star = 0$. Thus, $\theta^\star$ is strictly positive, and hence $\bar{x}$ is also strictly positive.

Summarizing, the solution takes the form

$$x(\theta) = \begin{cases} (1/N)F^{N-1}(\theta^\star) & \theta < \theta^\star \\ F^{N-1}(\theta) & \theta \geq \theta^\star \end{cases},$$

and $y(\theta) = (1/N)F^{N-1}(\theta^\star)$, for all θ. The functions x and y are easily seen to be feasible for the original (unrelaxed) problem. It is readily verified that the mechanism frame described in point *(a)* of Proposition OA.1 is a cutoff rule and corresponds to the reduced form mechanism described above.

Comparing case *(1)* and case *(2)*.

I have shown that the optimal mechanism is either the one from case *(1)* (corresponding to point (b) in Proposition OA.1) or the one from case *(2)* (corresponding to point (a) in Proposition OA.1). What remains to be shown is that the mechanism from case *(2)* is optimal under a regularity condition to be defined.

Given the optimal mechanism for case *(1)*, I will construct an alternative mechanism that is feasible and yields a strictly higher value of objective (OA.1.8) under a regularity condition. This will mean that the mechanism from case *(2)* must be optimal.

Fix the optimal mechanism in case *(1)* with $\beta > 0$. Consider an alternative mechanism, indexed by $\epsilon \geq 0$ with $y_\epsilon(\theta) = \epsilon$, for all θ, and

$$x_\epsilon(\theta) = \begin{cases} \epsilon & \theta < \beta_\epsilon \\ F^{N-1}(\theta) & \theta \geq \beta_\epsilon \end{cases},$$

where β_ϵ is defined by

$$\int_{\beta_\epsilon}^{1} (F^{N-1}(\theta) - \epsilon)\phi(\theta) f(\theta) d\theta = 0. \tag{OA.1.36}$$

At $\epsilon = 0$, $\beta(0) = \beta > 0$ (because β is defined by equation OA.1.25), so for small ϵ, there exists a strictly positive solution β_ϵ to equation (OA.1.36). Intuitively, I constructed a mechanism that takes a small step towards the optimal mechanism from case *(2)*. For small enough ϵ, constraint (OA.1.12) holds, and constraint (OA.1.11) is satisfied with equality given that equation (OA.1.36) holds. Thus, the pair (x_ϵ, y_ϵ) is feasible for small enough ϵ.

For $\epsilon = 0$, (x_0, y_0) is the optimal solution for case *(1)*. Therefore, it is enough to show that the objective function (OA.1.8) is strictly increasing in ϵ in the neighborhood of $\epsilon = 0$. Because the objective function is differentiable in ϵ (in particular, β_ϵ is differentiable in ϵ by the implicit function theorem), it is enough to show that the derivative is strictly positive at 0. Using the implicit function theorem to differentiate β_ϵ using equation (OA.1.36), the right derivative of (OA.1.8) under the mechanism $(x_\epsilon,\, y_\epsilon)$ at $\epsilon = 0$ can be shown to be

$$\lambda\left(vF(\beta) + v\frac{\int_\beta^1 \phi(\theta)f(\theta)d\theta}{\phi(\beta)} - (v-h)\int_0^1 \theta f(\theta)d\theta\right) \\ + (1-\lambda)\left(\int_0^\beta [\theta h + (1-\theta)l]f(\theta)d\theta + [\beta h + (1-\beta)l]\frac{\int_\beta^1 \phi(\theta)f(\theta)d\theta}{\phi(\beta)}\right) \quad \text{(OA.1.37)}$$

First, I argue that the term in brackets multiplying $(1-\lambda)$ is strictly positive. It is enough to show that

$$\frac{\int_\beta^1 \phi(\theta)f(\theta)d\theta}{\phi(\beta)} > 0$$

but this follows from the fact that since $\phi(\theta)$ is non-decreasing, (OA.1.36) for $\epsilon = 0$ implies that $\int_\beta^1 \phi(\theta)f(\theta)d\theta < 0$, and then clearly also $\phi(\beta) < 0$. Therefore, it is enough to prove that

$$vF(\beta) + v\frac{\int_\beta^1 \phi(\theta)f(\theta)d\theta}{\phi(\beta)} - (v-h)\int_0^1 \theta f(\theta)d\theta > 0. \quad \text{(OA.1.38)}$$

Equation (OA.1.36) defining β for $\epsilon = 0$ can be written as

$$\frac{v-h}{v-l} = 1 - \mathbb{E}[\max_{i\in\mathcal{N}} \tilde{\theta}_i |\, \max_{i\in\mathcal{N}} \tilde{\theta}_i \geq \beta].$$

Define $\theta_\beta^{(1)} \equiv \mathbb{E}[\max_{i\in\mathcal{N}} \tilde{\theta}_i |\, \max_{i\in\mathcal{N}} \tilde{\theta}_i \geq \beta]$. Since $l > 0$, we have $(v-h)/v < 1 - \theta_\beta^{(1)}$. Moreover, recalling that $\phi(\theta) \equiv (v-h) - (v-l)(1-\theta)$, we have $\phi(\theta) = (v-l)[\theta - \theta_\beta^{(1)}]$. Using these relations, to show (OA.1.38), it is enough to show the weak inequality

$$F(\beta) \geq \frac{\int_\beta^1 (\theta - \theta_\beta^{(1)})f(\theta)d\theta}{\theta_\beta^{(1)} - \beta} + (1-\theta_\beta^{(1)})\int_0^1 \theta f(\theta)d\theta.$$

Rearranging terms, we get

$$\theta_\beta^{(1)} - (\theta_\beta^{(1)} - \beta)(1-\theta_\beta^{(1)})\int_0^1 \theta f(\theta)d\theta - \beta F(\beta) \geq \int_\beta^1 \theta f(\theta)d\theta.$$

Using integration by parts, and rearranging again,

$$(1-\theta_\beta^{(1)})\left[1 + (\theta_\beta^{(1)} - \beta)\int_0^1 \theta f(\theta)d\theta\right] \leq \int_\beta^1 F(\theta)d\theta, \quad \text{(OA.1.39)}$$

where recall that $\theta_\beta^{(1)} \equiv \mathbb{E}[\max_{i\in\mathcal{N}} \tilde{\theta}_i |\, \max_{i\in\mathcal{N}} \tilde{\theta}_i \geq \beta]$. Thus, this equation depends solely on the primitive distribution f. If inequality (OA.1.39) holds for all $\beta \in [0, 1]$, I will say that the distribution satisfies the *regularity condition*. Under the regularity condition, I have shown that the mechanism from case *(1)* cannot be optimal, therefore the mechanism from case *(2)* must be optimal.

In the remainder of the proof, I show that $F(\theta) = \theta^\kappa$ satisfies the regularity condition for any $\kappa > 0$. I will show that a more restrictive inequality holds:

$$\int_\beta^1 F(\theta)d\theta - (1 - \theta^{(1)}_{\beta,2}) \left[1 + (1-\beta)\int_0^1 \theta f(\theta)d\theta\right] \geq 0,$$

where $\theta^{(1)}_{\beta,2}$ denotes the expectation of the first order statistic conditional on exceeding β when $N = 2$ (the smaller N, the harder it is to satisfy OA.1.39). By brute-force calculation, one can check that the left hand side of the above inequality is a concave function of β. Thus, it is enough to check that the inequality holds at the two endpoints. When $\beta = 0$, we have

$$\int_0^1 F(\theta)d\theta - (1 - \theta^{(1)}_{0,2}) \left[1 + \int_0^1 \theta f(\theta)d\theta\right] = \frac{1}{1+\kappa} - \left(1 - \frac{2\kappa}{2\kappa+1}\right)\left(1 + \frac{\kappa}{\kappa+1}\right) = 0.$$

On the other hand, for $\beta = 1$, we have $\theta^{(1)}_{\beta,2} = 1$, and the inequality is trivially satisfied.

OA.2 Additional results

OA.2.1 Robust implementation

In Section 5.1, I argued that the set of mechanism frames that can be implemented robustly, i.e., without the designer knowing the prior distribution of types and the exact form of the aftermarket, is contained within the set of cutoff rules. In this appendix, I show that there are indirect mechanisms that implement certain cutoff rules robustly. I maintain to assume that agents share a common prior belief about the environment. To simplify exposition, I assume that $N \geq 2$, agents are ex-ante identical, and there is a continuous distribution of types with cdf F and density f on $\Theta = [0, 1]$ (I drop the subscripts).

It is known (see for example Bergemann and Morris, 2013) that using direct mechanisms is not without loss of generality when agents have more information than the designer. For example, if agents are ex-ante identical and arbitrary indirect mechanisms are allowed, the designer can elicit information about the distribution by asking agents to report it, and punishing if reports disagree. Motivated by practical applicability, I focus on simple indirect mechanisms with a one-dimensional message space, such as standard auctions.

In general, it may be impossible to robustly implement a cutoff rule in an auction with a one-dimensional bid space. Even if the equilibrium bidding function is an injection on the set of types Θ, so that in principle there exists a mapping from bids into allocations and signals that implements the cutoff rule $(\boldsymbol{x}, \boldsymbol{\pi})$, the designer cannot invert the equilibrium bidding functions if she lacks knowledge about the prior f and the aftermarket A.

However, there are important cases in which inverting the bidding function is not necessary. Suppose that $\boldsymbol{x}$ allocates the object to the highest-value agent, and that the disclosure rule $\boldsymbol{\pi}$ takes one of the extreme forms, full or no disclosure, analyzed in Section 4.2.1. With this allocation rule and a continuous type space, the cutoff is equivalent to the second highest type. Define a function $v^{\boldsymbol{\pi}} : \Theta^2 \to \mathbb{R}$ by $v^{\boldsymbol{\pi}}(\theta, \hat{\theta}) = u(\theta; f^{\hat{\theta}}_{\boldsymbol{\pi}})$, where $f^{\hat{\theta}}_{\boldsymbol{\pi}}$ is the posterior belief over the type of the winner induced in a truthful equilibrium under the mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ when the second highest reported type was $\hat{\theta}$. When $\boldsymbol{\pi}$ is the no-disclosure rule, $f^{\hat{\theta}}_{\boldsymbol{\pi}}$ is the distribution of the first order statistic of N draws from f (and does not depend on $\hat{\theta}$). If $\boldsymbol{\pi}$ is the full-disclosure rule, $f^{\hat{\theta}}_{\boldsymbol{\pi}}$ is the truncation of f at $\hat{\theta}$.

The function $v^{\boldsymbol{\pi}}$ is similar to an object studied by Milgrom and Weber (1982) in the context of auctions with affiliated values. In the setting of Milgrom and Weber (1982), the value of the winner depends on the value of the second highest bidder $\hat{\theta}$ due to statistical correlation of types and the assumption of interdependent values. In my setting, the value of the winner depends on $\hat{\theta}$ because the bid of the second highest bidder influences the signal sent by the mechanism, and hence the continuation payoff of the winner. An important difference is that the payoff of the winner in my model depends on the reported type (or bid) rather than the actual type of the second highest bidder. In Milgrom and Weber (1982), affiliation of types implies that their analog of $v^{\boldsymbol{\pi}}(\theta, \theta)$ is non-decreasing in θ, a property necessary for existence of a monotone equilibrium in standard auctions. In my model, $v^{\boldsymbol{\pi}}(\theta, \hat{\theta})$ is non-decreasing in θ under the assumption of a monotone aftermarket. In general, when $\boldsymbol{\pi}$ is the full disclosure rule, there is no reason to expect monotonicity in $\hat{\theta}$, and hence $v^{\boldsymbol{\pi}}(\theta, \theta)$ may fail to be increasing in θ. A sufficient condition for monotonicity of $v^{\boldsymbol{\pi}}(\theta, \theta)$ is that posterior beliefs that are higher (e.g. in first-order stochastic dominance order) lead to a higher payoff for the winner in the aftermarket.

Proposition OA.2. *Suppose that $x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \mathbf{1}_{\{\theta_i \geq \boldsymbol{\theta}^{(1)}_{-i}\}}$, for all i, and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and $\boldsymbol{\pi}$ is the full-disclosure rule. If $v^{\boldsymbol{\pi}}(\theta, \theta)$ is strictly increasing in θ, then there exists a Bayesian Nash equilibrium implementing $(\boldsymbol{x}, \boldsymbol{\pi})$ for any prior f and monotone aftermarket A*

- *in a second price auction where the price paid by the winner is disclosed, or*
- *in a first price auction where the second highest bid is disclosed.*

The equilibrium in the second price auction is in dominant strategies.

Moreover, if $v^{\boldsymbol{\pi}}(\theta, \theta)$ is strictly positive for all θ, then there exists a Bayesian Nash equilibrium implementing $(\boldsymbol{x}, \boldsymbol{\pi})$ for any prior f and monotone aftermarket A in an all pay auction where the second highest bid is disclosed. The equilibrium in the second price auction is in dominant strategies.

If $\boldsymbol{\pi}$ is the no-disclosure rule, then there exists a Bayesian Nash equilibrium implementing $(\boldsymbol{x}, \boldsymbol{\pi})$ for any prior f and monotone aftermarket A in either the second, first, or all pay auction with no disclosure, as long as the aftermarket is strictly monotone.

I focus on the more interesting case when $\boldsymbol{\pi}$ is the full disclosure rule in the discussion. A candidate equilibrium bidding function is determined by the local (first-order) optimality condition. The bidding function must be strictly increasing to guarantee that full disclosure of the bid (or price) leads to full disclosure of the second highest type. In a SPA, the candidate bidding function in the full-disclosure case is $v^{\boldsymbol{\pi}}(\theta,\, \theta)$, and so $v^{\boldsymbol{\pi}}(\theta,\, \theta)$ must be strictly increasing in θ. In a FPA, it is enough that $(\int_0^\theta v^{\boldsymbol{\pi}}(\tau,\, \tau)dF^{N-1}(\tau))/(F^{N-1}(\theta))$ is strictly increasing in θ. This condition holds for all F if and only if $v^{\boldsymbol{\pi}}(\theta,\, \theta)$ is strictly increasing in θ. Finally, in an all pay auction, an agent with type θ bids $\int_0^\theta v^{\boldsymbol{\pi}}(\tau,\, \tau)dF^{N-1}(\tau)$. The bidding function is strictly increasing as long as $v^{\boldsymbol{\pi}}(\theta,\, \theta)$ is strictly positive for all θ, because, unlike in a first or second price auction, it is not obtained by conditioning on winning.

Despite the similarity to the interdependent value setting of Milgrom and Weber (1982), I am able to prove that the equilibrium in the SPA is in dominant strategies. This is because the value of the winner depends on the *bid* of the second highest bidder, rather than her *type*. Suppose that $\boldsymbol{\pi}$ is the full disclosure rule, and that aftermarket beliefs are formed based on the strictly increasing bidding function $\beta(\theta) = v^{\boldsymbol{\pi}}(\theta,\, \theta)$. Then, when the second highest bid is b, the aftermarket belief over the winner's type is the same as if the second highest type was $\hat{\theta} = \beta^{-1}(b)$. But this means that conditional on winning, $v^{\boldsymbol{\pi}}(\theta,\, \theta) \geq b$, the winner with type θ values winning $(v^{\pi}(\theta,\, \hat{\theta}))$ more than she pays $(b = v^{\boldsymbol{\pi}}(\hat{\theta},\, \hat{\theta}))$, and vice versa.

Proof of Proposition OA.2 I only prove the first part of the proposition because the proof of the second part (about implementation of the no-revelation rule) is analogous.

Suppose that $\boldsymbol{\pi}$ is the full-disclosure rule. In all three designs considered in Proposition OA.2, a necessary condition for robust implementation of $(\boldsymbol{x},\, \boldsymbol{\pi})$ is that there exists an equilibrium in strictly increasing bidding strategies – otherwise, disclosing the second highest bid does not correspond to disclosing the second highest type.

Consider a second price auction where the price paid by the winner is disclosed. I conjecture that $\beta^{SPA}(\theta) = v^{\boldsymbol{\pi}}(\theta,\, \theta)$ is the equilibrium bidding function. By assumption, $\beta^{SPA}(\theta)$ is strictly increasing, so disclosing the price b paid by the winner corresponds to disclosing the type $\hat{\theta}_b = (\beta^{SPA})^{-1}(b)$ (third parties in the aftermarket form beliefs based on the assumption that β^{SPA} is the equilibrium bid function). I must show that it is a dominant strategy for an agent with type θ to bid $\beta^{SPA}(\theta)$ in this auction (then, this bidding strategy profile is obviously also a Bayesian Nash equilibrium). Whenever the agent wins the auction against some bid b, her payoff is non-negative because the value from winning is $v^{\boldsymbol{\pi}}(\theta,\, \hat{\theta}_b)$ and the price paid is $b = v^{\boldsymbol{\pi}}(\hat{\theta}_b,\, \hat{\theta}_b)$. This follows from monotonicity of the aftermarket and the fact that $\theta \geq \hat{\theta}_b$ because $v^{\boldsymbol{\pi}}(\theta,\, \theta) = \beta^{SPA}(\theta) \geq b = v^{\boldsymbol{\pi}}(\hat{\theta}_b,\, \hat{\theta}_b)$. Moreover, her payoff would be the same with any alternative bid $\hat{b}$ that wins, and would be weakly lower if she lost. Conversely, when the agent loses with type θ and bid $\beta^{SPA}(\theta)$, it would not be optimal to increase the bid to win against the highest bid b, because $b = v^{\boldsymbol{\pi}}(\hat{\theta}_b,\, \hat{\theta}_b) \geq v^{\boldsymbol{\pi}}(\theta,\, \hat{\theta}_b)$ in this case.

Consider a first price auction. I conjecture an equilibrium bidding function

$$\beta^{FPA}(\theta) = \frac{\int_0^\theta v^{\boldsymbol{\pi}}(\tau,\, \tau)dF^{N-1}(\tau)}{F^{N-1}(\theta)}.$$

If $v^{\boldsymbol{\pi}}(\theta, \theta)$ is strictly increasing in θ, then the bidding function is strictly increasing for any full-support F. I have to show that the following optimality condition holds

$$\theta \in \text{argmax}_{\hat{\theta}} \int_0^{\hat{\theta}} \left(v^{\boldsymbol{\pi}}(\theta,\, \tau) - \beta^{FPA}(\hat{\theta}) \right) dF^{N-1}(\tau), \tag{OA.2.1}$$

for any $\theta \in \Theta$. We have

$$\int_0^{\hat{\theta}} \left(v^{\boldsymbol{\pi}}(\theta,\, \tau) - \beta^{FPA}(\hat{\theta}) \right) dF^{N-1}(\tau) = \int_0^{\hat{\theta}} \left(v^{\boldsymbol{\pi}}(\theta,\, \tau) - v^{\boldsymbol{\pi}}(\tau,\, \tau) \right) dF^{N-1}(\tau),$$

so the conclusion follows from the monotonicity of the aftermarket.

Finally, consider an all pay auction, with a candidate equilibrium bidding function

$$\beta^{APA}(\theta) = \int_0^{\theta} v^{\boldsymbol{\pi}}(\tau,\, \tau) dF^{N-1}(\tau).$$

Because $v^{\boldsymbol{\pi}}(\theta, \theta)$ is strictly positive for all θ, this bidding function is always strictly increasing. I have to argue that

$$\theta \in \text{argmax}_{\hat{\theta}} \left\{ \int_0^{\hat{\theta}} v^{\boldsymbol{\pi}}(\theta,\, \tau) dF^{N-1}(\tau) - \beta^{APA}(\hat{\theta}) \right\}, \tag{OA.2.2}$$

for any $\theta \in \Theta$. This is true by the monotonicity of the aftermarket.

OA.2.2 Indirect implementation of cutoff rules

In this section, I provide a characterization of cutoff rules as equilibria of simple dynamic auctions, called Generalized Clock Auctions (GCAs). These results formalize the discussion of indirect implementation of cutoff rules from Section 7.

I assume symmetry and a discrete type space for ease of exposition. A Generalized Clock Auction (GCA) is characterized by a (possibly random) sequence of prices and a disclosure rule. Let $\mathcal{T} = \{0,\, 1,\, 2,\, ...,\, T\}$ be the set of rounds. In round 0, agents simultaneously decide whether to participate or not. In every subsequent round $t \in \mathcal{T}$: (1) A price p^t is announced to bidders; (2) Bidders simultaneously decide to stay in the auction, or to exit; (3) The auctioneer observes bidders' decisions and implements the relevant outcome (to be specified); (4) The auctioneer announces to bidders the outcome of the round (whether the auction continues, the set of active bidders, the winner in case the auction ends). The outcome of a round is determined in the following way. If at least two bidders decide to stay (and $t < T$), the auction continues to the next round. Bidders who exited are declared inactive and do not participate in future rounds. If less than two bidders decide to stay, the auction terminates. There are two cases: If all remaining n active bidders drop out, the object is allocated uniformly at random among them, the winner pays p^t, and all bidders are declared inactive. If exactly one bidder stays (and $n-1$ bidders drop out), she wins the object at price p^{t+1}, and is declared inactive with probability $1/n$ (which is the probability she would have won the object by dropping out in that round). At $t = T$, all bidders must exit. After the object is allocated, a signal s is released publicly according to a disclosure rule (to be specified). The distinction between the winner being active or inactive at the end of the auction is irrelevant for the final allocation but will matter for the informational content of the signal.

Let H^t denote the public history of the bidding procedure described above up to and including round t, and let $\mathcal{H}^t$ be the set of all public histories. A public history in this context is identified with the sequence of announcements made by the auctioneer to the bidders during the auction. A Generalized Clock Auction (GCA) is a sequence of functions $\{(Y^t, P^t)\}_{t=1}^T$, where $Y^t : \mathcal{H}^t \to \times_{i\in\mathcal{N}}\Delta(\mathcal{S}_i)$, for some (finite) signal spaces $\mathcal{S}_i$, and $P^t : \mathcal{H}^{t-1} \to \Delta(\mathbb{R})$. In each round t, given a history H^{t-1}, a price p^t is drawn from the distribution $P^t(H^{t-1})$. If the auction ends in round t, the signal is drawn and announced according to distribution $Y^t(H^t)$. Hence, the signal s is an arbitrary garbling of the entire public history of the auction. The informational content of signals is determined by the equilibrium behavior of bidders.

Let N^t denote the number of active bidders at the end of round t. A GCA is called Markov, if P^t depends on H^{t-1} only through N^{t-1} and Y^t depends on H^t only through (N^{t-1}, N^t), the number of active bidders at the beginning and at the end of the last round. If the auction ends at t, N^t can be either 0 or 1, depending on whether the winner was declared active or inactive.

A pure strategy for an agent participating in a GCA is a mapping $a_i : \Theta \times \mathcal{T} \times \mathcal{H} \to \{0, 1\}$, i.e., for a type $\theta \in \Theta$, in round $t \in \mathcal{T}$, given a partial history $H^{t-1} \in \mathcal{H}^{t-1}$, $a_i(\theta, t, H^{t-1})$ specifies whether type θ exits in round t or not.[9] A strategy for agent i is monotone if for any two types $\theta > \hat{\theta}$, any $t \in \mathcal{T}$ and $H^{t-1} \in \mathcal{H}^{t-1}$, we have $a_i(\theta, t, H^{t-1}) \geq a_i(\hat{\theta}, t, H^{t-1})$. A strategy is Markov if $a_i(\theta, t, H^{t-1})$ depends on H^{t-1} only through N^{t-1}. Mixed strategies σ_i are defined in the usual way. I call a mixed-strategy σ_i monotone if it is a randomization over monotone pure strategies a_i.

An *equilibrium* is a Perfect Bayesian Equilibrium of the GCA with payoffs determined by the outcome of the auction and the aftermarket $A \equiv \{u(\theta; \bar{f}) : \theta \in \Theta, \bar{f} \in \Delta(\Theta)\}$. I assume that the aftermarket is monotone. Given a strategy profile σ, if an agent with type θ wins the auction and signal s is released, I denote the posterior belief over the winner's type by f_σ^s. In that case, the ex-post payoff of the winner is $u(\theta; f_\sigma^s)$.

In the statement of the result, I restrict attention to a subclass of allocation rules. This allows me to focus on simple GCAs. Define a hierarchical allocation rule $\boldsymbol{x}^{\kappa_1 \dots \kappa_k}(\boldsymbol{\theta})$ for any sequence $\kappa_1 < \dots < \kappa_k$, with $\kappa_m \in \Theta$ for all m, by

$$x_i^{\kappa_1\dots\kappa_k}(\theta_i, \boldsymbol{\theta}_{-i}) = \begin{cases} \frac{1}{|\{j\in\mathcal{N}:\, \kappa_m \leq \theta_j < \kappa_{m+1}\}|} & \text{if } \kappa_m \leq \theta_i < \kappa_{m+1} \text{ and, } \forall j,\ \theta_j < \kappa_{m+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where by convention $\kappa_{k+1} = \infty$. For example, if $\Theta = \{\theta_1, \dots, \theta_n\}$, then $\boldsymbol{x}^{\theta_1\dots\theta_n}(\boldsymbol{\theta})$ is the "efficient" allocation rule (highest type receives the good), $\boldsymbol{x}^{\theta_m\dots\theta_n}(\boldsymbol{\theta})$ excludes types $\theta_1, \dots, \theta_{m-1}$, and $\boldsymbol{x}^{\theta_1}(\boldsymbol{\theta})$ corresponds to a uniform lottery. A symmetric allocation rule $\boldsymbol{x}$ is called *decomposable* if it is a convex combination of hierarchical allocation rules. Decomposability is a mild restriction from a practical perspective and without loss of generality for optimal design. It rules out cases when the

[9] Given the set of feasible actions and the definition of public history, it is irrelevant whether bidders condition their strategies on private or public histories.

allocation for agent i depends on types of agents who never receive the good, regardless of what agent i does.[10]

I say that two mechanism frames $(\boldsymbol{x}, \boldsymbol{\pi})$ and $(\boldsymbol{x}', \boldsymbol{\pi}')$ are *Bayesian equivalent* under $\boldsymbol{f}$ if they induce the same reduced form (in particular, Bayesian equivalent mechanisms yield the same interim expected payoff to every agent conditional on any signal realization).

Theorem OA.1. *If $(\boldsymbol{x}, \boldsymbol{\pi})$ is a mechanism frame implemented by a monotone equilibrium of a GCA, then $(\boldsymbol{x}, \boldsymbol{\pi})$ is a cutoff rule. Conversely, if $\boldsymbol{x}$ is decomposable and $(\boldsymbol{x}, \boldsymbol{\pi})$ is a symmetric cutoff rule, then there exists a Bayesian equivalent $(\boldsymbol{x}, \boldsymbol{\pi}')$ that can be implemented in a pure-strategy equilibrium of a Markov GCA in which bidders have dominant strategies, and randomization over prices only happens in round 0 (subsequent prices are deterministic functions of the number of active bidders and the realization of the initial random price).*

Due to decomposability of $\boldsymbol{x}$, in order to implement a cutoff rule, it is enough to keep track of the number of active bidders in any round (the Markov property). This is because the allocation does not depend on the types of bidders who exited in previous rounds. The main difficulty in the proof is to show that decomposability of $\boldsymbol{x}$ implies existence of a signal distribution $\boldsymbol{\pi}'$, Bayesian equivalent to $\boldsymbol{\pi}$, that inherits this property. The original distribution $\boldsymbol{\pi}$ may depend on types of bidders who exited in previous rounds, and thus it cannot be implemented with a deterministic price path.

Prices do not have to change monotonically in a GCA.[11] Because the informational content of the signal in general depends on the termination time of the auction, it is as if a different good was offered for sale in every round. Prices may have to decrease when the current-round signal induces posterior beliefs that are less attractive for bidders.

Proof of Theorem OA.1

Proof of the direct part: I first introduce some notation. A deterministic price path $\boldsymbol{p} = (p^t)_{t\geq 1}$, a monotone pure-strategy profile a, and type profile $\boldsymbol{\theta}$, together pin down a unique time of exit for every agent. I let $\Gamma_i^{(y,\,p,\,a)}(\theta_i,\, \boldsymbol{\theta}_{-i})$ denote the exit time of agent i with type θ_i, when other types are $\boldsymbol{\theta}_{-i}$. Because strategies are assumed to be monotone, Γ_i is non-decreasing in θ_i. Let H_0^τ denote the history in which the winner becomes inactive in round τ, and let H_1^τ denote the history in which the winner remains active in round τ. For the tuple $(y,\, p,\, a)$, the corresponding allocation and

[10] For example, the following allocation rule is ruled out: If the third highest type is below some threshold $\theta^\star$ and strictly below the second highest type, then the good is allocated uniformly at random among the two highest types; otherwise it is allocated to the highest type.

[11] This is one key difference to obviously strategy-proof auctions considered by Li (2016).

revelation rules are given by

$$\pi_i^{(y,\,p,\,a)}(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i^{(y,\,p,\,a)}(\theta_i,\,\boldsymbol{\theta}_{-i})$$
$$= \begin{cases} \left(\frac{1}{N^{\tau-1}}\right) Y^\tau(H_0^\tau)(s) + \left(\frac{N^{\tau-1}-1}{N^{\tau-1}}\right) Y^\tau(H_1^\tau)(s) & \text{if } \Gamma_i^{(y,\,p,\,a)}(\boldsymbol{\theta}) > \tau \equiv \max_{j\neq i} \Gamma_j^{(y,\,p,\,a)} \\ \left(\frac{1}{N^{\tau-1}}\right) Y^\tau(H_0^\tau)(s) & \text{if } \Gamma_i^{(y,\,p,\,a)}(\boldsymbol{\theta}) = \tau \equiv \max_{j\neq i} \Gamma_j^{(y,\,p,\,a)} \\ 0 & \text{otherwise,} \end{cases} \tag{OA.2.3}$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $s \in \mathcal{S}_i$. In the above definition, τ denotes the round in which the auction ends. If agent i is the only agent who decides to stay in round τ (first case), she wins the auction, and there is a $1/N^{\tau-1}$ probability that the all bidders will be announced inactive, in which case the signal is drawn from the distribution conditional on history H_0^τ. Otherwise, the winner i is active, and the signal is drawn from distribution $Y^\tau(H_1^\tau)$. If all bidders, including agent i, decide to exit in round τ (second case), there is a $1/N^{(\tau-1)}$ probability that agent i receives the good, and the signal is drawn from $Y^\tau(H_0^\tau)$. Finally, if agent i exits before round τ (third case), she does not win the good.

For a Generalized Clock Auction $(Y,\,P) = \{(Y^t,\,P^t)\}_{t\geq 1}$, monotone mixed strategy profile σ, and type profile $\boldsymbol{\theta}$ we can define, for all i,

$$\pi_i(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i(\theta_i,\,\boldsymbol{\theta}_{-i}) = \mathbb{E}_{(y,\,p)\sim(Y,\,P),\,a\sim\sigma}\, \pi_i^{(y,\,p,\,a)}(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i^{(y,\,p,\,a)}(\theta_i,\,\boldsymbol{\theta}_{-i}).$$

Each $\pi_i^{(y,\,p,\,a)}(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i^{(y,\,p,\,a)}(\theta_i,\,\boldsymbol{\theta}_{-i})$ is non-decreasing in θ_i, for any $s \in \mathcal{S}_i$, by direct inspection of equation (OA.2.3). Therefore, $\pi_i(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i(\theta_i,\,\boldsymbol{\theta}_{-i})$ is also non-decreasing in θ_i. This means that condition (M) from Proposition 1 holds. By Proposition 1, $(\boldsymbol{x},\,\boldsymbol{\pi})$ is a cutoff rule, and thus the first part of Theorem OA.1 is proven.

Discussion: The conclusion of the first part of Theorem OA.1 relies on the distinction of whether the winner is active or inactive at the end of the auction (this information is included in the public history which determines the signal distribution). If the auction always disclosed the decision of the winner (whether she decided to exit or to stay in the last round), the implemented mechanism frame could fail to be a cutoff mechanism. To see that, fix $(y,\,p,\,a)$ and $\boldsymbol{\theta}_{-i}$, and note that the random cutoff representing the allocation for agent i in the GCA has a binary distribution on $\{\theta^\tau,\,\theta^{\tau+1}\}$ with probabilities $1/N^{\tau-1}$ and $(N^{\tau-1}-1)/N^{\tau-1}$, respectively, where θ^τ is the smallest type of agent i who exits in round τ, and $\theta^{\tau+1}$ is the smallest type of agent i who does not exit up to and including round τ. A cutoff mechanism only reveals the realization of the cutoff so the auctioneer cannot fully disclose whether the type of the winner is above or below $\theta^{\tau+1}$. By introducing the random determination of the status of the winner (active or inactive), I formally incorporated the cutoff into the definition of a GCA.

Proof of the converse part: In the first step of the proof, given a cutoff rule $(\boldsymbol{x},\,\boldsymbol{\pi})$ with a decomposable allocation rule, I construct a Bayesian equivalent mechanism frame $(\boldsymbol{x},\,\boldsymbol{\pi}')$. The equivalent disclosure rule $\boldsymbol{\pi}'$ will have the feature that the signal distribution only depends on the number of active bidders. In the second step, I show how to implement $(\boldsymbol{x}, \boldsymbol{\pi}')$ using a Markov GCA.

Because $\boldsymbol{x}$ is decomposable, it can be represented as a convex combination of hierarchical allocation rules (the convex combination is finite because there are finitely many hierarchical auctions when the type space is finite):

$$\boldsymbol{x}(\boldsymbol{\theta}) = \sum_{\alpha} \lambda^{\alpha} \boldsymbol{x}^{\kappa_1^{\alpha} \ldots \kappa_{k(\alpha)}^{\alpha}}(\boldsymbol{\theta}),$$

for some $\lambda^{\alpha} \geq 0$, $\sum_{\alpha} \lambda^{\alpha} = 1$, and hierarchy $\kappa_1^{\alpha} \ldots \kappa_{k(\alpha)}^{\alpha}$, for each α. Because $(\boldsymbol{x}, \boldsymbol{\pi})$ is a symmetric cutoff rule, for any i, there exists a signal function γ such that for all s, and $\boldsymbol{\theta}$,

$$\pi_i(s|\, \theta_i, \boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \sum_{c \leq \theta_i} \gamma(s|\, c, \boldsymbol{\theta}_{-i}) \Delta x_i(c, \boldsymbol{\theta}_{-i}),$$

For a hierarchy κ_1, ..., κ_k, define

$$\kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}) = \max\{\kappa_m :\, \kappa_m \leq max_{j \neq i} \theta_j\},$$

and

$$n^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}) = |\{j \in \mathcal{N} \setminus \{i\} :\, \theta_j \geq \kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})\}|.$$

That is, $\kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})$ is the highest of the thresholds $\kappa_1, \ldots, \kappa_k$ that at least one type in $\boldsymbol{\theta}_{-i}$ exceeds, and $n^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})$ is the number of types in $\boldsymbol{\theta}_{-i}$ that exceed $\kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})$. The vector $\nu^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}) \equiv (\kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}), n^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}))$ is a sufficient statistic for $\boldsymbol{\theta}_{-i}$ needed to implement $x_i^{\kappa_1 \ldots \kappa_k}(\theta_i, \boldsymbol{\theta}_{-i})$. Moreover, when $\theta_i \geq \kappa_{m+1}$ and $\kappa_m = \kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})$, then the allocation for i does not depend on $n^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})$.

Define a symmetric cutoff rule $(\boldsymbol{x}', \boldsymbol{\pi}')$ by

$$\pi_i'(s|\, \theta_i, \boldsymbol{\theta}_{-i}) x_i'(\theta_i, \boldsymbol{\theta}_{-i}) = \sum_{\alpha} \lambda^{\alpha} \sum_{c \leq \theta_i} \gamma'(s|\, c, \nu^{\kappa_1^{\alpha} \ldots \kappa_{k(\alpha)}^{\alpha}}(\boldsymbol{\theta}_{-i})) \Delta x_i^{\kappa_1^{\alpha} \ldots \kappa_{k(\alpha)}^{\alpha}}(c, \boldsymbol{\theta}_{-i}),$$

for all s, and $\boldsymbol{\theta}$, where the signal function γ' is defined by,

$$\gamma'(s|\, c, \nu) = \frac{\sum_{\{\boldsymbol{\theta}_{-i}:\, \nu = \nu^{\kappa_1^{\alpha}, \ldots, \kappa_{k(\alpha)}^{\alpha}}(\boldsymbol{\theta}_{-i})\}} \gamma(s|\, c, \boldsymbol{\theta}_{-i}) \boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i})}{\sum_{\{\boldsymbol{\theta}_{-i}:\, \nu = \nu^{\kappa_1^{\alpha}, \ldots, \kappa_{k(\alpha)}^{\alpha}}(\boldsymbol{\theta}_{-i})\}} \boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i})},$$

for any feasible vector ν. For any i, $x_i' = x_i$, and π_i' averages the signal distribution under π_i across all $\boldsymbol{\theta}_{-i}$ that lead to the same allocation rule for agent i, i.e., across all $\boldsymbol{\theta}_{-i}$ with the same vector ν. Thus, $(\boldsymbol{x}, \boldsymbol{\pi})$ and $(\boldsymbol{x}, \boldsymbol{\pi}')$ are Bayesian equivalent. Moreover, $(\boldsymbol{x}, \boldsymbol{\pi}')$ can be decomposed into hierarchical mechanism frames in such a way that the allocation and signal distribution depend on $\boldsymbol{\theta}_{-i}$ only through the sufficient statistic $\nu(\boldsymbol{\theta}_{-i})$.

In the second step of the proof, I show how to implement $(\boldsymbol{x}, \boldsymbol{\pi}')$ in dominant strategies in a

GCA. By definition of $(\boldsymbol{x},\, \boldsymbol{\pi}')$, it is enough to show that the hierarchical mechanism frame

$$\pi_i^{\kappa_1^\alpha \ldots \kappa_{k(\alpha)}^\alpha}(s|\, \theta_i,\, \boldsymbol{\theta}_{-i}) x_i^{\kappa_1^\alpha \ldots \kappa_{k(\alpha)}^\alpha}(\theta_i,\, \boldsymbol{\theta}_{-i}) = \sum_{c \le \theta_i} \gamma'(s|\, c,\, \nu^{\kappa_1^\alpha \ldots \kappa_{k(\alpha)}^\alpha}(\boldsymbol{\theta}_{-i})) \Delta x_i^{\kappa_1^\alpha \ldots \kappa_{k(\alpha)}^\alpha}(c,\, \boldsymbol{\theta}_{-i}),$$

can be implemented in a GCA with a price path that only depends on the number of active bidders, for any α. The claim of the second part of Theorem OA.1 can then be obtained by randomizing over α according to the distribution $\{\lambda^\alpha\}$ in round 0 of the GCA.[12] In the remainder of the proof, I fix α and omit it from the notation – I will denote the hierarchy to be implemented by $\kappa_1, \ldots, \kappa_k$. Intuitively, in the equilibrium I construct, bidders with types in $[\kappa_t,\, \kappa_{t+1})$ exit in round t.

First, I specify the signal distribution for every possible outcome of the bidding process. Without loss of generality, given the hierarchy $\kappa_1, \ldots, \kappa_k$, I can assume that the auction ends no later than in round k in equilibrium.[13] If the auction ends in round $\tau \le k$, there are two cases. Either *(i)* all $N^{\tau-1}$ bidders become inactive in round τ, or *(ii)* $N^{\tau-1}-1$ bidders become inactive and exactly one bidder remains active. In case *(i)*, the signal s is drawn from distribution $\gamma'(\cdot\,|\, \kappa_\tau,\, (\kappa_\tau,\, N^{\tau-1}-1))$. In case *(ii)*, the signal s is drawn from distribution $\gamma'(\cdot\,|\, \kappa_{\tau+1},\, (\kappa_\tau,\, N^{\tau-1}-1))$. In particular, the signal distribution depends on the public history of the auction only through $N^{\tau-1}$ and N^τ (the latter variable determines which case, *(i)* or *(ii)*, is used).

Second, I specify the price function P^t, for each $t \le k+1$. Because $(\boldsymbol{x}^{\kappa_1 \ldots \kappa_k},\, \boldsymbol{\pi}^{\kappa_1 \ldots \kappa_k})$ is a hierarchical cutoff rule in which all types in any interval $[\kappa_t,\, \kappa_{t+1})$ receive the same outcome that moreover depends on $\boldsymbol{\theta}_{-i}$ only through $\nu^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})$ (and only through $\kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i})$ when $\kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}) \equiv \kappa_m < \kappa_{m+1} \le \theta_i$), there exists a transfer rule $\boldsymbol{t}^{\kappa_1 \ldots \kappa_k}$ that makes truthful reporting a dominant strategy and has the same property. Let $\bar{t}_i^{\kappa_1 \ldots \kappa_k}(\theta_i,\, \boldsymbol{\theta}_{-i}) = t_i^{\kappa_1 \ldots \kappa_k}(\theta_i,\, \boldsymbol{\theta}_{-i}) / x_i^{\kappa_1 \ldots \kappa_k}(\theta_i,\, \boldsymbol{\theta}_{-i})$ for any i and $\boldsymbol{\theta}$ such that $x_i^{\kappa_1 \ldots \kappa_k}(\theta_i,\, \boldsymbol{\theta}_{-i}) > 0$. That is, $\bar{t}_i$ is the transfer that agent i pays conditional on winning.[14] Then, I can define $P^t(N^{t-1}) = \bar{t}_i^{\kappa_1 \ldots \kappa_k}(\theta_i,\, \boldsymbol{\theta}_{-i})$ for any $\theta_i \in [\kappa_t,\, \kappa_{t+1})$ and $\boldsymbol{\theta}_{-i}$ such that *(i)* $\nu^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}) = (\kappa_t,\, N^{t-1}-1)$ if $N^{t-1} > 1$, or *(ii)* $\kappa^{\kappa_1 \ldots \kappa_k}(\boldsymbol{\theta}_{-i}) = \kappa_{t-1}$ if $N^{t-1} = 1$. P^t is well-defined because the properties of the transfer function $\boldsymbol{t}^{\kappa_1 \ldots \kappa_k}$ established above ensure that $\bar{t}_i^{\kappa_1 \ldots \kappa_k}(\theta_i,\, \boldsymbol{\theta}_{-i})$ does not depend on which $(\theta_i,\, \boldsymbol{\theta}_{-i})$ is picked in the above definition. Note that P^t is only a function of the number of active bidders at the beginning of round t.

Third, I specify equilibrium strategies for bidders. A type θ in the interval $[\kappa_t,\, \kappa_{t+1})$ stays in the auction in any round $t' \le t$, and drops out otherwise, regardless of the number of active bidders observed.

Fourth, by the specification of the signal distribution and the fact that bidders with types $\theta \in [\kappa_t,\, \kappa_{t+1})$ exit in round t, if bidders follow the above strategies, the auction implements the desired mechanism frame $(\boldsymbol{x}^{\kappa_1 \ldots \kappa_k},\, \boldsymbol{\pi}^{\kappa_1 \ldots \kappa_k})$.

Fifth, I argue why the above behavior constitutes a dominant strategy for every bidder. Suppose that bidder i with type θ faces a profile of other agents' strategies that leads to some sequence $\boldsymbol{N}_{-i} \equiv N_{-i}^1,\, N_{-i}^2, \ldots, N_{-i}^k$ of active bidders in every round, excluding i (this sequence is sufficient to

[12] I implicitly assume that the mechanism designer informs the bidders about the realization of α in round 0, i.e., discloses which hierarchical auction will be used.

[13] It is enough to set the price to a prohibitively high level in the subsequent round.

[14] By the $(\boldsymbol{IR})$ constraint, $t_i(\theta_i,\, \boldsymbol{\theta}_{-i}) = 0$ whenever $x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) = 0$.

pin down i's payoff following any deviation). I will prove that agent i maximizes her payoff by exiting in round t with $\theta \in [\kappa_t, \kappa_{t+1})$. Choosing to exit in a different round $\hat{t}$ yields a payoff that agent i would obtain by reporting a type $\hat{\theta} \in [\kappa_{\hat{t}}, \kappa_{\hat{t}+1})$ to the direct mechanism $(\boldsymbol{x}^{\kappa_1 \dots \kappa_k}, \boldsymbol{\pi}^{\kappa_1 \dots \kappa_k}, \boldsymbol{t}^{\kappa_1 \dots \kappa_k})$ when other agents report any type profile $\boldsymbol{\theta}_{-i}$ that leads to the sequence $\boldsymbol{N}_{-i}$. Indeed, I have already shown that the allocation and disclosure are the same in the GCA and the direct mechanism. Moreover, the payment that the agent makes conditional on exiting in round $\hat{t}$ and winning, given the sequence $\boldsymbol{N}_{-i}$, is equal to $P^{\hat{t}}(N_{-i}^{\hat{t}-1}+1)$. By definition, $P^{\hat{t}}(N_{-i}^{\hat{t}-1}+1) = \bar{t}_i^{\kappa_1 \dots \kappa_k}(\hat{\theta}, \boldsymbol{\theta}_{-i})$ for all $\boldsymbol{\theta}_{-i}$ consistent with the sequence $\boldsymbol{N}_{-i}$. Thus, when agent i wins following the deviation, she pays the same transfer as she would pay in the direct mechanism. Since reporting $\hat{\theta}$ in the direct mechanism is not strictly profitable given any report $\boldsymbol{\theta}_{-i}$ of other agents, exiting in round $\hat{t}$ cannot be strictly profitable given any sequence $\boldsymbol{N}_{-i}$.

OA.2.3 What if the loser also interacts in the aftermarket?

In this appendix, I develop the extension described in Section 7. In the single-agent model ($N = 1$) with a finite type space Θ, I assume that the agent participates in a winner's aftermarket when she acquires the object, and in the loser's aftermarket in the opposite case.

The aftermarket is now a pair (A_l, A_w) with

$$A_j \equiv \{u_j(\theta; \bar{f}) : \theta \in \Theta, \bar{f} \in \Delta(\Theta)\},$$

for $j \in \{l, w\}$, where subscript l denotes the aftermarket for a "loser" (when the agent does not acquire the good), and w – the aftermarket for a "winner". Monotonicity of the aftermarket now means that both functions $u_j(\theta; \bar{f})$, $j \in \{l, w\}$, are non-decreasing in the type θ for any fixed $\bar{f}$. A mechanism frame in the extended setting is (x, π_l, π_w), where π_l is the signal distribution conditional on not allocating the object, and π_w is the signal distribution conditional on allocating the object. Because there is only one agent, only one of the signal structures is used ex-post. It is assumed that third-party players in the aftermarket know whether the agent acquired the good or not.

Definition OA.1. A mechanism frame (x, π_l, π_w) is a *cutoff rule* if $x(\theta)$ is non-decreasing in θ, and there exist signal functions $\gamma_l : C \to \Delta(\mathcal{S})$ and $\gamma_w : C \to \Delta(\mathcal{S})$ such that

$$\pi_l(s|\,\theta)(1 - x(\theta)) = \sum_{c > \theta} \gamma_l(s|\,c)\Delta x(c), \tag{OA.2.4}$$

$$\pi_w(s|\,\theta)x(\theta) = \sum_{c \le \theta} \gamma_w(s|\,c)\Delta x(c), \tag{OA.2.5}$$

for all $\theta \in \Theta$ and $s \in \mathcal{S}$.

Cutoff mechanisms are defined accordingly. Condition (OA.2.5) is analogous to condition (3.1) in the definition of cutoff rules from Section 3. Condition (OA.2.4) is a mirror image of condition (OA.2.5), with the sum indexed by all cutoffs exceeding the type of the agent.

Instead of formulating a richness condition similar to the one defined in Section 5, I simplify the analysis by requiring the mechanism frames to be implementable for all distributions f and all monotone aftermarkets $(A_l,\, A_w)$.

Claim OA.1. *A mechanism frame $(x,\, \pi_l,\, \pi_w)$ is implementable for any distribution f and any monotone aftermarkets $(A_l,\, A_w)$ if and only if x is a constant allocation rule (in which case no information can be revealed).*

The conclusion continues to hold when only the loser interacts, and the winner enjoys the utility of holding the object: $u_w(\theta;\, \bar{f}) = \theta$, for all θ, and $\bar{f} \in \Delta(\Theta)$.

In the absence of any conditions on the aftermarket, no information about the agent's type can be elicited in a cutoff mechanism. This is not surprising given that I have imposed no relationship between the winner's and the loser's aftermarkets. Suppose that in the loser's aftermarket, the same object is allocated as in the first-stage mechanism but with values doubled for every type. In this case, higher types have a relative preference for *not* acquiring the object in the first stage, which reverts the direction of single-crossing. To avoid the negative result, it is necessary to assume that the game played when the object is acquired is in some sense preferred to the game played when the object is not allocated.

Definition OA.2 (Single-crossing)**.** The winner's aftermarket A_w is *single-crossing-separated* from the loser's aftermarket A_l if for any $\theta > \hat{\theta}$, there exists $d(\theta,\, \hat{\theta}) > 0$, such that for all $\bar{f}$,

$$u_w(\theta;\, \bar{f}) - u_w(\hat{\theta};\, \bar{f}) \geq d(\theta,\, \hat{\theta}) \geq u_l(\theta;\, \bar{f}) - u_l(\hat{\theta};\, \bar{f}).$$

Single-crossing separation requires that the difference in utilities between any two types in A_w can be separated from the difference in utilities between these two types in A_l, uniformly in posterior beliefs. For example, the condition is satisfied (with $d(\theta,\, \hat{\theta}) = \theta - \hat{\theta}$) when A_l corresponds to buying an identical object from a different seller, and there is no winner's aftermarket, i.e. $u_w(\theta;\, \bar{f}) = \theta$, for all θ.

Proposition OA.3. *A mechanism frame $(x,\, \pi_l,\, \pi_w)$ is implementable for any distribution f and any monotone aftermarket $(A_l,\, A_w)$ such that A_w is single-crossing-separated from A_l if and only if $(x,\, \pi_l,\, \pi_w)$ is a cutoff rule.*

Using Proposition OA.3 and arguments analogous to the ones used in the main text, one can show that *(1)* for a fixed allocation function x, the problem of finding an optimal cutoff mechanism is a Bayesian persuasion problem, and *(2)* there always exists an optimal cutoff mechanism that reveals no information.

Proof of Claim OA.1 and Proposition OA.3 The proof closely resembles other proofs in the paper, so I omit some details.

In the extended setting, a necessary and sufficient condition for implementability is that for all $\theta > \hat{\theta}$, distributions f, and monotone aftermarkets A_l and A_w,

$$\sum_{s\in\mathcal{S}} \Big[u_l(\theta;\, f_l^s) - u_l(\hat{\theta};\, f_l^s)\Big] \Big[\pi_l(s|\,\theta)(1-x(\theta)) - \pi_l(s|\,\hat{\theta})(1-x(\hat{\theta}))\Big]$$
$$+ \sum_{s\in\mathcal{S}} \Big[u_w(\theta;\, f_w^s) - u_w(\hat{\theta};\, f_w^s)\Big] \Big[\pi_w(s|\,\theta)x(\theta) - \pi_w(s|\,\hat{\theta})x(\hat{\theta})\Big] \geq 0. \quad \text{(OA.2.6)}$$

In equation (OA.2.6), f_l^s denotes the posterior belief over the type of the agent conditional on not acquiring the object and signal s being sent,

$$f_l^s(\theta) = \frac{\pi_l(s|\,\theta)(1-x(\theta))}{\sum_\tau \pi_l(s|\,\tau)(1-x(\tau))},\ \theta \in \Theta,$$

and f_w^s denotes the posterior belief conditional on the agent acquiring the object and signal s being sent,

$$f_w^s(\theta) = \frac{\pi_w(s|\,\theta)x(\theta)}{\sum_\tau \pi_w(s|\,\tau)x(\tau)},\ \theta \in \Theta.$$

Proof of Claim OA.1. I will show that if condition (OA.2.6) holds for all distributions and monotone aftermarkets then

$$\pi_l(s|\,\theta)(1-x(\theta)) \text{ is non-decreasing,} \quad \text{(OA.2.7)}$$

and

$$\pi_w(s|\,\theta)x(\theta) \text{ is non-decreasing,} \quad \text{(OA.2.8)}$$

for all $s \in \mathcal{S}$. These two conditions imply that x has to be constant (by summing up over $s \in \mathcal{S}$, we see that x must be both non-decreasing and non-increasing). When x is constant, the random cutoff representing it is deterministic, and hence no information can be revealed by a cutoff mechanism. The rest of the proof establishes (OA.2.7) and (OA.2.8).

First, set $u_w(\theta;\, \bar{f}) = \theta$, for all $\bar{f}$ and θ. For a fixed $\theta > \hat{\theta}$, let the aftermarket A_l be such that $u_l(\theta;\, f_l^s) = u_l(\hat{\theta};\, f_l^s)$ for all $s \in \mathcal{S}_1$, where

$$\mathcal{S}_1 \equiv \{s \in \mathcal{S} :\ \pi_l(s|\,\theta)(1-x(\theta)) \geq \pi_l(s|\,\hat{\theta})(1-x(\hat{\theta}))\},$$

and $u_l(\theta;\, f_l^s) - u_l(\hat{\theta};\, f_l^s) = \alpha(\theta - \hat{\theta})$, for all $s \notin \mathcal{S}_1$, for some $\alpha > 0$.[15] Then, equation (OA.2.6) becomes

$$(\theta - \hat{\theta}) \left\{ \alpha \sum_{s\notin\mathcal{S}_1} \Big[\pi_l(s|\,\theta)(1-x(\theta)) - \pi_l(s|\,\hat{\theta})(1-x(\hat{\theta}))\Big] + x(\theta) - x(\hat{\theta}) \right\} \geq 0. \quad \text{(OA.2.9)}$$

If there exists $s \notin \mathcal{S}_1$, then by taking a sufficiently high α, we obtain a contradiction in the above

[15] Because implementability is required for all distributions f, it is always possible to choose an f such that distinct signals lead to distinct posteriors, making the above construction well-defined. The only exception is when two signals are indistinguishable but in this case they can be merged into one signal.

inequality. Thus, $\mathcal{S}_1 = \mathcal{S}$, and by definition of $\mathcal{S}_1$, $\pi_l(s|\,\theta)(1-x(\theta)) \geq \pi_l(s|\,\hat{\theta})(1-x(\hat{\theta}))$ for all $s \in \mathcal{S}$. Because $\theta > \hat{\theta}$ were arbitrary, condition (OA.2.7) holds.

To show condition (OA.2.8), I set $u_l(\theta;\,\bar{f}) = 0$, for all θ and $\bar{f}$. Then, it is as if the loser did not participate in the aftermarket, and hence the conclusion follows from the results in Section 5.

The other direction in Claim OA.1 holds trivially: if x is constant, it is implementable for all distributions and aftermarkets.

Finally, the second part of Claim OA.1 holds because *(i)* the proof of (OA.2.7) used a winner's aftermarket that gave the winner the utility of holding the object, $u_w(\theta;\,\bar{f}) = \theta$, and *(ii)* with such an aftermarket, the allocation rule $x(\theta)$ must be non-decreasing in θ (this follows from considering the case when there is no loser's aftermarket).

Proof of Proposition OA.3. I will show that condition (OA.2.6) is equivalent to

$$\pi_l(s|\,\theta)(1-x(\theta)) \text{ is non-increasing,} \tag{OA.2.10}$$

and

$$\pi_w(s|\,\theta)x(\theta) \text{ is non-decreasing,} \tag{OA.2.11}$$

for all $s \in \mathcal{S}$. The above conditions are analogous to the monotonicity condition (M) in Proposition 1. By the same (or analogous in case of OA.2.10) argument that was used to prove Proposition 1, (OA.2.10) and (OA.2.11) imply the cutoff representation. Conversely, if $(x,\,\pi_l,\,\pi_w)$ is a cutoff rule, conditions (OA.2.10) and (OA.2.11) are satisfied, by direct inspection of Definition OA.1.

First, assume that conditions (OA.2.10) and (OA.2.11) hold. I will show that (OA.2.6) holds. Under the condition that A_w is single-crossing-separated from A_l, we have

$$\begin{aligned}
&\sum_{s\in\mathcal{S}}\Big[u_l(\theta;\,f_l^s) - u_l(\hat{\theta};\,f_l^s)\Big]\Big[\pi_l(s|\,\theta)(1-x(\theta)) - \pi_l(s|\,\hat{\theta})(1-x(\hat{\theta}))\Big] \\
&\quad+\sum_{s\in\mathcal{S}}\Big[u_w(\theta;\,f_w^s) - u_w(\hat{\theta};\,f_w^s)\Big]\Big[\pi_w(s|\,\theta)x(\theta) - \pi_w(s|\,\hat{\theta})x(\hat{\theta})\Big] \\
&\geq d(\theta,\,\hat{\theta})\sum_{s\in\mathcal{S}}\Big[\pi_l(s|\,\theta)(1-x(\theta)) - \pi_l(s|\,\hat{\theta})(1-x(\hat{\theta}))\Big] + d(\theta,\,\hat{\theta})\sum_{s\in\mathcal{S}}\Big[\pi_w(s|\,\theta)x(\theta) - \pi_w(s|\,\hat{\theta})x(\hat{\theta})\Big] \\
&= d(\theta,\,\hat{\theta})\Big[(1-x(\theta)) - (1-x(\hat{\theta}))\Big] + d(\theta,\,\hat{\theta})\Big[x(\theta) - x(\hat{\theta})\Big] = 0
\end{aligned}$$

For the converse part, assume condition (OA.2.6). To show condition (OA.2.10), fixing $\theta > \hat{\theta}$, take A_w such that $u_w(\theta;\,\bar{f}) - u_w(\hat{\theta};\,\bar{f}) = d(\theta,\,\hat{\theta})$. Then, consider an aftermarket A_l that gives differences in payoffs $u_l(\theta;\,f_l^s) - u_l(\hat{\theta};\,f_l^s) = d(\theta,\,\hat{\theta})$ for $s \notin \mathcal{S}_2$ and $u_l(\theta;\,f_l^s) - u_l(\hat{\theta};\,f_l^s) = 0$ otherwise, where

$$\mathcal{S}_2 \equiv \{s \in \mathcal{S} :\, \pi_l(s|\,\theta)(1-x(\theta)) > \pi_l(s|\,\hat{\theta})(1-x(\hat{\theta}))\}.$$

Then, condition (OA.2.6) implies

$$x(\theta) - x(\hat{\theta}) \geq \sum_{s \notin \mathcal{S}_2} \left[\pi_l(s|\,\hat{\theta})(1 - x(\hat{\theta})) - \pi_l(s|\,\theta)(1 - x(\theta)) \right].$$

The above condition can be rewritten as

$$x(\theta) - x(\hat{\theta}) \geq \sum_{s \notin \mathcal{S}_2} \pi_l(s|\,\hat{\theta})(1 - x(\hat{\theta})) - \sum_{s \notin \mathcal{S}_2} \pi_l(s|\,\theta)(1 - x(\theta))$$

$$= \left(1 - \sum_{s \in \mathcal{S}_2} \pi_l(s|\,\hat{\theta}) \right)(1 - x(\hat{\theta})) - \left(1 - \sum_{s \in \mathcal{S}_2} \pi_l(s|\,\theta) \right)(1 - x(\theta)),$$

which implies

$$0 \geq \sum_{s \in \mathcal{S}_2} \left[\pi_l(s|\,\theta)(1 - x(\theta)) - \pi_l(s|\,\hat{\theta})(1 - x(\hat{\theta}) \right].$$

The definition of $\mathcal{S}_2$ together with the above equation imply that $\mathcal{S}_2 = \emptyset$. Because $\theta > \hat{\theta}$ were arbitrary, condition (OA.2.10) is proven.

To show condition (OA.2.11), take $u_l(\theta;\, \bar{f}) = 0$, for all θ and $\bar{f}$, and specify the aftermarket A_w (fixing $\theta > \hat{\theta}$) by $u_w(\theta;\, f_w^s) - u_w(\hat{\theta};\, f_w^s) = d(\theta,\, \hat{\theta}) > 0$ if $\pi_w(s|\,\theta)x(\theta) \geq \pi_w(s|\,\hat{\theta})x(\hat{\theta})$, and $u_w(\theta;\, f_w^s) - u_w(\hat{\theta};\, f_w^s) = \alpha d(\theta,\, \hat{\theta})$, for some $\alpha > 1$, otherwise. Then, for sufficiently high α condition (OA.2.6) is violated unless $\pi_w(s|\,\theta)x(\theta) \geq \pi_w(s|\,\hat{\theta})x(\hat{\theta})$, for all $s \in \mathcal{S}$.

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