# Mechanism Design with Aftermarkets: Cutoff Mechanisms

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**Abstract**

I study a mechanism design problem in which a designer allocates a single good to one of several agents, and the mechanism is followed by an *aftermarket* – a post-mechanism game played between the agent who acquired the good and third-party market participants. The designer has preferences over final outcomes, but she cannot design the aftermarket. However, she can influence its information structure by disclosing information elicited from the agents by the mechanism.

I introduce a class of allocation and disclosure rules, called *cutoff rules*, that disclose information about the buyer's type only by revealing information about the realization of a random threshold (cutoff) that she had to outbid to win the object. A rule is implementable regardless of the form of the aftermarket and the prior distribution of types if and only if it is a cutoff rule. I characterize aftermarkets for which restricting attention to cutoff mechanisms is without loss of generality in a subclass of all feasible mechanisms. Optimization within the class of cutoff mechanisms is tractable; I provide sufficient conditions for optimality of simple designs.

**Keywords:** Mechanism Design, Information Design, Aftermarkets, Implementation, Auctions

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# 1 Introduction

"The game is always bigger than you think." This phrase succinctly captures a prevalent feature of practical mechanism design problems – they can rarely be fully understood without the wider market context. When a seller designs an auction, she should not ignore future resale or bargaining opportunities which might influence bidders' endogenous valuations for the object. A dealer in a financial over-the-counter market understands that a counterparty in a transaction may not be the final holder of the asset. Yet, most theoretical models analyze the design problem in a vacuum.

In this paper, I revisit the canonical mechanism design problem of allocating an object to one of several agents endowed with one-dimensional private information. Unlike in the standard model, the mechanism is followed by an *aftermarket*, defined as a post-mechanism game played between the agent who acquired the object and other market participants (*third parties*). The aftermarket is beyond the control of the mechanism designer but she may have preferences over equilibrium outcomes of the post-mechanism game, either directly (e.g., when the designer wants to maximize efficiency) or indirectly through the impact on agents' endogenous valuations (e.g., when the designer wants to maximize revenue).

Although the mechanism designer is unable to design the aftermarket, she can influence its information structure by publicly releasing some of the information elicited by the mechanism. The design problem is therefore augmented with an additional variable – the *disclosure rule*. For example, if a bidder who wins an object engages in bargaining over acquisition of complementary goods after the auction, a disclosure rule impacts the bargaining position of the bidder in the aftermarket. Formally, I model the aftermarket as a collection of payoffs (for the agents and the designer) that depend on the true type of the agent who acquired the object but also on the *beliefs* about that agent's type induced by the mechanism.

The resulting structure of the problem can be described as a combination of mechanism and information design. The mechanism elicits information from the agents to determine the allocation and transfers, and subsequently discloses some of that information to other market participants in order to induce the optimal distribution of posterior beliefs in the aftermarket. The two parts of the problem interact non-trivially because disclosure influences the incentives of agents to reveal their private information to the mechanism.

Suppose that the designer considers some allocation and disclosure rule, that is, a mapping from agents' types to a probability distribution over mechanism outcomes: which agent receives the good and what signal is sent. Together with the exogenous aftermarket, the rule determines the final outcome and payoffs. By the revelation principle, implementing that rule is possible only if there exist transfers such that the resulting direct mechanism provides incentives for agents both to participate and to report truthfully. These incentives in the mechanism depend on the agents' values from acquiring the object which are influenced by payoffs from the aftermarket. Those, in turn, depend on the aftermarket protocol and the beliefs of aftermarket participants. As a result, the set of implementable allocation and disclosure rules varies with the aftermarket and the prior distribution of agents' types – the optimal mechanism is sensitive to details of the environment and difficult to find.

Consider, however, the following class of allocation and disclosure rules called *cutoff rules*. In order to receive the object, the agent must report a type that is above some (possibly random) threshold which I refer to as the *cutoff*. Depending on the allocation rule, the cutoff could be, for example, a report (bid) of another agent or a reserve price. I show that such a cutoff representation exists for any monotone allocation rule. Cutoff rules are then defined by a joint restriction on allocation and signals: The allocation rule is monotone, and the signal distribution should only depend on the realized cutoff of the winner. Formally, conditional on the cutoff, the signal from a cutoff mechanism does not depend on the type of the agent who acquires the good. For example, if the object is allocated to the highest bidder in an auction, the cutoff is the second highest bid; Conditional on the second highest bid, the message sent after the auction should not depend on the winner's type. Thus, a second-price auction with (full or partial) disclosure of the price paid by the winner implements a cutoff rule but a first-price auction with disclosure of the price does not.

The key property of cutoff mechanisms (mechanisms that implement cutoff rules) is that the report of the winner does not directly influence the signal. The signal is pinned down by the realization of the cutoff which is determined independently of the winner's report. Because the winner cannot manipulate the signal, cutoff mechanisms admit a truthful equilibrium regardless of the details of the environment. Formally, as long as a single-crossing condition holds (fixing the posterior belief in the aftermarket, any agent's payoff from winning the object is non-decreasing in her type), irrespective of the aftermarket protocol and the prior distribution of types, any cutoff rule can be implemented by some transfer scheme.

The paper focuses on the analysis of cutoff mechanisms and their properties in two steps. First, I develop methods for finding the optimal mechanism within the cutoff class and provide conditions for optimality of simple designs, assuming a general aftermarket. Second, I show that cutoff mechanisms are uniquely characterized by some properties that may be desirable in practical design problems, under appropriate restrictions on the aftermarket interactions. By introducing cutoff mechanisms and drawing a connection to information design, the paper contributes to the mechanism design literature by showing a way in which the economic effects of post-mechanism interactions can be analyzed in a tractable way – with the optimal cutoff mechanism often found in closed form. In practical applications that are well approximated by the assumptions of the characterization results, the paper offers insights about the optimal design of transparency in allocation mechanisms.

In the first part, I analyze *optimal cutoff mechanisms*, where optimality means that the mechanism maximizes some fixed objective function of the designer, such as revenue or total surplus. In a general mechanism, disclosure of information interacts with the incentive-compatibility constraints. However, disclosure of information in a cutoff rule does not: Regardless of what information about the cutoff is revealed, the agents want to report truthfully under appropriately chosen transfers. Therefore, finding the optimal disclosure rule in the cutoff class reduces to a standard information design problem where the cutoff plays the role of a state variable. Choosing the allocation rule corresponds to choosing a prior distribution of the state variable. In this way, the design problem can be decomposed into two independent steps, where each step can be solved using existing mechanism and information design techniques, respectively.

When the designer contracts with a single agent, a cutoff corresponds to a random reserve price. If the allocation rule is fixed, it may benefit the designer to disclose information about the cutoff. However, if the allocation and disclosure rule are chosen jointly, a strong conclusion holds: For any designer's objective function that depends on the final outcome, and regardless of the aftermarket protocol, there always exists an optimal cutoff mechanism that sends no signals. Intuitively, in single-agent problems, the designer has full discretion over the choice of the prior distribution of the cutoff – *any* distribution of the cutoff can be induced by choosing an appropriate non-decreasing allocation rule. Because the designer can directly *choose* the prior belief over the state variable (the cutoff), she need not send signals to induce optimal posterior beliefs.[1]

With more than one agent, it may be strictly optimal to disclose information also when the designer chooses both the allocation and the disclosure rule. This is because the designer might want to give up full control over the distribution of the cutoff to use competition between the agents. For example, if the designer decides to run an efficient auction, the distribution of the cutoff – the second highest bid – is exogenous and cannot be chosen. More generally, when the allocation depends on the ranking of agents' types, the designer is constrained in the choice of prior distributions of the cutoff. As a result, it may be beneficial to send signals to induce posterior beliefs that differ from the prior. I provide sufficient conditions for optimality of simple mechanisms, such as a second price auction with a reserve price that reveals the price paid by the winner.

[1] It may seem that when choosing the allocation rule the designer faces a trade-off between the consequences for *(i)* the allocation of the good and *(ii)* the posterior beliefs in the aftermarket. However, this trade-off can be incorporated into a single payoff function that maps posterior beliefs over the cutoff to overall payoffs to the designer (including the payoff from allocating the good).

The class of cutoff mechanisms often excludes the fully optimal mechanism. Thus, the second part of the paper attempts to provide a partial justification for restricting attention to cutoff mechanisms. I offer two characterizations of the class, with complementary roles. In the first characterization, I show that cutoff rules are uniquely pinned down by the property of being always implementable (regardless of the aftermarket and the prior distribution of types). This has a number of implications. First, cutoff rules provide a natural lower bound on the designer's objective in *any* design problem (and no mechanism outside of the class can serve as such a universal benchmark). Second, the property may be useful in settings where the designer has limited knowledge of the details of the environment. While cutoff rules are not fully robust because their transfers may depend on these details, implementability for all priors and aftermarkets is a *necessary* condition for robust implementation (I discuss this in detail in Section 5.1).

The second characterization result provides conditions on the aftermarket under which restricting attention to cutoff mechanisms is without loss of generality (and hence optimality) within a subclass of all feasible mechanisms. To define the subclass on which the characterization result holds, I first strengthen the notion of implementability to what I call *strong* dominant-strategy implementation: I require that all agents report truthfully regardless of what beliefs they hold about other agents' strategies *and* about the outcome of randomization devices used by the designer. In other words, the mechanism should remain incentive-compatible even if the designer could bias the coin toss that implements random outcomes. Because such a deviation of the designer cannot be detected by observing the final outcome of the mechanism, this form of robustness may be desirable in settings where agents do not fully trust the designer.[2] Cutoff rules can always be implemented in this stronger sense. The main result shows that among mechanisms that are strongly dominant-strategy implementable and satisfy a regularity condition, only cutoff mechanisms are feasible when the aftermarket is *submodular*. Informally, an aftermarket is submodular if lower types benefit more than high types (in relative terms) from a change in posterior beliefs that shifts more probability mass toward higher types.[3] Resale markets are submodular: Every agent benefits from inducing posterior beliefs that lead to a high resale price but low types have a higher willingness to pay for signals that improve beliefs – because they benefit more from resale. Submodular aftermarkets make information disclosure in the mechanism *difficult*: The direction of single-crossing between types and beliefs (treated as "goods" that are allocated to

[2] For a closely related point, see Dequiedt and Martimort (2015) and Akbarpour and Li (2018) – I explain the connection to these papers in Section 5.2.

[3] This is formalized using the monotone likelihood ratio order on beliefs: An aftermarket is submodular if the agent's aftermarket payoff function is submodular in her type and the posterior belief about her type. The regularity condition requires that posterior beliefs induced by the mechanism are ranked in that order.

agents) is opposite to the relationship between types and beliefs dictated by Bayes updating – limiting the informativeness of signals revealed by an incentive-compatible mechanism. By the previous characterization, information about the cutoff can always be revealed. Thus, the result says that disclosing information about the cutoff is the *only* feasible choice within the subclass in such informationally-challenging environments.

*Supermodular* aftermarkets are defined by reversing the direction of single-crossing between types and beliefs: High types benefit relatively more (than low types) from a change in posterior beliefs that shifts more probability mass toward higher types. This property allows the designer to disclose more information in the mechanism. Thus, while cutoff mechanisms retain their desirable properties in supermodular aftermarket, they are with loss of generality in this case (and likely with loss of optimality if the designer has preferences for disclosure).

The above analysis is complemented by a result in a companion paper Dworczak (2019) that studies a similar design problem in a more restricted setting (there is a single third party in the aftermarket that takes a binary action). There, I show that a version of submodularity of the aftermarket implies optimality of cutoff mechanisms in the class of all *Bayesian* incentive-compatible mechanisms.

The remainder of the paper is organized as follows. The next subsection discusses related literature. Section 2 introduces the baseline model. Section 3 defines cutoff mechanisms and Section 4 deals with the derivation of the optimal cutoff mechanism. In Section 5, I prove two characterization theorems for the cutoff class. Section 6 introduces the extension to continuous type spaces with some additional results on optimality of simple cutoff rules. In Section 7, I discuss other extensions, and Section 8 concludes.

## 1.1 Literature review

This paper combines mechanism design with information design. In a seminal paper, Myerson (1981) solves the problem of allocating a single asset in a mechanism design framework, where the designer is allowed to choose an arbitrary mechanism. In contrast, as surveyed by Bergemann and Morris (2016b), information design takes the mechanism (or game) as given and considers optimization over information structures. In my model, the principal designs the mechanism and the information structure jointly. My analysis makes use of the concavification argument first used by Aumann and Maschler (1995), and applied to the Bayesian persuasion model by Kamenica and Gentzkow (2011). A methodological contribution of the paper is to find a connection between the mechanism design problem and the concavification result via the introduction of cutoffs.[4]

[4] Kolotilin, Mylovanov, Zapechelnyuk and Li (2017) combine mechanism design with Bayesian persuasion in a different context by studying a model in which the agent reports private information to the designer

With regard to the structure of the problem, a closely related literature is a series of papers by Calzolari and Pavan (2006a,b, 2008, 2009) on sequential agency. In a sequential agency problem, the agent contracts with multiple principals, and an upstream principal decides how much information to reveal to downstream principals (which play a role analogous to the third parties in my aftermarket). The upstream principal designs the allocation and the disclosure rule jointly. Calzolari and Pavan (2006b) show in a two-stage sequential agency model with one agent that, under certain conditions, it is optimal to reveal no information in the upstream mechanism. This conclusion is similar to my result about optimality of no-revelation in single-agent problems. However, the results are not related otherwise: Calzolari and Pavan do not restrict attention to cutoff mechanisms; I do not impose any of the three economic assumptions of the main theorem of Calzolari and Pavan. For example, the upstream principal in Calzolari and Pavan has no direct preferences over the outcome of the second stage – I focus on exactly opposite cases when the principal cares about the final allocation (e.g. because she maximizes total surplus). Calzolari and Pavan (2006a) consider a model of a revenue-maximizing monopolist selling an object to an agent who can later resell to a third party. They study a simple setting with binary types which allows them to derive a closed-form solution. My model is more general in that it allows an arbitrary objective function, multiple agents, general second-stage game, and general type spaces. The approach and results of Calzolari and Pavan and mine are complementary: The paper by Calzolari and Pavan provides an example of a problem where using a cutoff mechanism is suboptimal under Bayesian implementability,[5] while my paper points out that implementability of their optimal non-cutoff mechanism relies on detailed knowledge of the setting and the ability of the designer to credibly commit to randomization in the mechanism.

A number of papers analyze the consequences of post-auction interactions between the bidders and third parties. Zhong (2002), Goeree (2003), Das Varma (2003), Katzman and Rhodes-Kropf (2008), and Hu and Zhang (2017) examine the effect of different bid announcement policies on revenue in standard auctions followed by Bertrand, Cournot, or other forms of competition. Giovannoni and Makris (2014) model the aftermarket through reduced-form reputational concerns (an additive component of the objective function depends on posterior beliefs). Back, Liu and Teguia (2018) study the effects of transparency on welfare and dealers' profits in financial over-the-counter markets. In all of these papers, enough assumptions are imposed on the aftermarket payoffs to guarantee existence of a revealing (monotone) equilibrium in the first stage, even when agents' reports (bids) are fully disclosed.[6] Roughly,

who then communicates her private information to the agent.

[5] Out of four mechanisms that can be optimal in the baseline model of Calzolari and Pavan (2006a) (depending on parameters), three are cutoff mechanisms, and one is not.

[6] With the exception of Hu and Zhang, 2017, these papers compare a small number of fixed auction

this assumption requires that higher types of agents have a higher willingness to pay for more favorable beliefs. In the terminology introduced by this paper, this is a feature of *supermodular* aftermarkets – such aftermarkets make information disclosure "easy". In contrast, the focus of this paper is on *submodular* aftermarkets (such as resale aftermarkets) that make information disclosure "difficult". The precise meaning of these statements is explained in the paper. Engelbrecht-Wiggans and Kahn (1991) and Dworczak (2015) explicitly construct non-monotone equilibria using a discrete type space in auctions followed by resale games (an example of a submodular aftermarket).

Overall, previous literature made progress on studying the consequences of aftermarket interactions with third parties in two cases: When the aftermarket has a special structure (such as supermodularity) under which full disclosure (and hence any intermediate disclosure) is feasible; or in the opposite case but under restrictive conditions on the type space, objective function, and the aftermarket interaction. By introducing cutoff mechanisms, this paper allows a tractable analysis of general aftermarkets, and moreover shows that the restriction to cutoff mechanisms has a justification precisely in the cases where progress has been hindered by lack of tractability (namely, with submodular aftermarkets).

A closely related problem is when bidders interact *with each other* after the mechanism. In general, such problems are significantly more complicated and yield different economic insights – this is primarily because agents in the first-stage mechanism consider not only the signaling effect of their behavior, but also how much they learn about others. A special case of such problems is auction design with inter-bidder resale (e.g. Gupta and Lebrun, 1999, Zheng, 2002, Haile, 2003, Hafalir and Krishna, 2008, Hafalir and Krishna, 2009, Zhang and Wang, 2013)[7]. In this literature, to circumvent the difficulty mentioned above, the disclosure rule is either *(i)* made redundant by assuming an information structure in the resale stage (e.g. types are revealed, as in Gupta and Lebrun, 1999)[8], *(ii)* fixed for the purpose of the analysis (as in Haile, 2003 who assumes that all bids are revealed), or *(iii)* only relevant to the extent that it permits implementing the optimal allocation in an equilibrium of the auction (as in Zheng, 2002, where the optimal allocation and payoff are known ex-ante, and no revelation rule can increase the payoff of the mechanism designer). In contrast, the disclosure rule plays an active role in my model, and in particular interacts non-trivially with the optimal allocation rule. Carroll and Segal (2018) consider a model where the auctioneer does not know the resale protocol and maximizes revenue in the worst case (the designer in

formats (e.g. first-price, second-price) and announcement rules (e.g. full revelation of bids, revelation of the winning bid).

[7] Calzolari and Pavan (2006a) consider an extension of their model to inter-bidder resale but they do not derive the optimal mechanism in that framework.

[8] In Zhang and Wang (2013), one of the two potential buyers in the mechanism has a known value.

my model maximizes a Bayesian objective function).[9]

Balzer and Schneider (2017) analyze a model in which two players try to resolve a conflict which (if unresolved) leads to an escalation game between the two sides. Because the behavior in the conflict management mechanism is informative of the payoff-relevant types of the players, a designer can influence payoffs in the escalation game by disclosing information in the mechanism. While the two problems are related on a high level, the models, and hence the techniques used to analyze them, are different.

The paper considers information disclosure *after* the auction, where outsiders learn about bidders' values. This complements a large literature on information disclosure *before* and *during* the auction, where information is controlled by the seller and refines bidders' estimates of their own values, as in Milgrom and Weber (1982), Eső and Szentes (2007), Bergemann and Wambach (2015), Li and Shi (2017), Smolin (2019) among many others. In these papers, there is no aftermarket. Lauermann and Virág (2012) consider a model where losing bidders exercise a common outside option after the auction, and the auctioneer can disclose information about the value of the outside option either before or after the auction.

The presence of aftermarkets has been cited as an important motivation for studying mechanisms with allocative and informational externalities, for example in Jehiel, Moldovanu and Stacchetti (1996) and Jehiel and Moldovanu (2001, 2006). The importance of the aftermarket and the disclosure rule for the behavior in the auction has been demonstrated experimentally by Fonseca, Giovannoni and Makris (2017).

## 2 Baseline model

A mechanism designer owns an indivisible good that she can allocate to one of $N$ agents. The designer chooses an allocation mechanism that specifies the probabilities with which agents receive the object, monetary transfers, and a signal distribution, as a function of agents' messages sent to the mechanism. The signal is publicly revealed after the mechanism. The agent who acquires the object in the mechanism (the "winner") participates in a post-mechanism game with third-party players. The mechanism designer cannot directly influence the post-mechanism game, and cannot contract with the third-party players. However, the signal revealed by the mechanism may be used to influence the equilibrium of the post-mechanism game by changing the beliefs over the winner's type. I describe the formal model next, deferring the discussion of the assumptions to the next subsection.

Let $\mathcal{N}$ denote the set of agents. Agent $i \in \mathcal{N}$ has a type $\theta_i \in \Theta_i$, where $\Theta_i$ is a finite subset of $\mathbb{R}_+$. Types are distributed according to a prior joint distribution with probability mass

[9] The designer in Carroll and Segal (2018) does not control the information leakage to the aftermarket.

function $\boldsymbol{f}$ on $\boldsymbol{\Theta} \equiv \times_{i\in\mathcal{N}}\Theta_i$ with independent marginals $f_i$ and cdf $F_i$. The prior distribution is common knowledge among all players. Throughout, bold symbols are used to denote vectors and products, in particular $\boldsymbol{\theta} \equiv (\theta_1, \theta_2, ..., \theta_N)$, $\boldsymbol{\theta}_{-i} \equiv (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_N)$, and $\boldsymbol{f}(\boldsymbol{\theta}) = \prod_{i\in\mathcal{N}} f_i(\theta_i)$, and a tilde is used to differentiate between random variables and their realizations, e.g., $\tilde{\theta}_i$ denotes a random variable that yields a realization denoted $\theta_i$.

Assuming that the mechanism designer has commitment power and is satisfied with partial implementation, the Revelation Principle will apply; thus, I restrict attention to direct mechanisms. I assume that the mechanism can send an arbitrary public signal once the good is allocated; thus, a direct mechanism is a tuple $(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{t})$, where $\boldsymbol{x} : \boldsymbol{\Theta} \to [0, 1]^N$ is an allocation rule with $\sum_{i\in\mathcal{N}} x_i(\boldsymbol{\theta}) \le 1$, for all $\boldsymbol{\theta}$; $\boldsymbol{\pi} : \boldsymbol{\Theta} \to \times_{i\in\mathcal{N}}\Delta(\mathcal{S}_i)$ is a signal function with a finite signal space $\mathcal{S}_i$ for each agent $i$, and $\boldsymbol{t} : \boldsymbol{\Theta} \to \mathbb{R}^N$ is a transfer function.[10] If agent $i$ reports $\hat{\theta}_i$, and other agents report truthfully, she receives the good with probability $x_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$ and pays $t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$. Conditional on allocating the good to agent $i$, the designer draws and publicly announces a signal $s \in \mathcal{S}_i$ according to distribution $\pi_i(\cdot|\, \hat{\theta}_i, \boldsymbol{\theta}_{-i})$. No other signals are sent but I assume that the identity of the winner is observed. Overall, the posterior belief over the winner's type $\tilde{\theta}_i$ induced by a mechanism $(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{t})$ conditional on signal realization $s$ (assuming truthful reporting) is given by (whenever defined)

$$f_i^s(\tau) = \frac{\sum_{\boldsymbol{\theta}_{-i}} \pi_i(s|\tau, \boldsymbol{\theta}_{-i})x_i(\tau, \boldsymbol{\theta}_{-i})\boldsymbol{f}(\tau, \boldsymbol{\theta}_{-i})}{\sum_{\boldsymbol{\theta}} \pi_i(s|\boldsymbol{\theta})x_i(\boldsymbol{\theta})\boldsymbol{f}(\boldsymbol{\theta})}, \forall \tau \in \Theta_i. \tag{2.1}$$

I do not explicitly model the third-party players in the aftermarket. Instead, the post-mechanism game is described in reduced form by the conditional expected payoffs it generates for the winner given the information revealed by the mechanism. Formally, an aftermarket $A$ is a collection of payoff functions $A \equiv \{u_i(\theta; \bar{f}) : \theta \in \Theta_i, \bar{f} \in \Delta(\Theta_i), i \in \mathcal{N}\}$, where $u_i(\theta; \bar{f})$ denotes the conditional expected payoff to agent $i$ with type $\theta \in \Theta_i$, when the posterior belief over the type $\tilde{\theta}_i$ is $\bar{f}$, conditional on agent $i$ holding the good. Importantly, the aftermarket is a primitive of the model in that its definition is independent of the mechanism chosen by the designer.

In the truthful equilibrium of the direct mechanism $(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{t})$, the expected payoff to agent $i$ with type $\theta_i$ who deviates to reporting $\hat{\theta}_i$ to the mechanism conditional on other agents reporting $\boldsymbol{\theta}_{-i}$ is $\sum_{s\in\mathcal{S}_i} u_i(\theta_i; f_i^s)\pi_i(s|\, \hat{\theta}_i, \boldsymbol{\theta}_{-i})x_i(s|\, \hat{\theta}_i, \boldsymbol{\theta}_{-i}) - t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$. The objective of the mechanism designer is to maximize

$$\sum_{i\in\mathcal{N}} \sum_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \sum_{s\in\mathcal{S}_i} V_i(\theta_i; f_i^s)\pi_i(s|\, \theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i})\boldsymbol{f}(\boldsymbol{\theta}), \tag{2.2}$$

[10] Because of finite type spaces, it is without loss of generality to assume that signal spaces $\mathcal{S}_i$ are finite.

where each $V_i : \Theta_i \times \Delta(\Theta_i) \to \mathbb{R}$ is assumed to be upper-semi continuous in the second argument. Thus, the payoff of the mechanism designer is normalized to zero when the good is not allocated, and is equal to $V_i(\theta_i;\, f_i^s)$ otherwise, where $V_i(\theta_i;\, f_i^s)$ is the payoff conditional on agent $i$ winning the object and belief $f_i^s$ being induced in the aftermarket.

### Comments on assumptions

The baseline model makes a number of strong assumptions on the environment in order to obtain all results in a single framework. However, many of these assumptions are not needed for a subset of results (see Section 7 for a discussion). In particular, three assumptions: *(i)* independence of types, *(ii)* the fact that only beliefs over the winner's type matter for payoffs, and *(iii)* restriction to public signals, are used in results on optimality but are not essential for results on implementation. Without assumption *(i)*, it is well known that optimal mechanisms may have unintuitive properties (e.g. full surplus extraction is possible if the designer uses a Crémer-McLean mechanism, see Crémer and McLean, 1988). Without assumptions *(ii)* and *(iii)*, the problem faced by the mechanism designer involves optimal disclosure of multi-dimensional information to multiple receivers which in itself is an intractable problem except for special cases. The restriction to public signals is realistic for many applications, while assumption *(ii)* is natural in private-value environments.

I assumed a discrete type space to simplify exposition – all subsequent results can be extended to continuous type spaces (see Section 6 for details).

The model allows for a general objecive function of the mechanism designer. The fact that the designer's payoff does not explicitly depend on transfers is essentially without loss of generality given that feasible mechanisms are required to be incentive-compatible and individually-rational: While there may be many transfer rules implementing any given allocation and disclosure rule, the set of implementing transfer rules is a complete sublattice with a largest element.[11] For example, revenue-maximizing expected transfers are uniquely pinned down by the allocation and disclosure rule, and thus formulation (2.2) allows for revenue maximization as the objective of the designer.

The assumption that only the winner interacts after the mechanism is both restrictive and crucial for all results. Some generalizations are possible and will be discussed in Section 7.

Many of the above assumptions become automatically satisfied when there is a single agent in the mechanism ($N = 1$) – this case will receive special attention in the paper.

[11] See, for example, Kos and Messner (2013). Dworczak and Zhang (2017) show that this result also follows from Shapley and Shubik (1971). With a continuous type space, expected transfers would be pinned down up to a constant, by the payoff equivalence theorem (see e.g. Milgrom, 2004).

## 2.1 The Aftermarket

I model the aftermarket as a "black-box" without explicitly defining the underlying post-mechanism game. This approach implicitly entails the following assumptions. A Bayesian game is played after the mechanism between agent $i$ who acquired the good (whose identity becomes known) and third-party players. Third-party players share the common prior $\boldsymbol{f}$ over the agents' types, observe the identity $i$ of the winner and the public signal $s$ sent by the mechanism. This leads to a posterior belief $f_i^s$ over the winner's type. Given belief $f_i^s$ and an aftermarket $A$, the corresponding game has a set of equilibria $EQ_i^A(f_i^s)$, where $EQ_i^A(\cdot)$ is an upper hemi-continuous correspondence mapping beliefs over the winner's type into equilibrium outcomes, where the equilibrium notion can be specified by the modeler. Then, fixing an equilibrium selection from $EQ_i^A$, $u_i(\theta; f_i^s)$ is the expected equilibrium payoff to type $\theta$ of agent $i$ conditional on $s$. A standard assumption in the mechanism design literature in such contexts is that the designer's preferred equilibrium is selected. However, the theory works for any selection, as long as it generates payoff functions $V_i(\theta; \bar{f})$ that are upper semi-continuous in $\bar{f}$. The above formulation allows for exogenous private information of third-party players.[12]

By assumption, the signal $s$ sent by the mechanism influences the aftermarket only through the posterior belief $f_i^s$ over the winner's type. Other roles of the signal (for example, as a coordination device) can be incorporated by considering an appropriate equilibrium concept (e.g. a version of correlated equilibrium, see Bergemann and Morris, 2016a). Consequently, I will not distinguish between two mechanisms that induce the same distribution of posterior beliefs for any prior.

The type $\theta_i$ is not necessarily the value of the object to agent $i$, and nothing so far implies that higher types have a higher willingness to pay for winning. Obtaining such a single-crossing property requires an assumption on aftermarket payoffs.

**Assumption 1** (Monotonicity)**.** An aftermarket $A$ is *monotone*, if for any agent $i \in \mathcal{N}$, and any belief $\bar{f} \in \Delta(\Theta_i)$, the aftermarket payoff $u_i(\theta; \bar{f})$ is non-decreasing in $\theta$.

If there is no aftermarket and the type is equal to the value, $u_i(\theta; \bar{f}) = \theta$, the assumption is trivially satisfied. With an aftermarket, the assumption says that types can be ranked by willingness to pay for the object *irrespective* of the posterior beliefs in the aftermarket. This is true in most applications where the type is interpreted as a value of the object to

[12] The payoff $u_i$ remains measurable with respect to the public belief $f_i^s$ because we can assume without loss that third-party players observe their exogenous signals after observing the public signal $s$ – the realizations of private signals can thus be integrated out in the payoff function $u_i$.

the agent.[13] An analogous assumption is made in all papers studying aftermarkets that are surveyed in Section 1.1.

I conclude with two examples of aftermarkets that will be used for illustration throughout.

**Example 1** (Resale)**.** [14] Suppose that with probability $\lambda > 0$ there is a single third party buyer in the aftermarket with some (potentially random) value $\tilde{v}$ for the object. (With the remaining probability, there is no aftermarket and the agent keeps the good obtaining her value $\theta$.) The third party bargains with the winner to repurchase the object. If the equilibrium price is equal to $p(\bar{f};\, v)$ when the belief over the winner's type is $\bar{f}$ and $\tilde{v} = v$, then we have

$$u_i(\theta;\, \bar{f}) = \lambda \mathbb{E}[\max\{\theta,\, p(\bar{f};\, \tilde{v})\}] + (1-\lambda)\theta$$

which is monotone in $\theta$. If the third party has full bargaining power, and the agent-preferred equilibrium is selected, then

$$p(\bar{f};\, v) = \max\left\{\operatorname{argmax}_p (v-p)\sum_{\theta \le p} \bar{f}(\theta)\right\}. \quad \blacksquare$$

**Example 2** (Ex-post binary types)**.** Unlike the previous example, this example is a class of simple aftermarkets that capture different economic applications in a tractable manner. Suppose that $\Theta_i \subset [0,\, 1]$, and interpret $\theta_i$ as the probability with which agent $i$ has an ex-post high type $h > 0$. With complementary probability $1-\theta_i$, the agent has a low type $l \in (0,\, h)$. The agent learns the ex-post type only after acquiring the object (but before the aftermarket game). If the payoffs of all players in the aftermarket only depend on the winner's ex-post type, the utility of the winner $i$ depends on the belief $\bar{f}$ over her ex-ante type $\tilde{\theta}_i$ only through its expectation $m(\bar{f}) \equiv \mathbb{E}_{\tilde{\theta}_i \sim \bar{f}}[\tilde{\theta}_i]$. Thus, denoting agent $i$'s aftermarket payoff by $\underline{u}_i(m)$ and $\bar{u}_i(m)$ when her ex-post type is high and low, respectively, we have

$$u_i(\theta;\, \bar{f}) = \theta \bar{u}_i(m(\bar{f})) + (1-\theta)\underline{u}_i(m(\bar{f})).$$

The aftermarket is monotone if $\bar{u}_i(m) \ge \underline{u}_i(m)$ for all posterior means $m$.

**(a)** [Resale] Consider the following resale aftermarket: An agent's ex-post type is her value for the good, the third party has full bargaining power and a value $v > h$, and the

[13] An example of an aftermarket violating monotonicity can be constructed, for example, by allowing a third party to observe exogenous signals (other than the ones sent by the mechanism) that depend on the agent's type. In a resale aftermarket, if the third party obtains a precise exogenous signal about high types but an imprecise exogenous signal about low and medium types, then medium types may enjoy a higher payoff in the aftermarket despite having a lower value for holding the good.

[14] This example generalizes the baseline model of Calzolari and Pavan (2006a).

aftermarket happens with probability $\lambda > 0$. Then, $\bar{u}_i(m) = h$, and

$$\underline{u}_i(m) = \begin{cases} \lambda h + (1-\lambda)l & (v-h) \geq (1-m)(v-l) \\ l & \text{otherwise.} \end{cases}.$$

**(b)** [Cournot competition][15] Suppose that the mechanism allocates a patent that allows an entrant to enter a market with an incumbent (the third party). Upon acquiring the patent, the winner learns her marginal cost of production which is $c < 1$ for the low type $l$ and $c - \Delta$ for the high type $h$, where $\Delta > 0$. The incumbent has cost $c$. Market demand is given by $Q(P) = 1 - P$, and the two firms compete *a la* Cournot. The equilibrium payoff for the agent in the aftermarket can be shown to be $\underline{u}_i(m) = \frac{1}{9}\left(1 - c + \frac{m\Delta}{2}\right)^2$ and $\bar{u}_i(m) = \frac{1}{9}\left(1 - c + \frac{m\Delta}{2} + \frac{3\Delta}{2}\right)^2$. The aftermarket is monotone.

**(c)** [Investment game] Consider again an aftermarket where an entrant interacts with an incumbent.[16] The type $\theta_i$ of the entrant is the probability that her business model succeeds, in which case a value $v = 1$ is generated (otherwise, the entrant gets a zero payoff). Before observing whether the entrant succeeds, the incumbent takes a costly investment $y$ that allows her to capture a fraction $\alpha(y)$ of the entrant's value in case the entrant is successful (and is a sunk cost otherwise): $y^\star(\bar{f}) \in \text{argmax}_y \mathbb{E}_{\bar{f}}[\tilde{\theta}_i \alpha(y) - y]$. Assume that $\alpha : \mathbb{R}_+ \to [0, 1]$ is strictly increasing and concave, differentiable, $\alpha'(0) = \infty$, and $\alpha^{-1}(1) \geq 1$ to guarantee an interior solution pinned down by the first-order condition. Then, the entrant's aftermarket payoff is given by $\bar{u}_i(m) = 1 - \alpha\left((\alpha')^{-1}(1/m)\right)$ and $\underline{u}_i(m) = 0$. ■

## 2.2 Implementability

I now formally define what it means for an outcome to be implementable. I will refer to $(\boldsymbol{x}, \boldsymbol{\pi})$, the allocation and disclosure rule, as the *mechanism frame*. The mechanism frame is the relevant first-stage outcome that pins down the designer's payoff.

**Definition 1.** A mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ is *dominant-strategy (DS) implementable* if there exist transfers $\boldsymbol{t}$ such that agents participate and report truthfully in the first-stage mechanism, taking into account the continuation payoff from the aftermarket:

$$\sum_{s \in \mathcal{S}_i} u_i(\theta_i;\, f_i^s)\pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) - t_i(\theta_i,\, \boldsymbol{\theta}_{-i}) \geq 0, \qquad (\boldsymbol{IR})$$

[15] This application was considered by Goeree (2003), Katzman and Rhodes-Kropf (2008), and Hu and Zhang (2017).

[16] The first stage could be any mechanisms that equips the entrant with something necessary to run her business, e.g., a license, patent, or funding.

$$\theta_i \in \underset{\hat{\theta}_i \in \Theta_i}{\operatorname{argmax}} \sum_{s \in \mathcal{S}_i} u_i(\theta_i;\, f_i^s)\pi_i(s|\,\hat{\theta}_i,\, \boldsymbol{\theta}_{-i})x_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) - t_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}), \qquad (\boldsymbol{IC})$$

for all $i \in \mathcal{N}$, $\theta_i \in \Theta_i$, and $\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$.

To appreciate the difficulty associated with adding the aftermarket, recall first that when there is no aftermarket, that is, $u_i(\theta_i;\, f_i^s) \equiv \theta_i$, then $(\boldsymbol{x},\, \boldsymbol{\pi})$ is DS-implementable if and only if $x_i(\theta,\, \boldsymbol{\theta}_{-i})$ is non-decreasing in $\theta$ for any $\boldsymbol{\theta}_{-i}$, that is, an *ex-post monotonicity* condition holds (of course, in this case the signal function $\boldsymbol{\pi}$ is irrelevant). In particular, the characterization of implementable outcomes is invariant to the details of the environment such as the distribution of types.

With the aftermarket, this is no longer the case. The existence of an aftermarket imposes restrictions on the set of implementable disclosure rules $\boldsymbol{\pi}$. For instance, in the resale aftermarket (see Example 1), it is generally impossible to implement full disclosure of the winner's type along with the efficient allocation rule.[17] More generally, the set of implementable mechanism frames is sensitive to changes in the prior distribution of types and the aftermarket. In the resale example, how much information can be disclosed about the winner's type depends on details such as the probability $\lambda$ that the aftermarket takes place, the value $\tilde{v}$ of the third party, and the bargaining protocol in the aftermarket.

As a consequence of the sensitivity of implementable outcomes to the details of the environment, the problem of finding the optimal mechanism is intractable in the absence of further restrictions on the prior distribution (as in Calzolari and Pavan, 2006a, who impose binary types) or the aftermarket (as in Goeree, 2003, Katzman and Rhodes-Kropf, 2008 or Hu and Zhang, 2017, who study a small set of aftermarkets that permit arbitrary disclosure in the mechanism). In the next section, I instead introduce a restriction on the class of mechanisms: I study a class of allocation and disclosure rules (cutoff rules) that can always be implemented, hence circumventing the above difficulty.

## 3 Cutoff mechanisms

To define cutoff mechanisms, I introduce the notion of a random-cutoff representation of an allocation rule.

[17] Intuitively, if this was possible, then the highest type would receive the highest resale price in the aftermarket. By mimicking the behavior of the highest type, any lower type would be able to deviate and secure the same payoff out-of-equilibrium. To deter that deviation, the transfer charged for reporting the highest type would have to eliminate any rents from getting the highest resale price for any lower type but that would also eliminate information rents enjoyed by the highest type. This is a contradiction because as long as lower types receive the good with positive probability, the highest type must receive strictly positive rents in equilibrium.

Assume first that there is one agent, $N = 1$, so that an allocation rule is a one-dimensional function $x(\theta)$ (I drop the subscripts). Let $\bar{c}$ be an arbitrary number strictly larger than $\max \Theta$, and let $C = \Theta \cup \{\bar{c}\}$. Suppose that $x(\theta)$ is non-decreasing on $\Theta$, and extend the definition to $C$ by assuming that $x(\bar{c}) = 1$. Then, $x$ can be seen as a cumulative distribution function, and we can define $\tilde{c}_x$ to be a random variable with realizations in $C$ with that distribution. By definition, $x(\theta) = \mathbb{P}(\theta \geq \tilde{c}_x)$. I will call $\tilde{c}_x$ a *random-cutoff representation* of $x$. Let $\Delta x$ denote the probability mass function of $\tilde{c}_x$.

The interpretation is as follows: The allocation rule $x(\theta)$ can be implemented by drawing a cutoff $c$ from the distribution of $\tilde{c}_x$, and giving the good to the agent if and only if the reported type $\theta$ is greater than the realized cutoff $c$. The set of cutoffs $C$ is equal to the set of types $\Theta$ except that some high cutoff $\bar{c}$ is included to allow for the possibility that the good is not allocated to any type.

Conversely, fix a random variable $\tilde{c}$ distributed according to some cdf $G$ supported on $C$. Then, $G(\theta)$ is a non-decreasing allocation rule on $\Theta$, and $\tilde{c} = \tilde{c}_G$. That is, $\tilde{c}$ is a random-cutoff representation of the allocation rule $G(\theta)$.

Summing up, there is a one-to-one correspondence between a subset of allocation rules on $\Theta$ and random cutoffs on $C$: Non-decreasing allocation rules are cdfs of random cutoffs. With slight abuse of notation, I will use $x$ to denote both an allocation rule and the corresponding cdf, with the interpretation determined by the context.

In the general model with $N$ agents, given an allocation rule $\boldsymbol{x}$, there is a separate set of cutoffs for every agent, $C_i = \Theta_i \cup \{\bar{c}_i\}$, and for any type profile $\boldsymbol{\theta}_{-i}$, $\Delta x_i(c; \boldsymbol{\theta}_{-i})$ denotes the pmf of the conditional cutoff distribution for agent $i$, induced by the interim allocation rule $x_i(\cdot, \boldsymbol{\theta}_{-i})$ (assuming it is non-decreasing). The interpretation remains the same: Fixing the reports $\boldsymbol{\theta}_{-i}$ of other agents, agent $i$'s allocation can be implemented by drawing cutoffs from the conditional distribution.

For example, consider the "efficient" allocation rule $x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \mathbf{1}_{\{\theta_i \geq \theta_{-i}^{(1)}\}}$, where $\theta_{-i}^{(1)} = \max_{j \neq i} \theta_j$, assuming for now that ties are broken in $i$'s favor. Then, the cutoff is equivalent to the highest competing type (and hence has a degenerate distribution conditional on $\boldsymbol{\theta}_{-i}$). If ties are broken uniformly at random, so that $x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \mathbf{1}_{\{\theta_i \geq \theta_{-i}^{(1)}\}} / (|\{j \in \mathcal{N} : \theta_j \geq \theta_{-j}^{(1)}\}|)$ is a two-step function, then the cutoff conditional on $\boldsymbol{\theta}_{-i}$ has a binary distribution, where the lower realization has probability equal to the probability that $i$ wins the tie-breaker.

A cutoff mechanism imposes two restrictions: *(i)* the allocation rule is non-decreasing, and *(ii)* the signal distribution is determined by the realization of the cutoff.

**Definition 2** (Cutoff rule and cutoff mechanism)**.** A mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ is a *cutoff rule* if $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ is non-decreasing in $\theta_i$ for all $\boldsymbol{\theta}_{-i}$, and there exists a signal function $\gamma_i$ :

$C_i \times \boldsymbol{\Theta}_{-i} \to \Delta(\mathcal{S}_i)$ such that

$$\pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) = \sum_{c \le \theta_i} \gamma_i(s|\,c,\, \boldsymbol{\theta}_{-i})\Delta x_i(c;\, \boldsymbol{\theta}_{-i}), \tag{3.1}$$

for all $\theta_i \in \Theta_i$, $\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$, $s \in \mathcal{S}_i$, and $i \in \mathcal{N}$. $(\boldsymbol{x},\, \boldsymbol{\pi},\, \boldsymbol{t})$ is a *cutoff mechanism* if $(\boldsymbol{x},\, \boldsymbol{\pi})$ is a cutoff rule.

In a cutoff rule $(\boldsymbol{x},\, \boldsymbol{\pi})$, each agent $i$ reports $\theta_i$. Conditional on other agents' reports $\boldsymbol{\theta}_{-i}$, the seller draws a cutoff $c$ from the distribution with pmf $\Delta x_i(\cdot\,;\, \boldsymbol{\theta}_{-i})$. If $\theta_i \ge c$, agent $i$ gets the good, and the designer draws and announces a signal from the distribution with pmf $\gamma_i(\cdot|\,c,\, \boldsymbol{\theta}_{-i})$. If $\theta_i < c$, agent $i$ does not receive the good.[18]

A signal that depends on the realized cutoff $c$ is informative about the type of the winner because third-party players condition on the event that the winning agent $i$ acquired the good, i.e., that $\theta_i \ge c$. Conditional on $i$ winning, a cutoff rule can also disclose information about $\boldsymbol{\theta}_{-i}$: Indeed, the signal distribution $\gamma_i$ is allowed to depend on $\boldsymbol{\theta}_{-i}$. For example, full disclosure of the losing agents' reports is a cutoff rule: It is enough to set $\mathcal{S}_i = \boldsymbol{\Theta}_{-i}$ and $\gamma_i(s|c,\, \boldsymbol{\theta}_{-i}) = \mathbf{1}_{\{s=\boldsymbol{\theta}_{-i}\}}$ for any $s \in \mathcal{S}_i$. However, a cutoff mechanism never fully reveals the type $\theta_i$ of the winner (I characterize distributions of posterior beliefs over the winner's type that can be induced by cutoff rules in Appendix A.4).

The amount of information that can be revealed by a cutoff mechanism depends on the allocation rule through the distribution of cutoffs that it induces. If all types of agent $i$ receive the good with the same probability (the allocation rule is constant), the cutoff for agent $i$ is degenerate and uninformative about her type. The "steeper" the allocation function, i.e., the higher the differences in probabilities of acquiring the good between high and low types, the more informative the realization of the cutoff is about the type of the winner.

It is important to note that the restriction on the allocation rule imposed by cutoff mechanisms is binding. With the aftermarket, dominant-strategy implementability does *not* imply that the allocation rule is non-decreasing (intuitively, a higher type might receive the good with lower probability if this is offset by a higher probability of a favorable signal).

[18] In order to implement a cutoff rule when $N > 1$ the designer must properly correlate the cutoffs for different agents to make sure the good is allocated to at most one agent ex-post. However, the joint distribution of cutoffs is irrelevant for payoffs (because only one agent interacts in the aftermarket) and implementability (all that matters is the marginal distribution for any agent), and thus the joint distribution need not be specified.

## 3.1 Implementability of cutoff rules

**Theorem 1.** *A cutoff rule is DS implementable for any prior distribution $\boldsymbol{f}$ and any monotone aftermarket $A$.*

*Proof.* The proof relies on the following monotonicity property of cutoff rules which follows directly from their definition (3.1):

$$\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) \text{ is non-decreasing in } \theta_i \text{ for all } s \in \mathcal{S}_i \text{ and } \boldsymbol{\theta}_{-i} \in \Theta_{-i}. \tag{M}$$

I give an intuitive argument for why condition (M) implies implementability for any prior distribution and any monotone aftermarket. (Appendix A.1 contains a formal but less intutive proof based on a version of Rochet (1987)'s cyclic monotonicity condition.) We can think of signal realizations as defining distinct goods allocated by the seller. Then, condition (M) says that for each of these goods, indexed by $s$, the allocation rule is non-decreasing. Moreover, a monotone aftermarket guarantees that a single-crossing property holds between the types of each agent and each of the goods. Thus, for every $s \in \mathcal{S}_i$ and every fixed $\boldsymbol{\theta}_{-i}$, there exists a transfer rule $t_i^s(\theta_i, \boldsymbol{\theta}_{-i})$ that implements the allocation rule $\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i})$ of good $s$. Defining $t_i(\theta_i, \boldsymbol{\theta}_{-i}) = \sum_{s\in\mathcal{S}_i} t_i^s(\theta_i, \boldsymbol{\theta}_{-i})$ finishes the proof. □

The economic intuition for Theorem 1 is straightforward: Under a cutoff rule, the report of the winner does not directly influence the signal sent by the mechanism, and thus the winner cannot manipulate the aftermarket belief over her type. Losing agents can manipulate posterior beliefs but this is irrelevant since they do not participate in the aftermarket. This is reminiscent of why VCG mechanisms (such as second price auctions) are truthful. In a VCG mechanism, the report of an agent does not influence the transfer the agent pays, except when it changes the allocation. In a cutoff mechanism, the report does not influence the signal, except when it changes the allocation. While the agent can change the outcome by affecting the probability with which she acquires the good, monotonicity of the aftermarket implies that such a deviation can be deterred by appropriately chosen transfers.

The transfer function implementing a cutoff rule will in general depend on the prior $f$ and the aftermarket $A$. This is a consequence of the setting rather than a feature of cutoff rules: With the aftermarket, the prior $f$ and the aftermarket $A$ directly influence the values $u_i(\theta_i;\, f_i^s)$ that agent $i$ has for winning. In the analogy developed by the above proof sketch, prices of goods indexed by $s \in \mathcal{S}_i$ must depend on how valuable they are to agent $i$.

As argued in the proof of Theorem 1, cutoff rules satisfy a strengthening of the ex-post monotonicity condition– they are monotone in the type *for every signal realization $s \in \mathcal{S}_i$*. This property plays a key role in the analysis – it is in fact a defining property of cutoff rules, as I show next.

**Proposition 1.** *A mechanism frame* $(\boldsymbol{x}, \boldsymbol{\pi})$ *is a cutoff rule if and only if condition* (M) *holds for all* $i \in \mathcal{N}$.

*Proof.* I only have to prove the "if" direction. Let $(\boldsymbol{x}, \boldsymbol{\pi})$ be a mechanism frame. Because both the property (M) and the definition of a cutoff rule are checked for every $i \in \mathcal{N}$ separately, I fix an agent $i$ and a profile $\boldsymbol{\theta}_{-i}$, and suppress them from the notation (so that, for example, $x(\theta)$ stands for $x_i(\theta, \boldsymbol{\theta}_{-i})$, and so on). Let $\beta_s(\theta) \equiv \pi(s|\,\theta)x(\theta)$. By condition (M), $\beta_s(\theta)$ is a non-decreasing function on $\Theta$, for any $s$. Summing over $s \in \mathcal{S}$, we get that $x(\theta)$ is non-decreasing. Let $\underline{\theta} = \min(\Theta)$, and let $\theta^-$ be the largest type in $\Theta$ smaller than $\theta$, for any $\theta > \underline{\theta}$. Because $\beta_s(\theta)$ is non-decreasing, it induces a positive additive (not necessarily probabilistic) measure with pmf $\mu_s$ on $C$ defined by $\mu_s(\underline{\theta}) = \beta_s(\underline{\theta})$, and $\mu_s(\theta) = \beta_s(\theta) - \beta_s(\theta^-)$ for any $\theta > \underline{\theta}$. The pmf $\mu_s$ is absolutely continuous with respect to the pmf $\Delta x$ of the cutoff representing the allocation rule $x$:

$$\mu_s(\theta) \leq \sum_{s' \in \mathcal{S}} \mu_{s'}(\theta) = \Delta x(\theta).$$

By the Radon-Nikodym Theorem, there exists a positive function $g_s$ on $C$ that is a density of $\mu_s$ with respect to $\Delta x$. In particular,

$$\pi(s|\,\theta)x(\theta) = \beta_s(\theta) = \mu_s(\{\tau : \tau \leq \theta\}) = \sum_{c \leq \theta} g_s(c)\Delta x(c), \tag{3.2}$$

for all $\theta$ and $s \in \mathcal{S}$. Moreover, we have, for any $\theta$,

$$x(\theta) = \sum_{c \leq \theta} \sum_{s \in \mathcal{S}} g_s(c)\Delta x(c) \implies \sum_{c \leq \theta} \left( \sum_{s \in \mathcal{S}} g_s(c) - 1 \right) \Delta x(c) = 0.$$

It follows that $\sum_s g_s(c) = 1$, for all $c$ with $\Delta x(c) > 0$. I can now define the measure $\gamma : C \to \Delta(\mathcal{S})$ by

$$\gamma(s|\,c) = g_s(c),$$

for all $c$ with $\Delta x(c) > 0$ (and in an arbitrary way for $c$ which have probability zero under $x$). Because $\sum_s g_s(c) = 1$, $\gamma$ is a well defined signal function. Moreover, equation (3.2) implies that the equality (3.1) from Definition 2 of cutoff rules holds for all $s$ and $\theta$. $\square$

## 4 Optimal cutoff mechanisms

In this section, I consider optimization in the class of cutoff mechanisms. I first focus on the single-agent case which produces a particularly sharp result and simplifies exposition. Then, I show how to generalize the solution method to multi-agent mechanisms.

### 4.1 Optimal cutoff mechanisms with a single agent

In this subsection, I assume that $N = 1$ (and thus drop the subscripts $i$ in the notation).

I say that a disclosure rule $\pi$ *reveals no information* if every signal realization $s$ is uninformative about the type of the agent: $\pi(s|\,\theta) = \pi(s|\,\hat{\theta})$ for all $\theta$, $\hat{\theta} \in \Theta$, $s \in \mathcal{S}$. Importantly, even when $\pi$ reveals no information, the posterior belief in the aftermarket may differ from the prior because the fact that the agent participates in the aftermarket is informative of her type when the allocation rule $x$ is non-constant. The following result establishes a strong conclusion about optimal cutoff mechanisms in the single-agent model.

**Theorem 2.** *With $N = 1$, the problem of maximizing (2.2) subject to $(x, \pi)$ being a cutoff rule has an optimal solution $(x^\star, \pi^\star)$ where $\pi^\star$ reveals no information.*

The conclusion of Theorem 2 holds regardless of the objective function. The type of the objective may influence the shape of the optimal allocation rule $x^\star$ but never requires the designer to make explicit announcements via $\pi^\star$. I prove the theorem in two steps: First, I consider optimization over disclosure rules for any fixed allocation rule $x$, and then I show that at the optimal allocation rule $x^\star$ the corresponding optimal $\pi^\star$ reveals no information. The first step will provide an important building block for the multi-agent model, while the second step is specific to the case of a single agent.

**Proof of Theorem 2. Step 1: Optimization over disclosure rules.** I fix a non-decreasing allocation rule $x$, and optimize over disclosure rules $\pi$ subject to $(x, \pi)$ being a cutoff rule. The proof strategy is as follows: As discussed in Section 3, any non-decreasing allocation rule $x$ can be represented by a random cutoff $\tilde{c}_x$. In a cutoff mechanism, the signal only depends on the realization of $\tilde{c}_x$. By Theorem 1, any disclosure of the cutoff is compatible with both (***IR***) and (***IC***) constraints. Thus, the mechanism design problem becomes a pure communication problem in which the designer chooses a disclosure policy of the random cutoff $\tilde{c}_x$ in order to induce the optimal distribution of posterior beliefs – this is the Bayesian persuasion problem of Kamenica and Gentzkow (2011) where the relevant state is the cutoff.

The prior distribution of the cutoff $\tilde{c}_x$ (the state variable) is given by the cdf $x$. I will use $G$ to denote the cdf of a posterior distribution of the cutoff. The aftermarket belief over the winner's type can be derived in two steps in a cutoff rule: *(i)* conditional on a signal realization, the cdf of the posterior belief over the cutoff is $G$, *(ii)* conditional on the agent acquiring the object, the corresponding posterior belief over that agent's type is given by

$$f^G(\theta) \equiv \mathbb{P}_{\tilde{c}\sim G}(\tilde{\theta} = \theta |\, \tilde{\theta} \geq \tilde{c}) = \frac{G(\theta)f(\theta)}{\sum_\tau G(\tau)f(\tau)}. \tag{4.1}$$

The above derivation uses the fact that the order of conditioning does not matter, and that in a cutoff rule the signal is independent of the winner's type conditional on the cutoff, so that in step *(ii)* the belief over the winner's type depends on the signal only indirectly through the belief over the cutoff. Using the equivalence between non-decreasing allocation rules and beliefs over cutoffs, $f^G$ can also be interpreted as the aftermarket belief over the type of the agent who acquired the good that would arise if the designer implemented the allocation rule $G(\theta)$ (and disclosed no further information). Next, let

$$\mathcal{V}(G) = \sum_{\theta\in\Theta} V(\theta;\, f^G)G(\theta)f(\theta) \tag{4.2}$$

be the expected payoff to the mechanism designer conditional on the signal inducing a posterior cdf $G$ of the cutoff and the agent acquiring the object in the mechanism. Equivalently, $\mathcal{V}(G)$ is the expected payoff to the mechanism designer that would arise if the allocation function were $G$ (instead of the actual $x$) and the mechanism revealed no additional information to the third party. It now follows from Theorem 1 and Kamenica and Gentzkow (2011) that we can optimize over distributions of posterior beliefs over the cutoff (this is immediate but I provide a formal proof in Appendix A.2).

**Lemma 1.** *With $N = 1$, for every non-decreasing allocation rule $x$, the problem of maximizing (2.2) over $\pi$ subject to $(x, \pi)$ being a cutoff rule is equivalent to*

$$\max_{\varrho\in\Delta(\Delta(C))} \mathbb{E}_{G\sim\varrho}\mathcal{V}(G) \tag{4.3}$$

$$s.t.\ \mathbb{E}_{G\sim\varrho}G(\theta) = x(\theta),\ \forall\theta\in\Theta. \tag{4.4}$$

The mechanism designer seeks to maximize her expected payoff over distributions $\varrho$ of posterior beliefs over the cutoff (equation 4.3). Condition (4.4) is the Bayes-plausibility constraint – the induced posterior beliefs over the cutoff must average out to the prior belief (with beliefs represented by cdfs).

Lemma 1 implies that the concavification approach of Aumann and Maschler (1995) and Kamenica and Gentzkow (2011) can be applied to the current setting. Let $\mathcal{X}$ be the set of all non-decreasing allocation rules on $\Theta$.

**Corollary 1.** *With $N = 1$, for a fixed allocation rule $x$, the maximal expected payoff to the mechanism designer is equal to the concave closure of $\mathcal{V}$ at $x$: $co\mathcal{V}(x) \equiv \sup\{y : (x, y) \in CH(graph(\mathcal{V}))\}$, where $CH$ denotes the convex hull, and $graph(\mathcal{V}) \equiv \{(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathbb{R} : \hat{y} = \mathcal{V}(\hat{x})\}$.*

**Step 2: Optimization over allocation rules.** By Corollary 1, the value to the designer at an optimal solution, now involving optimizing over $x$ as well, is $\sup_{x\in\mathcal{X}} \mathrm{co}\mathcal{V}(x)$. By definition of the concave closure, $\sup_{x\in\mathcal{X}} \mathrm{co}\mathcal{V}(x) = \sup_{x\in\mathcal{X}} \mathcal{V}(x)$, that is, the value of the function and its concave closure coincide at the supremum. An optimal solution $x^\star$ exists because $\mathcal{V}$ is an upper semi-continuous function on a compact set. This finishes the proof of Theorem 2: $\mathcal{V}(x^\star)$ is the expected payoff to the mechanism designer when $x^\star$ is the allocation rule and the disclosure rule reveals no information. $\square$

The proof provides a simple intuition for Theorem 2: When choosing an optimal cutoff rule, the problem of the designer is to choose a prior distribution over cutoffs (the allocation rule), and then optimally disclose information about the realized cutoff. Thus, the designer is a Sender who can *choose* the prior distribution of the state. When the posterior belief can be chosen directly by choosing the prior, there is no need to reveal information about the state. In the design of the optimal cutoff mechanism, there is no need to reveal information about the cutoff because the optimal posterior distribution can be induced directly by choosing the prior, that is, the allocation rule.

To illustrate the above results, I apply them to solve a simple version of Example 1.

**Example 3.** Consider Example 1 with $N = 1$, $\lambda = 1$, and $\Theta = \{l, h\}$ with $f(l) = f(h)$. There is a single third party with a constant value $v \in (h, 2h - l)$ that makes a take-it-or-leave-it offer to the agent in the aftermarket. The designer maximizes total surplus: $V(\theta; \bar{f}) = \theta \mathbf{1}_{\{\theta > p(\bar{f})\}} + v\mathbf{1}_{\{\theta \le p(\bar{f})\}}$, where $p(\bar{f})$ denotes the optimal offer made by the third party under posterior belief $\bar{f}$ (with ties broken in the designer's favor).

It is clear that $x(h) = 1$ in the optimal solution. Hence, the set of allocation rules is a one-dimensional family indexed by the probability $x(l)$ that the low type $l$ gets the object. The cutoff representation $\tilde{c}_x$ of $x$ is a binary random variable on $C = \{l, h\}$ with probability mass function $\Delta x$ given by $\Delta x(l) = x(l)$ and $\Delta x(h) = 1 - x(l)$. A posterior belief with cdf $G$ of the cutoff corresponds to a belief $f^G(h) = 1/(1 + G(l))$ that the type of the agent is high conditional on participation in the aftermarket (see 4.1). The third party offers a high price $h$ when she believes that the probability of the high *type* in the aftermarket is at least $(h - l)/(v - l)$, or, that the probability of the high *cutoff* is at least $\alpha^\star \equiv 2 - (v - l)/(h - l)$.

Thus, (4.2) becomes

$$\mathcal{V}(G) = \begin{cases} vG(l)f(l) + hf(h) & \text{if } 1 - G(l) < \alpha^\star \\ vG(l)f(l) + vf(h) & \text{if } 1 - G(l) \geq \alpha^\star \end{cases}.$$

By Corollary 1, optimal disclosure for any fixed allocation rule $x$ yields the concave closure of $\mathcal{V}$. The function $\mathcal{V}$ and its concave closure are depicted in Figure 4.1. When $1 - x(l) < \alpha^\star$ (as is the case in Figure 4.1), so that the third party would offer a low price when no signals are sent, it is optimal to disclose information about the cutoff in the form of a binary signal: $s \in \{s_L, s_H\}$. The designer sends $s_L$ when the cutoff is low with probability $\eta$ and sends $s_H$ in all other cases. The probability $\eta$ is chosen so that conditional on $s_H$, the third party is indifferent between offering the high and the low price (and offers the high price).[19] When $1 - x(l) \geq \alpha^\star$, the third party already offers a high price under the prior; $\mathcal{V}$ coincides with its concave closure, and the designer makes no announcement in the optimal mechanism.

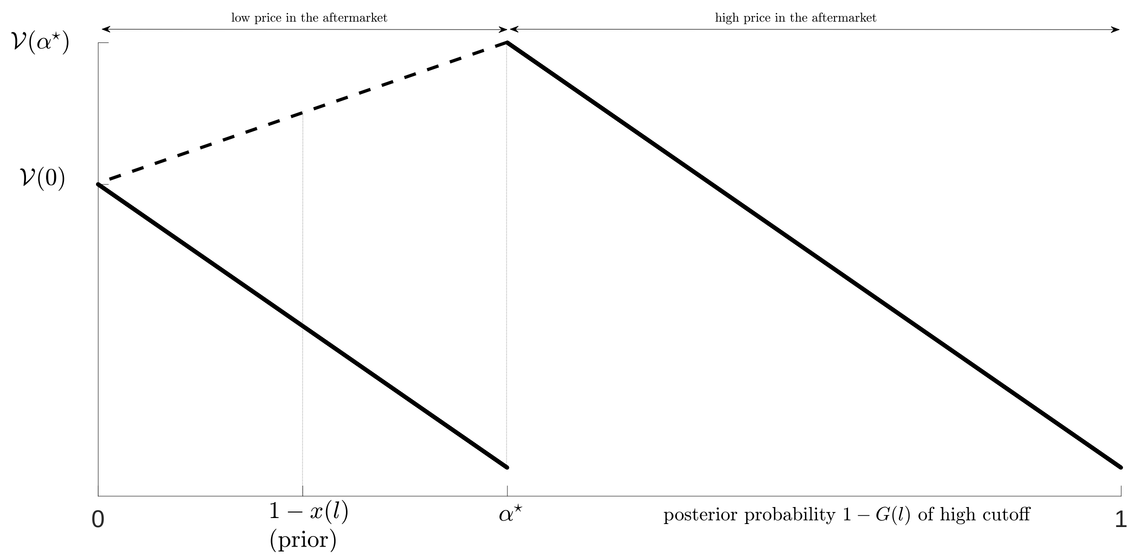


Fig. 4.1: Function $\mathcal{V}$ (solid line) and its concave closure (dashed line).

Next, suppose that the designer can optimize over both the allocation and the disclosure rule, that is, she can additionally choose $x(l)$. In Figure 4.1, the designer can choose an arbitrary point on the $x$-axis to maximize the concave closure of $\mathcal{V}$. Assuming that $v(v-h) \geq h(h-l)$ (as is the case in Figure 4.1), the expected payoff to the mechanism designer is maximized at $1 - x(l) = \alpha^\star$. At that prior, the function $\mathcal{V}$ coincides with its concave closure. Thus, the optimal mechanism is to allocate with probability $1 - \alpha^\star$ to the low type, with probability 1 to the high type, and reveal no information (which leads to a high price offered

[19] That is, $\eta$ solves $\alpha^\star = (1 - x(l))/(1 - x(l) + x(l)(1 - \eta))$.

in the aftermarket with probability one conditional on allocation).

When $v(v-h) < h(h-l)$, the value of co$\mathcal{V}$ is higher at 0 than at $\alpha^\star$, and thus it is optimal to choose $1 - x(l) = 0$, resulting in a constant allocation rule $x \equiv 1$ and again no information revelation. In this case, a low price is offered in the aftermarket. ■

I conclude with a few remarks based on the above example. First, since no information disclosure is always optimal with one agent, the allocation rule (equivalently, the distribution of cutoffs) is chosen to optimally trade-off "allocative efficiency" against the quality of beliefs in the aftermarket. The trade-off can be seen in Figure 4.1: A higher distribution of cutoffs induces higher beliefs in the aftermarket (the third party's optimal price jumps up at $\alpha^\star$) at the cost of excluding the low type from trading with higher probability (the function $\mathcal{V}$ is decreasing for a fixed price in the aftermarket). Second, Theorem 2 implies that no information disclosure is optimal only at the *optimal* allocation rule. As Example 3 shows, information disclosure can be optimal when the allocation rule is chosen suboptimally.

## 4.2 Optimal cutoff mechanisms with multiple agents

In this subsection, I consider the model with $N$ agents. The analysis of single-agent cutoff mechanisms will be useful because the general problem can be reduced to one-dimensional optimization over disclosure rules. In principle, a multi-agent cutoff mechanism allows the designer to disclose information about the cutoff and all losing agents' types. However, because I assumed that only the belief over the winner's type is payoff-relevant, this highly-dimensional information is redundant – it suffices to disclose information about the "interim expected" cutoff defined by the interim expected allocation rule. For example, consider the allocation rule $x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \mathbf{1}_{\{\theta_i \geq \theta_{-i}^{(1)}\}}$, where for simplicity the tie is broken in $i$'s favor. When $i$ is the winner, a cutoff mechanism can disclose information about the losing agents' types (an $N-1$ dimensional vector), and $i$'s cutoff has a degenerate distribution conditional on $\boldsymbol{\theta}_{-i}$. In this case, however, the highest competing type $\theta_{-i}^{(1)}$ is a sufficient statistic for the belief about $i$'s type when she is the winner, and thus it is enough to consider disclosure rules that only depend on the one-dimensional variable $\theta_{-i}^{(1)}$. The variable $\theta_{-i}^{(1)}$ is exactly the cutoff representing the interim expected allocation rule $x_i^{\boldsymbol{f}}(\theta_i) \equiv \prod_{j \neq i} F_j(\theta_i)$.

There are two difficulties with this approach. First, letting $\bar{x}_i : \Theta_i \to [0, 1]$ denote a generic interim expected allocation rule for agent $i$, we must ensure that the $N$-tuple $(\bar{x}_1, ..., \bar{x}_N)$ is feasible, i.e., induced by some joint allocation rule $\boldsymbol{x}$ under $\boldsymbol{f}$ (for example, choosing $\bar{x}_i \equiv 1$, for all $i$, is clearly not feasible). This problem can be addressed by applying tools from the literature on reduced-form auctions that formulates a necessary and sufficient condition for feasibility (the so-called Matthews-Border condition, see Matthews, 1984, and Border, 1991). Second, interim expected allocation rules are not sufficient to express dominant-strategy implementability – the reduced form of a mechanism can only be used to establish Bayesian implementability.[20] To circumvent this problem, I show that in the class of cutoff mechanism there is no gap between Bayesian and dominant-strategy implementation – the argument relies on proof techniques from the literature on BIC-DIC equivalence (see Manelli and Vincent, 2010, and Gershkov *et al.*, 2013).

Definitions (4.1) and (4.2) are directly generalized to the multi-agent setting by putting back the subscripts $i$ denoting respective agents. The posterior belief with cdf $G_i$ in $\mathcal{V}_i(G_i)$ is now interpreted as the belief over agent $i$'s (interim expected) cutoff. Equivalently, $\mathcal{V}_i(G_i)$ is the designer's expected payoff from interacting with agent $i$ that would arise if $G_i$ were the interim expected allocation rule for agent $i$ and the mechanism revealed no additional information. Let $\mathcal{X}_i$ denote the set of one-dimensional non-decreasing allocation rules on $\Theta_i$.

**Theorem 3.** *The following problem is equivalent to maximizing (2.2) over cutoff rules:*

$$\max_{\{\bar{x}_i \in \mathcal{X}_i\}_{i \in \mathcal{N}}} \sum_{i \in \mathcal{N}} co\mathcal{V}_i(\bar{x}_i) \tag{4.5}$$

*subject to the Matthews-Border condition:*

$$\sum_{i \in \mathcal{N}} \sum_{\theta_i > \tau_i} \bar{x}_i(\theta_i) f_i(\theta_i) \leq 1 - \prod_{i \in \mathcal{N}} F_i(\tau_i), \ \forall \boldsymbol{\tau} \in \mathbb{R}^N. \tag{M-B}$$

*Formally, any cutoff mechanism that maximizes (2.2) is payoff-equivalent to a cutoff mechanism whose reduced form solves the problem (4.5) subject to (M-B). Conversely, given any solution to problem (4.5) subject to (M-B), there exists a cutoff mechanism that maximizes (2.2) in the set of cutoff mechanisms and induces that solution as its reduced form.*

Theorem 3 implies that to solve the general problem, it is enough to solve $N$ one-dimensional persuasion problems – corresponding to finding the concave closures of each $\mathcal{V}_i$ – and then maximize over interim expected allocation rules subject to condition (M-B). In the symmetric case, i.e., when agents are ex-ante identical, it is without loss to look at symmetric mechanisms, and the problem takes a simpler form (subscripts can be dropped):

$$N \max_{\bar{x} \in \mathcal{X}} \mathrm{co}\mathcal{V}(\bar{x}) \quad \text{subject to} \quad \sum_{\theta > \tau} \bar{x}(\theta) f(\theta) \leq \frac{1 - F^N(\tau)}{N}, \ \ \forall \tau \in \mathbb{R}. \tag{4.6}$$

Apart from the complications discussed above, the proof of the theorem follows the same steps as the derivation of the optimal mechanism in Section 4.1 for the single-agent case (the

[20] See Appendix A.3 for a formal definition of a reduced form in the current context.

proof can be found in Appendix A.3). However, the last step in the proof of Theorem 2 (establishing optimality of no disclosure) cannot be taken if there is more than one agent. When $N = 1$, the Matthews-Border condition (M-B) holds vacuously and hence the optimality of no information disclosure follows from unconstrained maximization of the concavified objective: $\max_{x \in \mathcal{X}} \mathrm{co}\mathcal{V}(x) = \max_{x \in \mathcal{X}} \mathcal{V}(x)$. In contrast, when $N \geq 2$, the Matthews-Border condition is non-redundant, and it may be optimal to disclose information. To see why, consider the symmetric case (4.6). The concave closure of $\mathcal{V}$ is taken in the space of *all* non-decreasing interim allocation rules (equivalently, all posterior beliefs over the cutoff), while the actual rule $\bar{x}$ must be chosen from a strictly smaller subset of rules that satisfy the Matthews-Border condition (M-B). It might be optimal to induce posterior beliefs over the cutoff that do not correspond to an interim allocation rule satisfying (M-B).

For intuition, note that in the single-agent model, the optimal mechanism reveals no information because the seller can freely choose the prior distribution of the state variable (the cutoff) used for persuasion. With multiple agents, the seller faces a trade-off between full control over the distribution of the state variable (the cutoff) and inducing competition between the agents. Indeed, the Matthews-Border condition imposes constraints on interim allocation rules when there is positive probability that that reports of agents other than $i$ influence the allocation of agent $i$. For example, when the object is allocated to the highest type, the distribution of the second-order statistic (the cutoff) is pinned down by the prior distribution $\boldsymbol{f}$ and hence the designer has no control over it. Typically the designer finds it optimal to condition the allocation on all agents' reports: Thus, she gives up full control over the *prior* distribution of cutoffs, and instead uses signals to affect the *posterior* beliefs. I give examples of optimal mechanisms sending informative signals in subsequent sections.

#### 4.2.1 Optimality of simple disclosure rules

I conclude the section by providing sufficient conditions for optimality of full and no disclosure of the cutoff. A simple sufficient condition is, respectively, convexity or concavity of the functions $\mathcal{V}_i$. However, $\mathcal{V}_i$'s are derived objects that connect beliefs over cutoffs to expected payoffs. In applications, it may be easier to work with beliefs over agents' *types* directly.

In Appendix A.4, I prove that a conditional distribution of beliefs over the winner's type is feasible, that is, induced by some cutoff rule, if and only if *(i)* an appropriate Bayes-plausibility condition holds, and *(ii)* each posterior belief over the winner's type likelihood-ratio (LR) dominates the prior belief.[21] Condition *(i)* is natural in the context of information design (see Kamenica and Gentzkow, 2011), while condition *(ii)* is a consequence of monotonicty of cutoff rules - regardless of the signal, higher types receive the good with higher

[21] A pmf $g$ *likelihood-ratio dominates* a full-support pmf $f$ if $g(\theta)/f(\theta)$ is non-decreasing.

probability, so a posterior belief over the winner's type must be higher than the prior. Define

$$\mathcal{W}_i(\bar{f}) = \sum_{\theta \in \Theta_i} V_i(\theta;\, \bar{f})\bar{f}(\theta) \tag{4.7}$$

as the expected payoff to the designer conditional on agent $i$ winning and posterior belief $\bar{f}$ over $i$'s type. Let $M_{f_i}$ be the set of distributions over $\Theta_i$ that likelihood-ratio dominate the prior $f_i$, and let $f_i^{\bar{x}_i}$, defined by (4.1), be the posterior belief over $i$'s type given the interim expected allocation rule $\bar{x}_i$, when $i$ is the winner and no other information is revealed.

**Proposition 2.** *The maximal expected payoff to the mechanism designer optimizing (2.2) over cutoff rules for a fixed interim allocation rule $\bar{\boldsymbol{x}}$ is equal to*

$$\sum_{i \in \mathcal{N}} \left( \sum_{\theta_i \in \Theta_i} \bar{x}_i(\theta_i) f_i(\theta_i) \right) co^{M_{f_i}}\, \mathcal{W}_i(f_i^{\bar{x}_i}) \tag{4.8}$$

*where $co^{M_{f_i}}\, \mathcal{W}_i(f_i^{\bar{x}_i}) \equiv \sup\left\{y : \,(f_i^{\bar{x}_i},\, y) \in CH\Big(graph(\mathcal{W}_i)|_{M_{f_i}}\Big)\right\}$, and $graph(\mathcal{W}_i)|_{M_{f_i}}$ is the graph of $\mathcal{W}_i$ restricted to domain $M_{f_i}$.*

Objectives (4.5) and (4.8) are analogous except for two important details. First, in (4.8), $\mathcal{W}_i$ is concavified in the subspace $M_{f_i} \subsetneq \Delta(\Theta_i)$, while in (4.5) the concave closure of $\mathcal{V}_i$ is taken in the entire space $\Delta(C_i)$. This is because a cutoff rule can induce an arbitrary belief over the cutoff but can only induce beliefs over the winner's type that LR dominate the prior. Second, in (4.8) the concavified objective is multiplied by an additional term $\sum_{\theta_i \in \Theta_i} \bar{x}_i(\theta_i) f_i(\theta_i)$ – the ex-ante probability of allocating the good to agent $i$. This is because the distribution of beliefs over the winner's type is a conditional distribution (conditional on allocating the good to agent $i$), so the conditional expected payoff must be converted into an ex-ante expected payoff. As a corollary of Proposition 2, I obtain the following result.[22]

**Corollary 2.** *If $\mathcal{W}_i$ is convex on its domain, the optimal cutoff mechanism fully discloses $i$'s cutoff when $i$ is the winner. If $\mathcal{W}_i$ is concave, the optimal cutoff mechanism reveals no information when $i$ is the winner.*

Corollary 2 can be used to provide simple examples showing that full disclosure is uniquely optimal.[23] I conclude with such an example.

[22] Molnar and Virág (2008) establish a similar result in a model where all information structures are implementable, and full or no disclosure pertains to the type of the winner rather than the cutoff.

[23] When $\mathcal{W}$ is convex in the single-agent setting, a consequence of Corollary 2 and Theorem 2 is that the optimal cutoff distribution must be degenerate (then, and only then, full disclosure and no disclosure coincide). This corresponds to the optimal allocation rule being a threshold rule: $x(\theta) = \mathbf{1}_{\{\theta \geq r\}}$ for some $r$.

**Example 4.** Consider the Cournot competition model (case b) from Example 2. Suppose that there are $N$ ex-ante symmetric potential entrants competing for a single patent, and the mechanism designer chooses a disclosure rule in an auction to maximize total surplus (defined as the area under the demand curve minus the costs of production). Dropping subscripts (due to symmetry), we obtain

$$V(\theta;\, \bar{f}) = \theta \bar{V}(\mathbb{E}_{\bar{f}}[\tilde{\theta}]) + (1-\theta)\underline{V}(\mathbb{E}_{\bar{f}}[\tilde{\theta}]),$$

where $\bar{V}(m)$ and $\underline{V}(m)$ denote the total surplus conditional on the winner's type being high or low, respectively, when the aftermarket belief about the winner's type has expectation $m$. From this, we get that

$$\mathcal{W}(\bar{f}) = \sum_{\theta \in \Theta} V(\theta;\, \bar{f})\bar{f}(\theta) = \mathbb{E}_{\bar{f}}[\tilde{\theta}]\bar{V}(\mathbb{E}_{\bar{f}}[\tilde{\theta}]) + (1 - \mathbb{E}_{\bar{f}}[\tilde{\theta}])\underline{V}(\mathbb{E}_{\bar{f}}[\tilde{\theta}]) \equiv W(\mathbb{E}_{\bar{f}}[\tilde{\theta}]).$$

The objective function $\mathcal{W}(\bar{f})$ depends on the posterior belief over the winner's type only through its expectation. By direct calculation, $W(m)$ is a quadratic function of $m$, with coefficient $(17/72)\Delta^2$ on $m^2$. It follows that $\mathcal{W}(\bar{f})$ is a convex function of $\bar{f}$. By Corollary 2, full disclosure of the cutoff is optimal in the class of cutoff rules. When $\Delta > 0$, this is the unique optimal cutoff rule.[24] For example, if the designer uses a second-price auction to allocate the patent, then disclosure of the price after the auction is optimal. In Section 6, I discuss this example further, solve for the optimal allocation rule (showing that full disclosure continues to be uniquely optimal), and relate it to existing results in the literature. ■

# 5 Characterizations of cutoff rules

So far, I have not provided a reason for restricting attention to the class of cutoff mechanisms. In this section, I attempt to fill that gap by proving two characterization theorems. The first one establishes the converse to Theorem 1 – only cutoff rules can be implemented regardless of the (monotone) aftermarket and the prior distribution of types. The second one provides a restriction on the aftermarkets under which cutoff mechanisms are the only feasible choice within a subclass of mechanisms satisfying a strong notion of implementability and a regularity condition.

[24] This follows from strict convexity of $W(m)$ in the mean $m$: If any information about the cutoff was pooled, it would be possible to reveal additional information and raise total surplus.

## 5.1 Cutoff rules are implementable for all distributions and aftermarkets

I let $\mathcal{A}$ and $\mathcal{F}$ denote an abstract set of possible aftermarkets and prior distributions, respectively. For example, $\mathcal{A}$ may include various versions of a post-mechanism game differing in parameters of the bargaining protocol and characteristics of third-party players, or different equilibria of the same aftermarket game.

**Definition 3** (Flexibility)**.** A mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ is flexible with respect to $(\mathcal{F}, \mathcal{A})$, if $(\boldsymbol{x}, \boldsymbol{\pi})$ is DS implementable for any prior distribution $\boldsymbol{f} \in \mathcal{F}$ and any aftermarket $A \in \mathcal{A}$.

Theorem 1 shows that cutoff rules are flexible with respect to all distributions and monotone aftermarkets. To prove a converse, I define an appropriate Richness condition.

**Definition 4** (Richness)**.** The pair $(\mathcal{F}, \mathcal{A})$ satisfies *Richness* if for any mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$, $i \in \mathcal{N}$, $\theta_i > \hat{\theta}_i$ and $\boldsymbol{\theta}_{-i}$, there exists a prior distribution $\boldsymbol{f} \in \mathcal{F}$ and an aftermarket $A \in \mathcal{A}$ such that

$$\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) < \pi_i(s|\hat{\theta}_i, \boldsymbol{\theta}_{-i})x_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \implies u_i(\theta_i;\, f_i^s) > u_i(\hat{\theta}_i;\, f_i^s), \tag{5.1}$$

$$\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) > \pi_i(s|\hat{\theta}_i, \boldsymbol{\theta}_{-i})x_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \implies u_i(\theta_i;\, f_i^s) = u_i(\hat{\theta}_i;\, f_i^s). \tag{5.2}$$

**Theorem 4.** *Suppose that a mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ is flexible with respect to $(\mathcal{F}, \mathcal{A})$ that satisfies the Richness condition. Then, $(\boldsymbol{x}, \boldsymbol{\pi})$ is a cutoff rule.*

*Thus, if $(\mathcal{F}, \mathcal{A})$ satisfies Monotonicity and Richness, flexibility is a defining property of cutoff mechanisms.*

The set of all prior distributions and all monotone aftermarkets trivially satisfies the Richness condition (it is enough to carefully examine Definition 4).

**Corollary 3.** *A mechanism frame is implementable for all prior distributions and all monotone aftermarkets if and only if it is a cutoff rule.*

Before commenting on Theorem 4, I discuss the intuition for its simpler version, Corollary 3. Recall from Section 3 that we can think of different signal realizations $s \in \mathcal{S}_i$ as different goods allocated by the designer to agent $i$. For any fixed prior distribution and aftermarket, incentive-compatibility requires that these goods are allocated with probability that is non-decreasing in any agent's type *on average* across $s$. However, as we consider all possible priors and aftermarkets, the allocation probability must be monotone in *each* good $s$ separately – this is the only way to guarantee that the average allocation probability is monotone regardless of the valuations $u_i(\theta_i;\, f_i^s)$ for different goods $s$. By Proposition 1, this is exactly what defines cutoff mechanisms.

Theorem 4 is stronger than Corollary 3 in that it only requires that the mechanism frame is implementable for a sufficiently large set of prior distributions and aftermarkets. I now prove Theorem 4, and then discuss the meaning of the Richness condition using an example.

*Proof of Theorem 4.* By Proposition 1, it is enough to show that if $(\mathcal{F},\, \mathcal{A})$ satisfies Richness and a mechanism frame $(\boldsymbol{x},\, \boldsymbol{\pi})$ is flexible with respect to $(\mathcal{F},\, \mathcal{A})$, then $(\boldsymbol{x},\, \boldsymbol{\pi})$ satisfies condition (M).

Fix a mechanism frame $(\boldsymbol{x},\, \boldsymbol{\pi})$, $i \in \mathcal{N}$, $\theta_i > \hat{\theta}_i$ and $\boldsymbol{\theta}_{-i}$. Since $(\boldsymbol{x},\, \boldsymbol{\pi})$ is assumed DS implementable, condition (***IC***) has to hold for $\theta_i$ and $\hat{\theta}_i$. In particular, type $\theta_i$ cannot find it profitable to report $\hat{\theta}_i$, and vice versa. Summing up the two resulting inequalities, we can cancel out transfers, and obtain

$$\sum_{s\in\mathcal{S}_i} \Big[u_i(\theta_i;\, f_i^s) - u_i(\hat{\theta}_i;\, f_i^s)\Big] \Big[\pi_i(s|\, \theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) - \pi_i(s|\, \hat{\theta}_i,\, \boldsymbol{\theta}_{-i})x_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i})\Big] \geq 0. \quad (5.3)$$

Denote $\beta_s(\tau) \equiv \pi_i(s|\, \tau,\, \boldsymbol{\theta}_{-i})x_i(\tau,\, \boldsymbol{\theta}_{-i})$. By the Richness condition, there exist $\boldsymbol{f} \in \mathcal{F}$, and $A \in \mathcal{A}$ such that conditions (5.1) and (5.2) hold. Under these $\boldsymbol{f}$ and $A$, inequality (5.3) becomes

$$\sum_{\{s\in\mathcal{S}_i:\, \beta_s(\theta_i)<\beta_s(\hat{\theta}_i)\}} \Big[u_i(\theta_i;\, f_i^s) - u_i(\hat{\theta}_i;\, f_i^s)\Big] \Big[\beta_s(\theta_i) - \beta_s(\hat{\theta}_i)\Big] \geq 0,$$

with $u_i(\theta_i;\, f_i^s) > u_i(\hat{\theta}_i;\, f_i^s)$ for each signal $s$ in the summation, by condition (5.1). We have thus obtained that a sum of strictly negative terms is non-negative. This is only possible when the set of indices in the sum is empty: $\{s \in \mathcal{S}_i : \beta_s(\theta_i) < \beta_s(\hat{\theta}_i)\} = \emptyset$. Because $\theta_i > \hat{\theta}_i$ and $\boldsymbol{\theta}_{-i}$ were arbitrary, this shows that condition (M) holds, thereby finishing the proof. $\square$

The proof of the theorem is simple because the Richness condition is tailored toward the result. The difficulty often lies in proving that a certain set of priors and aftermarkets satisfy the Richness condition. I go through one such example next. The example also illustrates the fact that the set $\mathcal{A}$ need not be very large if $\mathcal{F}$ is large.

**Example 5.** Consider the resale aftermarket from Example 1 assuming for now that $\lambda = 1$ (the aftermarket happens with probability one), the third party has a constant value $v$ larger than the highest type of any agent, and makes a take-it-or-leave-it offer with indifference broken in the agent's favor ($|\mathcal{A}| = 1$). Let $\mathcal{F}$ be the set of all possible prior distributions.

I prove that $(\mathcal{F},\, \mathcal{A})$ satisfies Richness. Consider first the single agent case $N = 1$. Fix any mechanism frame $(x,\, \pi)$ and $\theta > \hat{\theta}$. Consider a distribution with pmf $f$ supported on the set $\{\hat{\theta},\, \theta\}$. The optimal price offered by the third party is either $\hat{\theta}$ or $\theta$. Following a signal $s$, the third party Bayes-updates beliefs (see equation 2.1), and offers price $\hat{\theta}$ if

$$(\theta - \hat{\theta})\pi(s|\,\hat{\theta})x(\hat{\theta})f(\hat{\theta}) > (v - \theta)\pi(s|\,\theta)x(\theta)f(\theta). \tag{5.4}$$

Price $\theta$ is uniquely optimal following signals $s$ under which the opposite strict inequality holds. Define $f$ as the unique pmf supported on $\{\hat{\theta},\, \theta\}$ such that $f(\hat{\theta})/f(\theta) = (v-\theta)/(\theta-\hat{\theta})$. That is, in the absence of additional information, the third party is indifferent between offering price $\theta$ and $\hat{\theta}$.

Suppose that the premise of condition (5.1) holds: $\pi(s|\,\theta)x(\theta) < \pi(s|\,\hat{\theta})x(\hat{\theta})$. Then, by choice of $f$, condition (5.4) must hold, and therefore the price $\hat{\theta}$ is uniquely optimal for the third party. It follows that type $\theta$ rejects the offer and receives $u(\theta;\, f_i^s) = \theta$, while type $\hat{\theta}$ accepts the offer and receives $u(\hat{\theta};\, f_i^s) = \hat{\theta}$. Thus, (5.1) holds. Now suppose that the premise of condition (5.2) holds: $\pi(s|\,\theta)x(\theta) > \pi(s|\,\hat{\theta})x(\hat{\theta})$. Then, price $\theta$ is uniquely optimal for the third party, and both types resell, getting utility $\theta = u(\theta;\, f_i^s) = u(\hat{\theta};\, f_i^s)$. Thus, (5.2) holds. Therefore, Richness is satisfied in the single-agent case.

To extend this result to an arbitrary $N$, notice that fixing any $i \in \mathcal{N}$ and $\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$, we can choose (Dirac delta) distributions $f_j(\tau) = \mathbf{1}_{\{\tau=\theta_j\}}$ for each $j \neq i$. Then, we can choose $f_i$ in the same way as above, and all conclusions hold (the proof is identical, with $x(\theta)$ and $\pi(s|\,\theta)$ replaced by $x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$ and $\pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i})$, respectively).

Finally, notice that when $(\mathcal{F},\, \mathcal{A})$ satisfies Richness, then all supersets of $\mathcal{F}$ and $\mathcal{A}$ also satisfy Richness. This means that the set of aftermarkets described by Example 1 with no restrictions on parameters satisfies Monotonicity and Richness. ■

The example serves as an illustration for the intuition behind the Richness condition. Aftermarkets differ in the sensitivity of induced payoffs to the information revealed by the mechanism. The Richness condition requires that among possible priors and aftermarkets we can always find some that make payoffs particularly sensitive to information. The premise in condition (5.1) can be interpreted as "bad news" about the agent's type – after observing a signal $s$ that satisfies the left-hand side inequality (for a fixed $\boldsymbol{\theta}_{-i}$), the posterior probability of the lower type $\hat{\theta}_i$ increases. Under some prior distribution $\boldsymbol{f}$ and aftermarket $A$, the expected payoff of the higher type $\theta_i$ has to strictly exceed the expected payoff of the lower type $\hat{\theta}_i$ following "bad news". On the other hand, when the mechanism sends "good news" (condition 5.2), the expected payoffs of the two types should be equal. In Example 5, for any two types $\theta_i$ and $\hat{\theta}_i$, there exists a prior $f$ under which the third party is indifferent between a high and a low price in the aftermarket. Therefore, any "bad news" (a signal realization that is more likely under the low type) will tilt the price to be low, leading to a gap between the payoffs of the high and the low type. On the other hand, any "good news" will tilt the price to be high, in which case both types resell and enjoy the same payoff.

Conditions (5.1) and (5.2) resemble submodularity of the agent's payoff in the type and posterior belief: Higher types enjoy higher payoffs than low types for any aftermarket belief but the difference in payoffs decreases (to zero) when the belief over the winner's type "increases" (after good news). In Section 5.2, I show that this is not a coincidence: I formally define submodular aftermarkets and argue that they make information disclosure in the mechanism difficult; in such cases, disclosing the cutoff may be the only feasible option (as we already know, information about the cutoff can always be disclosed).

### 5.1.1 Flexibility vs robustness

Flexibility of cutoff rules, that is, implementability for all possible prior distributions and aftermarkets, implies that cutoff mechanisms are a natural benchmark that can be used to establish a lower bound on the value of the objective function in *any* design problem with a monotone aftermarket. Corollary 3 states that, moreover, cutoff rules are the *largest* class that can serve this purpose: Any rule outside of the class cannot be implemented in at least some cases, therefore, it cannot serve as a universal lower bound. The result also implies that full disclosure of the cutoff is a tight lower bound on how much information (in the Blackwell sense) can be disclosed in a mechanism followed by an aftermarket.

Flexibility can also be a useful property in practical design problems due to its connection to robustness. In general, flexible implementation allows transfers to be a function of the distribution of types and the form of the aftermarket, and therefore cutoff mechanisms are not a detail-free design. However, certain simple cutoff rules can be implemented with no knowledge of the environment by the designer when transfers are pinned down in equilibrium of an indirect implementation of a cutoff rule (see the Online Appendix for formal results). Moreover, flexibility is a *necessary condition* for robust implementation – certainly, if the designer hopes to implement a mechanism frame without knowing the details of the environment, then there must exist transfers that implement that frame in each possible case. Theorem 4 implies that a designer interested in robust implementation of a mechanism frame has no reason to look beyond the class of cutoff mechanisms.

When the designer does not know the distribution of types and the aftermarket, it is no longer without loss of generality to restrict attention to direct mechanisms. The designer might instead fix an indirect mechanism, allowing the allocation and disclosure rule to be determined endogenously in equilibrium as the distribution and the aftermarket vary. Optimizing over all indirect mechanisms appears intractable, and it is known that optimal mechanism may exhibit unintuitive properties (when agents have superior knowledge about the environment, and there are no restrictions on the message spaces, the designer can often elicit that information at no cost, see Bergemann and Morris, 2013).

## 5.2 Cutoff rules satisfy strong dominant-strategy implementability

The goal of this section is to derive conditions on the aftermarket under which restricting attention to cutoff rules is without loss of generality. Because cutoff rules are a relatively narrow class that imposes constraints on the allocation and disclosure rule for all type profiles and signals, these conditions are also relatively restrictive. In particular, the main result characterizes cutoff rules as the unique feasible class within mechanism frames that *(i)* satisfy a strong notion of implementability, and *(ii)* induce posterior beliefs that can be ranked in a certain way. While mathematically limiting, conditions *(i)* and *(ii)* have no bite when there is no aftermarket, and are likely to hold in practical design problems. Under *(i)* and *(ii)*, I show that cutoff rules are without loss of generality under *submodular aftermarkets*. An aftermarket is submodular if lower types benefit more (in relative terms) than higher types from a shift in the posterior beliefs that puts more mass on higher types. A formal definition requires introducing an order on posterior beliefs so that agents' payoffs in the aftermarket can be specified as submodular functions of types and beliefs. First, however, I introduce a strengthening of dominant-strategy implementability and show that cutoff rules are implementable in this stronger sense with no further assumptions on the aftermarket.

### 5.2.1 Strong dominant-strategy implementation

I say that a mechanism frame $(\boldsymbol{x},\, \boldsymbol{\pi})$ is deterministic if for all $i$, $\boldsymbol{\theta}$, and $s$, we have $x_i(\boldsymbol{\theta}),\ \pi_i(s|\,\boldsymbol{\theta}) \in \{0,\, 1\}$. Randomization in the mechanism can be captured by allowing the designer to have a type $\theta_0$ drawn (independently of agents' types) from some auxiliary type space $\Theta_0$, and letting the mechanism depend deterministically on the extended type profile $(\boldsymbol{\theta},\, \theta_0) \in \boldsymbol{\Theta} \times \Theta_0$.

**Observation 1.** *For any mechanism frame $(\boldsymbol{x},\, \boldsymbol{\pi})$, there exists a measurable space $\Theta_0$ and a distribution with cdf $F_0 \in \Delta(\Theta_0)$ such that*

$$x_i(\boldsymbol{\theta}) = \int_{\Theta_0} \hat{x}_i(\boldsymbol{\theta};\, \theta_0) dF_0(\theta_0), \tag{5.5}$$

$$\pi_i(s|\,\boldsymbol{\theta}) = \int_{\Theta_0} \hat{\pi}_i(s|\,\boldsymbol{\theta};\, \theta_0) dF_0(\theta_0), \tag{5.6}$$

*where $(\hat{\boldsymbol{x}}(\cdot\,;\, \theta_0),\, \hat{\boldsymbol{\pi}}(\cdot\,;\, \theta_0))$ is a deterministic mechanism for any $\theta_0 \in \Theta_0$. I call $(\hat{\boldsymbol{x}},\, \hat{\boldsymbol{\pi}})$ the deterministic decomposition of $(\boldsymbol{x},\, \boldsymbol{\pi})$.*

When randomization is modeled as an endogenous type of the mechanism designer, it is natural to extend dominant-strategy implementation by requiring truthful reporting regardless of the beliefs held by agents over the extended type profile.

**Definition 5.** A mechanism frame $(\boldsymbol{x},\, \boldsymbol{\pi})$ is *strongly dominant-strategy* (SDS) implementable if there exists a dominant-strategy (DS) implementable deterministic decomposition of $(\boldsymbol{x},\, \boldsymbol{\pi})$.[25]

SDS implementation is equivalent to requiring that a mechanism can be represented as a randomization over deterministic DS mechanisms. Thus, from the perspective of any single agent, the mechanism is SDS-IC if the designer could disclose the outcome of any randomization *prior* to the agent reporting her type and still satisfy the IC constraints for that agent. Consequently, the agent should report truthfully regardless of what beliefs she holds about how the designer is going to randomize.

SDS implementation is desirable in contexts where agents do not fully trust the mechanism designer. Arguably, as long as the designer implements an outcome that lies within the support of the distribution, it is difficult to prove that randomization was not correctly conducted. All other deviations by the designer, such as changing the payments or the allocation as a function of reports, can be directly detected if agents share their information after participating in the mechanism. If agents think that the designer has limited commitment in that she might disobey the rules of the mechanism as long as this cannot be detected, they might not want to report truthfully even if the mechanism was DS-IC. However, they would want to report truthfully if the mechanism was SDS-IC. Akbarpour and Li (2018)[26] discuss similar concerns to motivate their class of credible mechanisms – in their work, the designer can consider deviations that cannot be detected by any single agent. SDS-IC is a milder restriction as the designer is only allowed to consider deviations that would not be detected even if all agents could freely share information.

The concept of SDS implementation strengthens DS implementation but in a weak way: It has no bite in the private-value model without the aftermarket. For example, the optimal Myerson auction can always be implemented in an SDS way. With a monotone aftermarket, SDS mechanisms include cutoff mechanisms.

**Proposition 3.** *When the aftermarket is monotone, any cutoff rule is strongly dominant-strategy implementable.*

*Proof.* By definition of a cutoff rule, we have for any $i$, $s$ and $\boldsymbol{\theta}$:

$$\pi_i(s|\,\boldsymbol{\theta})x_i(\boldsymbol{\theta}) = \sum_{c\leq\theta_i} \gamma_i(s|\,c,\,\boldsymbol{\theta}_{-i})\Delta x_i(c;\,\boldsymbol{\theta}_{-i}) = \sum_{c\in C_i}\sum_{s'\in\mathcal{S}_i} \left(\mathbf{1}_{\{s'=s\}}\mathbf{1}_{\{\theta_i\geq c\}}\right)\gamma_i(s|\,c,\,\boldsymbol{\theta}_{-i})\Delta x_i(c;\,\boldsymbol{\theta}_{-i}).$$

[25] Implementing transfers are allowed to depend on $\theta_0$ and $s$.

[26] See also Dequiedt and Martimort (2015).

This, however, is a representation of a cutoff rule as randomization over deterministic DS implementable mechanism frames, where the DS property follows from monotonicity in $\theta_i$ for any $c \in C_i$ and $s' \in \mathcal{S}_i$ (and monotonicity of the aftermarket). $\square$

The intuition is as simple as the proof: In a cutoff rule, the cutoff captures the randomization in the mechanism from the perspective of any given agent. Moreover, fixing any agent $i$, the designer could in principle reveal other agents' reports, the cutoff realization, and even the signal realization to agent $i$ *before* asking her to report her type. This is because in a cutoff mechanism, the allocation remains monotone in the type even conditional on a cutoff and signal realization (recall property (M)). Next, I formulate a property of the aftermarket that allows me to prove a partial converse to Proposition 3.

### 5.2.2 Submodular aftermarkets

To avoid situations where players care about the "label" of a belief rather than about its implication for the payoff-relevant outcome, I make the following assumption which is automatically satisfied when the payoffs in the aftermarket are derived from optimal choices of actions by Bayesian agents: If for some $\bar{f},\, \bar{g} \in \Delta(\Theta_i)$, $u_i(\theta_i;\, \bar{f}) = u_i(\theta_i;\, \bar{g})$ for all $\theta_i \in \Theta_i$ (beliefs $\bar{f}$ and $\bar{g}$ have the same payoff consequences), then also $u_i(\theta_i;\, \bar{f}) = u_i(\theta_i;\, \alpha\bar{f} + (1-\alpha)\bar{g})$ for any $\alpha \in (0,\, 1)$ (their convex combination has the same payoff consequences). The same property is assumed about the designer's payoff $V_i$, for all $i$. More substantially, I impose submodularity of the agent's payoff in her type and beliefs – implying that the willingness to pay for "high" beliefs is decreasing in the type of the agent. I use the monotone likelihood ratio order on beliefs defined in Section 4.2.1,[27] denoted $\succeq^{LR}$.

**Definition 6.** An aftermarket $A$ is *submodular* if for any $i \in \mathcal{N}$, $\bar{f},\, \bar{g} \in \Delta(\Theta_i)$,

$$\bar{f} \succeq^{LR} \bar{g} \implies u_i(\theta;\, \bar{f}) - u_i(\theta;\, \bar{g}) \text{ is non-increasing in } \theta.$$

An aftermarket is *strictly submodular* if additionally

$$\bar{f} \succeq^{LR} \bar{g} \implies u_i(\theta;\, \bar{f}) - u_i(\theta;\, \bar{g}) \text{ is strictly decreasing in } \theta$$

whenever $u(\theta_i;\, \bar{f}) \neq u(\theta_i;\, \bar{g})$ for some type $\theta_i \in \Theta_i$.

An aftermarket is submodular if lower types have a higher willingness to pay for an upward shift in beliefs. For example, if all types of the agent prefer to be perceived as a high

[27] Formally, to allow for the possibility of disjoint supports, I say that $\bar{g} \succeq^{LR} \bar{f}$ if there exist full-support $\bar{g}_\epsilon$ and $\bar{f}_\epsilon$ such that $\bar{f}_\epsilon \to \bar{f}$, $\bar{g}_\epsilon \to \bar{g}$, and for small enough $\epsilon > 0$, $\bar{g}_\epsilon(\theta)/\bar{f}_\epsilon(\theta)$ is non-decreasing in $\theta$.

type, this means that any improvement in posterior beliefs is valued more by lower types. This is the case in resale aftermarkets because lower types benefit more (relative to keeping the good) from a high resale price than higher types. In particular, the resale aftermarkets from Example 1 and Example 2 (a) satisfy submodularity because beliefs higher in the LR order lead to (weakly) higher resale prices.

Simple resale aftermarkets are typically not *strictly* submodular – this is because two types $\theta > \hat{\theta}$ differ in their willingness to pay for a resale price $p$ only if that price is accepted by $\hat{\theta}$ but rejected by $\theta$. It can be shown that a resale market becomes strictly submodular if either *(i)* every price happens with positive probability conditional on any given signal (for example, because the value of the third party is stochastic) or *(ii)* the type of the agent in the resale stage is stochastic conditional on the type in the mechanism, all types have positive probability following any type in the mechanism, but higher types in the mechanism induce a higher distribution of types in the resale stage.

Submodularity may also be consistent with agents preferring to be perceived as low types. In Example 2, submodularity requires that $\bar{u}_i(m) - \underline{u}_i(m)$ is decreasing in the posterior mean $m$. The investment game from Example 2 (c) induces a strictly submodular aftermarket because $\bar{u}_i(m)$ is strictly decreasing and $\underline{u}_i(m) = 0$: Each type benefits from being perceived as a low type but lower types are hurt less by an increase in the posterior mean.

An example of an aftermarket that does *not* satisfy submodularity is the Cournot model from Example 2 (b). Here, the agent wants to be perceived as a high type (that is, as having a low cost), and high types benefit more from more favorable beliefs – the aftermarket is in fact *supermodular* (with a formal definition analogous to Definition 6).

A key observation is that it is *difficult* to disclose information about agents' types under a submodular aftermarket: Indeed, submodularity implies that the direction of single-crossing is opposite to the one dictated by Bayesian updating. If beliefs are thought of as goods allocated by the mechanism, then submodularity of the aftermarket implies that high beliefs (beliefs that put more mass on higher types) must be "allocated" to lower types. Bayesian updating requires the opposite: On average, high beliefs must be associated with high types. This tension implies that an incentive-compatible mechanism can only disclose coarse information when the aftermarket is submodular.

An aftermarket is *strictly monotone* if $u_i(\theta; \bar{f})$ is strictly increasing in $\theta$ for any $\bar{f} \in \Theta_i$ and $i \in \mathcal{N}$. Strict monotonicity is satisfied in Example 2 (b) and (c) and holds for any original monotone aftermarket if we add an arbitrarily small probability that the aftermarket does not take place (in which case the agent keeps the good and receives a value equal to her type), as in Example 1 for any $\lambda < 1$.

### 5.2.3 Main result

Having defined a strengthening of dominant-strategy implementation and a restriction on the aftermarkets, I introduce a regularity condition on the mechanisms that allows me to derive the consequences of these concepts in a clean way.

**Definition 7.** A mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ is *regular* if for every $i \in \mathcal{N}$, the posterior beliefs $\{f_i^s\}_{s \in \mathcal{S}_i}$ over $i$'s type can be completely ranked in the likelihood ratio order.

The regularity condition is mathematically restrictive. However, three comments are in place. First, the regularity condition only restricts the reduced form of feasible mechanisms and thus it can be seen as an exogenous constraint. Importantly, given the upcoming result, regularity does not in itself rule out signals that directly reveal the winning agent's type. Second, regularity holds, for example, when the disclosure rule has an arbitrary monotone partitional structure in either the type of the agent or the cutoff. Monotone partitional signals are appealing from a practical perspective and received special attention in the theoretical literature.[28] Third, regularity automatically holds in the special case of a binary type space which is assumed in many papers studying optimal information design, including Calzolari and Pavan (2006a,b).

**Theorem 5.** *Suppose that the aftermarket is strictly monotone and strictly submodular. Then, any regular SDS implementable mechanism frame is payoff-equivalent to a cutoff rule.*

The assumption of *strict* monotonicity and submodularity can be dropped if the mechanism is robust to how the agents break ties between reports when they are indifferent.

**Definition 8.** A mechanism frame is *strict-dominant-strategy* implementable if there exists a transfer rule $\boldsymbol{t}$ such that for each $i$, $\boldsymbol{\theta}_{-i}$, agent $i$ *strictly* prefers to report her true type $\theta_i$ to reporting any other type that receives a different outcome:

$$\operatorname{argmax}_{\hat{\theta}_i} \sum_{s \in \mathcal{S}_i} u_i(\theta_i;\, f_i^s)\pi_i(s|\,\hat{\theta}_i,\, \boldsymbol{\theta}_{-i})x_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) - t_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) = \mathcal{I}_{\boldsymbol{\theta}_{-i}}(\theta_i),$$

where $\mathcal{I}_{\boldsymbol{\theta}_{-i}}(\theta_i) = \{\hat{\theta}_i : \forall s,\, \pi_i(s|\,\hat{\theta}_i,\, \boldsymbol{\theta}_{-i})x_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) = \pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i})\}$. A mechanism frame is *strongly strict-dominant-strategy* (SSDS) implementable if there exists a strict-dominant-strategy implementable deterministic decomposition of $(\boldsymbol{x}, \boldsymbol{\pi})$.

Strict dominant-strategy implementability requires that for some transfer rule each agent *strictly* prefers to receive the outcome that she obtains by reporting truthfully. This has no

[28] See Ivanov (2015), Dworczak and Martini (2018), Mensch (2018), Kolotilin and Zapechelnyuk (2018).

bite without the aftermarket: If different types receive the good with different probabilities, there exists a transfer rule that makes truthful reporting a *unique* dominant strategy.[29] Failure of this property implies that the mechanism relies on all types breaking the indifference in the direction preferred by the designer. In this sense, the property is equivalent to the designer considering the worst case over all dominant-strategy equilibria of the mechanism.

**Theorem 5'.** *Suppose that the aftermarket is monotone and submodular. Then, any regular SSDS implementable mechanism frame is payoff-equivalent to a cutoff rule.*

**Corollary 4.** *If $|\Theta_i| = 2$ for all $i$, then any SSDS mechanism followed by a monontone and submodular aftermarket is payoff-equivalent to a cutoff mechanism.*

The proofs of Theorems 5 and 5' can be found in Appendices A.5 and A.6. I show that incentive-compatibility of the mechanism implies that if two types receive the same allocation (conditional on some profile of types of other agents), then lower types must be assigned to signals that lead to higher posterior beliefs (in the likelihood-ratio order). This is a consequence of the single-crossing property in types and beliefs induced by a submodular aftermarket. On the other hand, Bayesian updating implies the opposite relationship between types and beliefs. The resulting conflict between incentive compatibility and Bayes plausibility limits the informativeness of signals that can be sent in a feasible mechanism. Information about the cutoff can always be disclosed (Theorem 1), and the proof demonstrates that this lower bound on informativeness is achieved.

Strong dominant-strategy implementation plays in important role in the proof because it allows me to apply the above reasoning for any player $i$, fixing the types of all other players, including the endogenous type of the mechanism designer. Under weaker solution concepts, it would be possible to use randomization in the mechanism in order to disclose additional information about the winner's type. For example, in the single-agent binary-types model of Calzolari and Pavan (2006a), the aftermarket is a resale game and is therefore submodular. However, using Bayesian implementation, Calzolari and Pavan show that in one of four cases it is optimal to use a non-cutoff mechanism (in particular, a non-cutoff mechanism is feasible). Corollary 4 implies that the incentives to report truthfully in their optimal mechanism crucially rely on providing a random outcome to the low type. If the agent did not trust the designer to correctly randomize, she would not report truthfully.

The assumption of a submodular aftermarket is crucial for the result. Under the opposite case of supermodularity (which is satisfied by the Cournot aftermarket), it is much easier

[29] Of course, such a transfer rule would not in general guarantee the same payoff to the designer. However, it could guarantee an arbitrarily close approximation of that payoff. While an optimal mechanism satisfying the strict-dominant-strategy property might not exist, this concept may be applied by requiring that an optimal mechanism is approximated by a sequence of mechanisms with this property.

to disclose information. In that case, the relationship between types and beliefs implied by incentive-compatibility and Bayes plausibilty is aligned: Higher types are associated with higher beliefs. It is thus possible to support truthful disclosure of the type by using transfers, even when all types receive the same allocation. Indeed, Goeree (2003) and Hu and Zhang (2017) show that the optimal mechanism for a Cournot aftermarket is to fully disclose the *type* of the winner. In those cases, as seen in Example 4, an optimal cutoff mechanism fully discloses the *cutoff*. Thus, restricting attention to cutoff mechanism is likely to yield a suboptimal mechanism when the aftermarket is supermodular.

## 6 Continuous distributions of types

In this section, I extend the definition of cutoff mechanisms to continuous type spaces. I then use the additional tractability of the continuous model to establish sufficient conditions for optimality of simple cutoff mechanisms. As an application, I characterize optimal cutoff mechanisms for a subclass of problems introduced by Example 2. The formal proofs of all results can be found in the Online Appendix.

### 6.1 Definitions

I assume that the product distribution of types is continuous, i.e., it admits a density $\boldsymbol{f}$ on some compact convex $\boldsymbol{\Theta}$. A mechanism $(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{t})$ is defined as before, except that it is assumed that all functions are measurable, and the signal spaces $\mathcal{S}_i$ are allowed to be arbitrary (possibly infinite) measurable spaces. I will equate mechanisms that differ on a measure-zero set of type profiles: $(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{t})$ and $(\boldsymbol{x}', \boldsymbol{\pi}', \boldsymbol{t}')$ are treated as the same mechanism if $\boldsymbol{x}(\boldsymbol{\theta}) = \boldsymbol{x}'(\boldsymbol{\theta})$, $\boldsymbol{\pi}(\cdot\,|\boldsymbol{\theta}) = \boldsymbol{\pi}'(\cdot\,|\boldsymbol{\theta})$, and $\boldsymbol{t}(\boldsymbol{\theta}) = \boldsymbol{t}'(\boldsymbol{\theta})$, for almost all $\boldsymbol{\theta}$. Consequently, all statements of the form "for all types" should be interpreted as "for almost all types", and profitable deviations are allowed for a measure-zero set of types of any agent.

The payoffs $u_i(\theta_i;\, \bar{f})$ and $V_i(\theta_i;\, \bar{f})$ are assumed to be bounded and measurable. $V_i(\theta_i;\, \bar{f})$ is additionally upper semi-continuous in $\bar{f}$ (in the weak$^\star$ topology), for any $i$.[30]

The definition of implementability remains identical, except that the sum operator $\sum_{s\in\mathcal{S}_i}$ is replaced by an integral $\int_{\mathcal{S}_i}$ with respect to the measure induced by $\pi_i(s|\,\cdot)$.

A cutoff mechanism is defined as follows. Suppose that the interim allocation rule $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ is non-decreasing in $\theta_i$ for any $\boldsymbol{\theta}_{-i}$. A non-decreasing function is continuous almost everywhere, and thus there exists a non-decreasing, right-continuous $x_i'(\theta_i, \boldsymbol{\theta}_{-i})$ which differs from $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ on a measure-zero set of types $\theta_i$. Because I equate mechanisms that

[30] The representation of posterior beliefs by a density $\bar{f}$ is only justified within the class of cutoff mechanisms; outside of the class, posterior beliefs might not be represented by a continuous distribution.

differ on measure-zero set of types, I can without loss of generality assume that $x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$ is right-continuous. Thus, $x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$ can be extended to a cumulative distribution function on $C_i = \Theta_i \cup \{\bar{\theta}_i\}$ by defining $x_i(\bar{\theta}_i,\, \boldsymbol{\theta}_{-i}) = 1$. The random variable defined by this cdf is the random-cutoff representation of the allocation rule $x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$. For any measurable function $g$ on $C_i$, $\int g(c)dx_i(c,\, \boldsymbol{\theta}_{-i})$ denotes the Lebesgue integral of $g$ with respect to the distribution of the cutoff induced by the allocation rule $x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$ on $C_i$.

**Definition 9** (Cutoff rule with continuous types)**.** A mechanism frame $(\boldsymbol{x},\, \boldsymbol{\pi})$ is a *cutoff rule* if $x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$ is non-decreasing in $\theta_i$ for all $\boldsymbol{\theta}_{-i}$, and $\pi_i$ can be represented as

$$\pi_i(S|\, \theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \gamma_i(S|\, c,\, \boldsymbol{\theta}_{-i})dx_i(c,\, \boldsymbol{\theta}_{-i}), \tag{6.1}$$

for some measurable signal function $\gamma_i : C_i \times \boldsymbol{\Theta}_{-i} \to \Delta(\mathcal{S}_i)$, for all $\theta_i \in \Theta_i$, $\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$, measurable $S \subset \mathcal{S}_i$, and $i \in \mathcal{N}$.

The only difference in the definition is that (6.1) must be expressed for all measurable subsets of $\mathcal{S}_i$ rather than for all elements $s \in \mathcal{S}_i$. Similarly, condition (M) becomes

$$\pi_i(S|\theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) \text{ is non-decreasing in } \theta_i \text{ for all measurable } S \in \mathcal{S}_i \text{ and all } \boldsymbol{\theta}_{-i} \in \Theta_{-i}. \tag{6.2}$$

With these definitions, all properties of cutoff mechanisms derived in Sections 3 and 4 continue to hold (see the Online Appendix for details).

## 6.2 Optimality of simple cutoff mechanisms

The continuous type space allows me to apply calculus to fully characterize the optimal cutoff mechanisms in some simple cases, under the assumption that the payoff in the aftermarket depends on the posterior belief only through its mean. This assumption is satisfied in Example 2 which I will use for illustration of the results derived below. To simplify exposition, I assume that agents are symmetric (hence drop the subscripts), and normalize $\Theta \equiv [0,\, 1]$.

Let $M(\bar{f}) \equiv \int_0^1 \theta \bar{f}(\theta)d\theta$, and assume that $\mathcal{W}(\bar{f}) = W(M(\bar{f}))$ for some function $W : [0,\, 1] \to \mathbb{R}_+$, where $\mathcal{W}$ is defined by (4.7). I also let $m(c) \equiv \int_c^1 \theta f(\theta)d\theta/(1 - F(c))$ denote the expected value of $\tilde{\theta}$ under the prior conditional on $\tilde{\theta} \geq c$, and let $w(c) \equiv W(m(c))$, for any $c \in [0,\, 1]$. Thus, $w(c)$ is the expected payoff to the mechanism designer conditional on allocating the good and inducing a belief that the type of the winner is above $c$.

**Proposition 4.** *Suppose that $f$ is a continuous density with cdf $F$, fully-supported on* $[0,\, 1]$.

1. *If $W$ is concave and non-decreasing, it is optimal to allocate the good to the highest type if it exceeds $r^\star$ (and to no one otherwise), and to reveal no information, where*

$$r^\star \in \underset{r\in[0,1]}{argmax} \left(1 - F^N(r)\right) W\left(\frac{\int_r^1 \theta dF^N(\theta)}{1-F^N(r)}\right). \tag{6.3}$$

2. *If $W$ is concave and decreasing, it is optimal to allocate the object uniformly at random and reveal no information.*

3. *Assume that $W$ is differentiable, and let $J_w(c) \equiv w(c) - w'(c)\frac{1-F(c)}{f(c)}$ . If (i) $W$ is convex, and (ii) $J_w(c)$ is non-positive for $c \leq \underline{r}$, and positive non-decreasing for $c \geq \underline{r}$, then it is optimal to allocate the good to the highest type if it exceeds $\underline{r}$ (and to no one otherwise), and to disclose the second highest type (if the second highest type is below $\underline{r}$, it is enough to announce that the second highest type was below $\underline{r}$). A sufficient condition for property (ii) is that $W$ is increasing and log-concave.*

If $W$ is concave and increasing, it is optimal not to disclose any information, and the allocation rule is designed to maximize the posterior expected type of the winner by allocating to the highest bidder. The mechanism can additionally raise the expectation by excluding types below $r$ from trading. This incurs a utility cost because the good is not always allocated. The $r^\star$ that solves equation (6.3) optimally trades-off these two effects.

Second, if $W$ is concave and decreasing, it is optimal to allocate the good randomly, with no disclosure. In this case, the designer wants to minimize the expectation of the type of the winner. However, it is not incentive-compatible to allocate to low types more often than to high types – hence the use of a uniform lottery.

Third, if $W$ is convex, full disclosure of the cutoff is optimal. The optimal allocation rule is determined by the properties of the function $J_w(c)$ which captures the local trade-off between allocative efficiency (as captured by the term $w(c)$) and the induced information structure (as captured by the term $-w'(c)(1 - F(c))/f(c)$. Allocating the good with smaller probability conditional on realization $c$ lowers surplus if $w(c)$ is positive but increases posterior beliefs over the winner's type conditional on allocating. The function $J_w(c)$ is similar to the virtual surplus function which captures the trade-off between allocative efficiency and information rents in the revenue-maximization problem. In the regular case, the virtual surplus function is increasing, and the seller does not introduce randomization to the revenue-maximizing mechanism. Analogously, if $J_w(c)$ is increasing, the designer does not use randomization in the allocation rule to optimally influence beliefs in the aftermarket.

I apply Proposition 4 to solve three examples based on the binary model of Example 2.

**Example 6.** Consider first the investment game (**case c**) from Example 2. Let $y(m) = \text{argmax}_y\{m\alpha(y) - y\}$ be the optimal investment for the incumbent when the expected type of the agent is $m$. Consider a designer who maximizes total surplus in the aftermarket. Because the value generated if the entrant is successful is split between the entrant and the incumbent, maximizing surplus is equivalent to minimizing the cost of the (socially wasteful) investment $y(m)$: $W(m) = m - y(m)$. By the envelope formula we have $W'(m) = 1 - \alpha(y(m)) \geq 0$, and $W''(m) = -\alpha'(y(m))y'(m) \leq 0$. Thus, $W$ is non-decreasing and concave. By Proposition 4 point *(1)*, the optimal mechanism in the first stage is an auction with a reserve price and no information disclosure.

Next, consider the Cournot model (**case b**) from Example 2. We already know from Example 4 that for any fixed allocation rule, it is optimal (for total surplus) to fully disclose the cutoff. Using Proposition 4, we can also pin down the optimal allocation rule. First, $W(m)$ is a convex, non-decreasing function on [0, 1]. Moreover, because $W(m)$ is a quadratic function, it is log-concave. It follows from point *(3)* of Proposition 4 that the optimal mechanism is to run a second price auction with some reserve price $r$ and reveal the price paid by the winner. The optimal reserve price is determined by the point at which the function $J_w(c)$ crosses zero. For example, when the prior distribution is uniform, $J_w(c)$ is a convex quadratic function whose both zeros are negative. Therefore, it is optimal to have no reserve price in the auction.

Finally, consider the resale model (**case a**) from Example 2. Suppose that $\lambda \geq (h - l)/(v - l)$, $N > 1$, and the designer maximizes total surplus. Proposition 4 is not applicable because the function $W(m)$ has a discontinuity at the posterior mean $(h-l)/(v-l)$ at which the third party is indifferent between quoting the high and the low price in the aftermarket. However, we can use the fact that the third party takes a binary action – implying that it is enough to send a binary signal in the mechanism. I describe the optimal mechanism informally below; the formal description and proof are relegated to the Online Appendix.

Under a regularity condition on the prior distribution, the surplus-maximizing cutoff mechanism admits the following indirect implementation: The designer runs a (second price) auction with a minimum bid $b^\star$, with the object allocated to one of the bidders uniformly at random if no one bids above $b^\star$. A binary signal, high or low, is sent. The low signal is sent for sure if no one bid above $b^\star$, and with probability $1/N$ conditional on the event that only the winner bid above $b^\star$ (the high signal is sent otherwise). The minimum bid $b^\star$ is set in such a way that conditional on announcing the high signal, the third party is indifferent between the high and low price (and quotes the high price).[31]

[31] This is a cutoff mechanism: Conditional on the second highest bid being below $b^\star$, the cutoff for the winner is equal to 0 with probability $1/N$ and $b^\star$ with probability $1 - 1/N$; the signal is low when the cutoff

Intuitively, the high signal reveals that the type of the winner is high to induce a high resale price in the aftermarket; however, regardless of the type of the winner, there is always a strictly positive probability that mechanism sends a low signal and hence induces a low resale price – this is necessary to prevent lower types from bidding high and reselling for sure at a high price. To understand the allocation rule, note that conditional on the event that no one exceeded the reserve price, a low resale price will be offered in the aftermarket with probability one. Thus, to maximize the probability of resale the designer should allocate to the lowest type. However, incentive-compatibility constraints make it impossible to allocate to low types more often than to high types. Therefore, the mechanism allocates the object by a uniform lottery in this case. ■

It follows from the analysis of Goeree (2003) and Hu and Zhang (2017) that with the Cournot aftermarket (case b) the optimal cutoff mechanism from Example 6 is *not* optimal overall – the optimal mechanism discloses the *type* of the agent rather than the cutoff.[32] Disclosing the type is incentive-compatible because the Cournot aftermarket is supermodular. On the other hand, restricting attention to the case when the aftermarket happens with probability one, I show in a companion paper Dworczak (2019) that the optimal cutoff mechanism for the resale aftermarket (case a) is optimal among all Bayesian implementable mechanisms. It is easy to show that the optimal cutoff mechanism for the investment-game aftermarket (case c) is optimal overall. This is because these aftermarkets are submodular: Such environments make information disclosure difficult, and hence disclosing the cutoff is sufficient for optimality.

# 7 Extensions

**Correlated types.** I ruled out correlation of types by assuming that the prior $\boldsymbol{f}$ is a product distribution. However, the implementation results of Sections 3 and 5 immediately extend to a more general case of an arbitrary joint distribution of types: Cutoff rules are defined in exactly the same way and they remain to be implementable for all (correlated) distributions of types as long as the aftermarket is monotone. Theorem 4 extends with virtually no change to the argument. As a consequence, we can conclude that implementability for all aftermarkets and prior distributions is sufficient to avoid the paradoxical result that all decision rules are implementable under correlated types (Crémer and McLean, 1988).

On the other hand, my results on optimality of mechanisms have limited scope when types

realization is 0.

[32] Goeree (2003) and Hu and Zhang (2017) focus on revenue but their results can be easily modified to apply to total surplus maximization.

are allowed to be correlated. It is no longer without loss of generality to exclude explicit transfers from the objective function of the designer, as in (2.2), because the allocation and disclosure rule do not pin down expected transfers. In fact, since cutoff rules only impose a restriction on the allocation and disclosure rule, transfers in a cutoff mechanism can be used for surplus extraction *a la* Crémer and McLean.

**Private signals and beliefs about losing agents' types.** Assuming a public signal and irrelevance of beliefs about losing agents' types allowed me to characterize the payoffs in the aftermarket as a function of a single posterior belief. Instead, I could assume that aftermarket payoffs depend on the *vector* of beliefs (one belief for each realization of a different private signal) about the entire type *profile*. Then, in a cutoff rule, the distributions of all these signals would be required to only depend on the cutoff and the losing agents' reports. With an analogous definition of monotonicity of the aftermarket, cutoff rules would still be the unique class satisfying implementability for all aftermarkets and prior distributions.

In terms of optimality, for a fixed allocation rule, the problem still reduces to pure information design but with a multi-dimensional state (the cutoff and losing agents' reports) and private signals. One can then apply tools such as the Bayes correlated equilibrium of Bergemann and Morris (2016a). Nevertheless, even within the class of cutoff mechanisms, such a design problem would not be tractable in general.

**Losing agents interacting in the aftermarket.** In many cases, agents that do not acquire the object in the mechanism may also engage in post-mechanism interactions. For example, a loser may purchase a similar object in the aftermarket, or negotiate to gain access to an object owned by another market participant. Allowing for the possibility that a loser also participates in the aftermarket is easy only in the case of a single agent: In the Online Appendix, I propose a model in which the agent participates in either the winner's or the loser's aftermarket, depending on the outcome in the mechanism – hence, a mechanism now includes a pair of signal functions. Because only one signal is sent ex-post, all results from the single-agent model generalize easily under an appropriate single-crossing assumption. The extension to multiple players is significantly more complicated and outside the scope of this paper.

**Indirect implementation.** All results of this paper are phrased in terms of direct mechanisms. However, it can be shown that cutoff rules admit a simple and intuitive indirect implementation via a dynamic procedure that I call a *generalized clock auction* (GCA): In every round the designer announces a price (according to a pre-specified, not necessarily monotone, price path), and agents simultaneously decide whether to stay or exit. The auction continues until there is one bidder remaining (who becomes the winner). Next, the designer publicly announces an arbitrary garbling of the bidding history (for example the final price or the set of active bidders in every round). In the Online Appendix, I show that under certain conditions, any monotone equilibrium of a GCA must be a cutoff rule, and conversely, every cutoff rule can be implemented as an equilibrium of some GCA.

# 8 Conclusions

In this paper, I studied mechanism design in a setting where the mechanism is followed by an aftermarket, i.e., a post-mechanism game played between the agent who acquired the object and third-party market participants. Existence of an exogenous aftermarket creates a new tool in the design problem – the disclosure rule. By disclosing information elicited by the mechanism, the designer influences the information structure of the aftermarket. I introduced a tractable class of cutoff rules that are characterized by being always implementable – regardless of the aftermarket and the prior distribution of types. Under a strong notion of implementability and regularity, cutoff rules coincide with the set of feasible outcomes in cases when the aftermarket satisfies a submodularity condition.

The approach taken to mechanism design in this paper is non-standard. Instead of looking for the optimal mechanism that can depend on fine details of the model, I proposed a class of allocation and disclosure rule with a certain robustness property (implementability in the "worst case"). Within the class, the designer maximizes a Bayesian objective function – this distinguishes this approach from studies that look for the mechanism with the highest payoff guarantee (optimality in the "worst case"). An interesting direction for future research is to apply this approach to other design problems.

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# A Proofs

## A.1 Proof of Theorem 1

To formally prove the theorem, I use a condition for implementability in arbitrary type and allocation spaces from Dworczak and Zhang (2017) which is a version of Rochet (1987)'s classic cyclic monotonicity condition: Given a set of types and their final allocations, the assignment is implementable if and only if the matching between types and final allocations is efficient (see Dworczak and Zhang for a formal definition[33]). A monotone aftermarket guarantees that for any $s \in \mathcal{S}_i$ the payoff of each agent is non-decreasing in her type. Thus, matching efficiency is implied by pairwise stability – it is enough to show that joint surplus cannot be increased by swapping the allocations of any pair of types. This amounts to checking that for any $i$, $\boldsymbol{\theta}_{-i}$, and $\theta_i > \hat{\theta}_i$,

$$\sum_{s \in \mathcal{S}_i} \Big[ u_i(\theta_i;\, f_i^s) - u_i(\hat{\theta}_i;\, f_i^s) \Big] \Big[ \pi_i(s|\theta_i,\, \boldsymbol{\theta}_{-i}) x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) - \pi_i(s|\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) x_i(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i}) \Big] \geq 0.$$

[33] Although Dworczak and Zhang consider single-agent mechanisms, checking DS implementability in a model with multiple agents boils down to checking conditions ($\boldsymbol{IR}$) and ($\boldsymbol{IC}$) for any fixed $\boldsymbol{\theta}_{-i}$.

This holds because both bracketed terms are non-negative – the first one by monotonicity of the aftermarket, and the second one by condition (M).

## A.2 Proof of Lemma 1

Consider the problem of maximizing

$$\sum_{\theta\in\Theta}\sum_{s\in\mathcal{S}} V(\theta;\, f^s)\pi(s|\,\theta)x(\theta)f(\theta)$$

over $\pi$ subject to $(x,\,\pi)$ being a cutoff rule. By definition of a cutoff rule, there exists a function $\gamma : C \to \Delta(\mathcal{S})$ such that $\pi(s|\,\theta)x(\theta) = \sum_{c\le\theta}\gamma(s|\,c)\Delta x(c)$. Thus, the problem becomes

$$\begin{aligned}\max_{\gamma}&\sum_{\theta\in\Theta}\sum_{s\in\mathcal{S}} V(\theta;\, f^s)\sum_{c\le\theta}\gamma(s|\,c)\Delta x(c)f(\theta)\\ &= \max_{\gamma}\sum_{s\in\mathcal{S}}\underbrace{\left(\sum_{c}\gamma(s|\,c)\Delta x(c)\right)}_{\varsigma_s}\sum_{\theta\in\Theta}V(\theta;\, f^s)\underbrace{\left(\frac{\sum_{c\le\theta}\gamma(s|\,c)\Delta x(c)}{\sum_{c}\gamma(s|\,c)\Delta x(c)}\right)}_{G^s(\theta)}f(\theta). \quad \text{(A.1)}\end{aligned}$$

In the above expression, $\varsigma_s$ is the unconditional probability of sending signal $s$, and the remaining expression is equal to $\mathcal{V}(G^s)$, as defined in (4.2), where $G^s$ is the posterior cdf of the cutoff conditional on signal $s$. Thus, the objective function can be written as $\mathbb{E}_{s\sim\varsigma}\mathcal{V}(G^s)$. To confirm that $\mathcal{V}$ depends solely on the posterior belief over the cutoff, note that $f^s = f^{G^s}$ by (2.1) and (4.1), so that

$$\mathcal{V}(G^s) = \mathbb{E}_{\tilde{c}\sim G^s}\sum_{\theta\in\Theta}V(\theta;\, f^{G^s})\mathbf{1}_{\{\theta\ge\tilde{c}\}}f(\theta).$$

Thus, the problem is formally equivalent to the Bayesian persuasion problem of Kamenica and Gentzkow (2011). Instead of optimizing over distributions $\varsigma$ of signals, we can optimize over distributions of posterior beliefs $\varrho\in\Delta(\Delta(C))$ subject to a Bayes-plausibility constraint. This yields equations (4.3) and (4.4). Equation (4.4) is the Bayes-plausibility constraint on posterior beliefs over the cutoff expressed in terms of cdfs.

## A.3 Proof of Theorem 3

Given $(\boldsymbol{x},\, \boldsymbol{\pi})$, its *reduced form* under distribution $\boldsymbol{f}$, denoted $(\boldsymbol{x}^{\boldsymbol{f}},\, \boldsymbol{\pi}^{\boldsymbol{f}})$, is defined by

$$x_i^{\boldsymbol{f}}(\theta_i) = \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} x_i(\theta_i,\, \boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}),$$

$$\pi_i^{\boldsymbol{f}}(s|\,\theta_i)x_i^{\boldsymbol{f}}(\theta_i) = \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} \pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}),$$

for all $s \in \mathcal{S}_i$, $\theta_i \in \Theta_i$, and $i \in \mathcal{N}$.

The designer's and the agents' expected payoffs, as well as the posterior beliefs $f_i^s$, depend only on the reduced form of a mechanism (see equations 2.1 and 2.2). However, the definition of a cutoff rule relies on properties of $i$'s allocation and disclosure rule that hold conditional on any given profile of other agents' reports $\boldsymbol{\theta}_{-i}$. To work with reduced forms, I must know which reduced forms correspond to cutoff rules. The lemma below answers this question.

**Lemma 2.** *A pair $(\bar{\boldsymbol{x}},\, \bar{\boldsymbol{\pi}})$, where $\bar{x}_i : \Theta_i \to [0,\, 1]$ and $\bar{\pi}_i : \Theta_i \to \Delta(\mathcal{S}_i)$, for all $i$, is a reduced form of a cutoff rule under prior distribution $\boldsymbol{f}$ if and only if,*

1. *The interim allocation rule $\bar{x}_i(\theta_i)$ is non-decreasing in $\theta_i$, for all $i \in \mathcal{N}$;*
2. *The interim signal function $\bar{\pi}_i$ can be represented as*

$$\bar{\pi}_i(s|\,\theta_i)\bar{x}_i(\theta_i) = \sum_{c\le\theta_i} \gamma_i(s|\,c)\Delta\bar{x}_i(c), \tag{A.2}$$

*for some signal function $\gamma_i : C_i \to \Delta(\mathcal{S}_i)$, for all $i \in \mathcal{N}$, $\theta_i$, and $s \in \mathcal{S}_i$;*

3. *Interim expected allocation rules are jointly feasible under $\boldsymbol{f}$:*

$$\sum_{i\in\mathcal{N}}\sum_{\theta_i>\tau_i} \bar{x}_i(\theta_i)f_i(\theta_i) \le 1 - \prod_{i\in\mathcal{N}} F_i(\tau_i),\ \forall\boldsymbol{\tau} \in \mathbb{R}^N. \tag{M-B}$$

*Proof of Lemma 2.* **"Only if":** In this part of the proof, I show that a reduced form $(\boldsymbol{x}^{\boldsymbol{f}},\, \boldsymbol{\pi}^{\boldsymbol{f}})$ of any cutoff rule $(\boldsymbol{x},\, \boldsymbol{\pi})$ satisfies conditions *1-3*. Condition *1* holds because $x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$ is non-decreasing in $\theta_i$ for every $\boldsymbol{\theta}_{-i}$, and thus also when expectation is taken with respect to $\boldsymbol{\theta}_{-i}$. Condition *3* must hold whenever $\boldsymbol{x}$ is feasible, $\sum_{i\in\mathcal{N}} x_i(\boldsymbol{\theta}) \le 1$ for all $\boldsymbol{\theta}$; indeed Border (2007) (Theorem 3) and Mierendorff (2011) (Theorems 2 and 3) show that the interim expected allocation rules must satisfy the generalized (asymmetric) Matthews-Border (M-B) in this case. Finally, to show that condition *2* holds as well, notice that since $\pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i})$ is non-decreasing in $\theta_i$ for each $\boldsymbol{\theta}_{-i}$ (by definition of cutoff rules), $\pi_i^{\boldsymbol{f}}(s|\,\theta_i)x_i^{\boldsymbol{f}}(\theta_i)$ is also non-decreasing in $\theta_i$, for any $s \in \mathcal{S}_i$. A reduced form can be formally treated as a single-agent mechanism since $x_i^{\boldsymbol{f}}$ and $\pi_i^{\boldsymbol{f}}$ are mappings from individual type spaces $\Theta_i$ into allocations and signals, respectively. It follows from Proposition 1 from Section 3 that $(x_i^{\boldsymbol{f}},\, \pi_i^{\boldsymbol{f}})$, viewed as a single-agent mechanism, is a cutoff rule, and in particular satisfies condition 2 of Lemma 2.

**"If":** Given a reduced form $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\pi}})$ satisfying conditions *1 – 3*, I will show existence of a cutoff rule $(\boldsymbol{x}, \boldsymbol{\pi})$ such that $(\boldsymbol{x^f}, \boldsymbol{\pi^f}) = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{\pi}})$. By Theorems 2 and 3 in Mierendorff (2011), condition (M-B) (along with the fact that each $\bar{x}_i$ is monotone) implies that there exists a joint allocation rule $\boldsymbol{x}$ such that $\boldsymbol{x^f} = \bar{\boldsymbol{x}}$. Define $\boldsymbol{\pi} : \boldsymbol{\Theta} \to \times_{i\in\mathcal{N}}\Delta(\mathcal{S}_i)$ by

$$\pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i}) = \bar{\pi}_i(s|\,\theta_i),$$

for all $s \in \mathcal{S}_i, \theta_i \in \Theta_i, \boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$, $i \in \mathcal{N}$. Then, $(\boldsymbol{x}, \boldsymbol{\pi})$ is a well-defined mechanism frame such that $(\boldsymbol{x^f}, \boldsymbol{\pi^f}) = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{\pi}})$. The goal is to modify $(\boldsymbol{x}, \boldsymbol{\pi})$ in order to obtain a cutoff rule $(\boldsymbol{x}^\star, \boldsymbol{\pi}^\star)$ that induces the same reduced-form. Intuitively, this modification is closely analogous to how a Bayesian IC mechanism can be modified to produce a payoff-equivalent dominant-strategy IC mechanism, in an approach pioneered by Manelli and Vincent (2010) and developed further by Gershkov *et al.* (2013).[34]

To apply the results of Gershkov *et al.* (2013), I introduce the following notation. Let $\mathcal{K} = (\mathcal{N} \cup \{0\}) \times (\bigcup_i \mathcal{S}_i)$ be the set of social alternatives, where an outcome $k = (i,\, s)$ is interpreted as player $i$ getting the object ($i = 0$ denotes the mechanism designer) and signal $s$ being sent. An allocation rule in this setting is defined as an element of the set $\mathcal{Y} = \{\{y^{i,\,s}\} : y^{i,\,s}(\boldsymbol{\theta}) \geq 0, \forall (i,\, s) \in \mathcal{K}, \sum_{i\in\mathcal{N},\, s\in\mathcal{S}_i} y^{i,\,s}(\boldsymbol{\theta}) \leq 1, \forall \boldsymbol{\theta}\}$, where $\{y^{i,\,s}\}$ is a shorthand notation for $\{y^{i,\,s} : i \in \mathcal{N} \cup \{0\},\, s \in \mathcal{S}_i\}$. That is, $y^{i,\,s}(\boldsymbol{\theta})$ is the probability of implementing outcome $(i,\, s)$ conditional on type profile $\boldsymbol{\theta}$. Define an allocation rule

$$x^{i,\,s}(\boldsymbol{\theta}) = \pi_i(s|\,\theta_i,\, \boldsymbol{\theta}_{-i})x_i(\theta_i,\, \boldsymbol{\theta}_{-i}),$$

for all $i \in \mathcal{N}$, and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, as the probability that outcome $\{i,\, s\}$ is implemented in the mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ (where $x^0$ is defined as the residual probability). Clearly, $\{x^{i,\,s}\} \in \mathcal{Y}$. The following result follows directly from Lemma 3 in Gershkov *et al.* (2013).

**Lemma 3** (Gershkov, Goeree, Kushnir, Moldovanu and Shi, 2013)**.** *Suppose that for allocation* $\{x^{i,\,s}\}$, $\sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} x^{i,\,s}(\theta_i,\, \boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i})$ *is non-decreasing in* $\theta_i$*, for all* $i \in \mathcal{N}$, $s \in \mathcal{S}_i$*. Define* $\{y^{i,\,s}\}$ *as the solution to the problem (the solution always exists):*

$$\min_{\{y^{i,\,s}\}\in\mathcal{D}} \sum_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \sum_{i\in\mathcal{N},\, s\in\mathcal{S}_i} (y^{i,\,s}(\boldsymbol{\theta}))^2, \ \textit{where}$$

$$\mathcal{D} = \left\{ \{y^{i,\,s}\} \in \mathcal{Y} : \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} y^{i,\,s}(\theta_i,\, \boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}) = \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} x^{i,\,s}(\theta_i,\, \boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}), \forall i,\, \theta_i,\, s \right\}.$$

*Then,* $y^{i,\,s}(\theta_i,\, \boldsymbol{\theta}_{-i})$ *is non-decreasing in* $\theta_i$*, for all* $\boldsymbol{\theta}_{-i}$*, and all* $i \in \mathcal{N}$, $s \in \mathcal{S}_i$*.*

[34] I use the proof technique of Gershkov *et al.* (2013) but not their main result which is stated in terms of interim expected utilities: Because in my problem the allocation rule is monotone not only for every $i \in \mathcal{N}$ but also for all $s \in \mathcal{S}_i$, I am able to show the equivalence in terms of interim expected allocations.

The allocation function $\{x^{i,s}\}$ satisfies the assumption of Lemma 3 because

$$\sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} x^{i,s}(\theta_i,\,\boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}) = \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} \bar{\pi}_i(s|\,\theta_i)x_i(\theta_i,\,\boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}) = \bar{\pi}_i(s|\,\theta_i)\bar{x}_i(\theta_i),$$

and the last expression is non-decreasing in $\theta_i$ because $(\bar{\boldsymbol{x}},\,\bar{\boldsymbol{\pi}})$ satisfies condition *2* in Lemma 2 (which clearly implies monotonicity). Given the allocation $\{y^{i,s}\}$ produced from $\{x^{i,s}\}$ by Lemma 3, I now define a mechanism $(\boldsymbol{x}^\star,\,\boldsymbol{\pi}^\star)$ by

$$x_i^\star(\boldsymbol{\theta}) = \sum_{s\in\mathcal{S}_i} y^{i,s}(\boldsymbol{\theta}),$$

$$\pi_i^\star(s|\,\boldsymbol{\theta}) = \frac{y^{i,s}(\boldsymbol{\theta})}{x_i^\star(\boldsymbol{\theta})},$$

with $\pi_i^\star(s|\,\boldsymbol{\theta})$ defined in an arbitrary way for $x_i^\star(\boldsymbol{\theta}) = 0$.

To show that $(\boldsymbol{x}^\star,\,\boldsymbol{\pi}^\star)$ is a cutoff rule it is enough to invoke Proposition 1 found in Section 3 – because $\pi_i^\star(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i^\star(\theta_i,\,\boldsymbol{\theta}_{-i}) \equiv y^{i,s}(\theta_i,\,\boldsymbol{\theta}_{-i})$ is non-decreasing in $\theta_i$, for all $s\in\mathcal{S}_i$ and $\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}$, it must be a cutoff rule.

Finally, $(\boldsymbol{x}^{\star f},\,\boldsymbol{\pi}^{\star^f}) = (\bar{\boldsymbol{x}},\,\bar{\boldsymbol{\pi}})$ follows from the fact that $\{y^{i,s}\}\in\mathcal{D}$:

$$\sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} \pi_i^\star(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i^\star(\theta_i,\,\boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}) = \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} y^{i,s}(\theta_i,\,\boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i})$$
$$= \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} x^{i,s}(\theta_i,\,\boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}) = \sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} x_i(\theta_i,\,\boldsymbol{\theta}_{-i})\pi_i(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}),$$

for all $i$, $s$, and $\theta_i$. The same calculation can be done for $\boldsymbol{x}^\star$ by summing over $s$. □

I call a reduced-form mechanism satisfying conditions *1-3* of Lemma 2 a *reduced-form cutoff rule*. By Lemma 2, optimization over cutoff mechanisms can be performed in the space of reduced-form cutoff mechanisms. For a fixed allocation $\boldsymbol{x}$ and distribution $\boldsymbol{f}$, a reduced-form cutoff mechanism for agent $i$ is formally equivalent to a single-agent cutoff mechanism from Subsection 4.1. Moreover, the disclosure problem for any agent $i$ can be solved independently from the disclosure problems for all other agents $j \neq i$ because ex-post there is only one participant in the aftermarket. Thus, we can use the proof of Lemma 1 to establish the following result.

**Lemma 4.** *For every non-decreasing allocation rule $\boldsymbol{x}$, the problem of maximizing (2.2) over $\boldsymbol{\pi}$ subject to $(\boldsymbol{x},\,\boldsymbol{\pi})$ being a cutoff rule is equivalent to solving, for every $i\in\mathcal{N}$,*

$$\max_{\varrho_i\in\Delta(\Delta(C_i))} \mathbb{E}_{G_i\sim\varrho_i}\mathcal{V}_i(G_i) \tag{A.3}$$

*subject to*

$$\mathbb{E}_{G_i\sim\varrho_i}G_i(\theta_i) = x_i^{\boldsymbol{f}}(\theta_i),\ \forall\theta_i\in\Theta_i. \tag{A.4}$$

Applying Corollary 2 in Kamenica and Gentzkow (2011), I obtain the concave-closure characterization of the optimal payoff.

**Corollary 5.** *The maximal expected payoff to the mechanism designer in the problem (A.3)-(A.4) is equal to*

$$\sum_{i\in\mathcal{N}} co\mathcal{V}_i(x_i^{\boldsymbol{f}}) \equiv \sum_{i\in\mathcal{N}} \sup\{y:\ (x_i^{\boldsymbol{f}},\, y) \in CH(graph(\mathcal{V}_i))\},$$

*where* $graph(\mathcal{V}_i) \equiv \{(\bar{x}_i,\, \bar{y}) \in \mathcal{X}_i \times \mathbb{R} :\ \bar{y} = \mathcal{V}_i(\bar{x}_i)\}$.

Theorem 3 follows directly from Lemma 2, Lemma 4, and Corollary 5.

## A.4 Proof of Proposition 2 and supplementary material for Subsection 4.2.1

In this appendix, I formalize the result stated in Subsection 4.2.1 about feasible distributions of posterior beliefs over the winner's type induced by cutoff mechanisms, and prove Proposition 2.

For a fixed (interim) allocation rule $\bar{x}_i$, I call $f_i^{\bar{x}_i}$, defined by (4.1), the *no-communication posterior*:

$$f_i^{\bar{x}_i}(\theta) = \frac{\bar{x}_i(\theta) f_i(\theta)}{\sum_{\tau\in\Theta_i} \bar{x}_i(\tau) f_i(\tau)}, \forall\, \theta \in \Theta_i.$$

The no-communication posterior is the belief over the type of the winner held by the third party when the interim allocation rule is $\bar{x}_i$, and the mechanism reveals no information (other than the identity of the winner). Recall that a pmf $g$ *likelihood-ratio dominates* a full-support pmf $f$ (denoted $g \succ^{LR} f$) if $g(\theta)/f(\theta)$ is non-decreasing.

**Lemma 5.** *A distribution of beliefs $\eta_i \in \Delta(\Delta(\Theta_i))$ over $i$'s type conditional on $i$ being the winner is induced by a cutoff mechanism with interim allocation rule $\bar{x}_i$ if and only if*

$$\bar{f}_i \succ^{LR} f_i,\ \forall \bar{f}_i \in supp(\eta_i) \tag{A.5}$$

*and*

$$\mathbb{E}_{\bar{f}_i\sim\eta_i} \bar{f}_i(\theta) \equiv \int_{supp(\eta_i)} \bar{f}_i(\theta) d\eta_i(\bar{f}_i) = f_i^{\bar{x}_i}(\theta),\ \forall \theta \in \Theta_i. \tag{A.6}$$

Condition (A.6) is the standard Bayes-plausibility constraint, except that posterior beliefs must average out to the no-communication posterior, instead of the prior. This is because the distribution of beliefs is conditional on agent $i$ being the winner. Condition (A.5) is an additional constraint on posterior belief – each posterior must LR dominate the prior.

*Proof of Lemma 5.* Because the Lemma is stated for some fixed $i$, I drop the subscript $i$ to simplify notation. I first show that every ex-ante (unconditional) distribution $\varrho \in \Delta(\Delta(C))$

of beliefs over the cutoff for some agent $i$ that is feasible under allocation $\bar{x}$ defines a posterior (conditional) distribution $\eta \in \Delta(\Delta(\Theta))$ of beliefs over $i$'s type conditional on $i$ being the winner that satisfies conditions (A.5)-(A.6).

Because $\varrho$ is a feasible distribution of beliefs over the cutoff, it satisfies the Bayes-plausibility condition (see equations (4.4) and (A.4)) which states that

$$\mathbb{E}_{G\sim\varrho}G(\theta) = \bar{x}(\theta),\ \forall\theta \in \Theta. \tag{A.7}$$

For every $G \in \text{supp}(\varrho)$, let $f^G$, defined by (4.1), be the corresponding posterior belief over the type of the winner. Each $f^G$ satisfies condition (A.5) because $G$ is a non-decreasing function. Given the ex-ante distribution $\varrho$ for agent $i$, define the posterior distribution $\bar{\varrho}$ conditional on $i$ being the winner:

$$\bar{\varrho}(\mathcal{G}) = \frac{\int_{\mathcal{G}} \left(\sum_\Theta G(\theta)f(\theta)\right) d\varrho(G)}{\int_{\text{supp}(\varrho)} \left(\sum_\Theta G(\theta)f(\theta)\right) d\varrho(G)},\ \text{ for any measurable } \mathcal{G} \subseteq \Delta(\Delta(C)). \tag{A.8}$$

Conditional on $i$ becoming the winner, there is higher probability that the cutoff for $i$ was drawn from a distribution that puts relatively more mass on low cutoff realizations. That is why the posterior distribution $\bar{\varrho}$ puts more weight on distributions $G$ that allocate the good with higher probability. Define the corresponding posterior distribution $\eta \in \Delta(\Delta(\Theta))$ of beliefs over the type of the winner by

$$\eta(\mathcal{F}) = \bar{\varrho}\left(\{G \in \Delta(\Delta(C)) : f^G \in \mathcal{F}\}\right),\ \text{for any measurable } \mathcal{F} \subseteq \Delta(\Delta(\Theta)).$$

To show that condition (A.6) holds, note that because $\varrho$ is a feasible ex-ante distribution, it satisfies condition (A.7), and hence

$$\int_{\text{supp}(\varrho)} \left(\sum_\Theta \hat{G}(\theta)f(\theta)\right) d\varrho(\hat{G}) = \sum_\Theta \left(\int_{\text{supp}(\varrho)} \hat{G}(\theta)d\varrho(\hat{G})\right) f(\theta) = \sum_\Theta \bar{x}(\theta)f(\theta).$$

Then, we have

$$\begin{aligned} \int_{\text{supp}(\eta)} \bar{f}(\theta)d\eta(\bar{f}) &= \int_{\text{supp}(\varrho)} f^G(\theta)d\bar{\varrho}(G) \\ = \int_{\text{supp}(\varrho)} \frac{G(\theta)f(\theta)}{\sum_\Theta G(\tau)f(\tau)} &\frac{\sum_\Theta G(\tau)f(\tau)}{\int_{\text{supp}(\varrho)} \left(\sum_\Theta \hat{G}(\tau)f(\tau)\right) d\varrho(\hat{G})} d\varrho(G) \\ = \frac{\left(\int_{\text{supp}(\varrho)} G(\theta)d\varrho(G)\right) f(\theta)}{\sum_\Theta \bar{x}(\tau)f(\tau)} &= \frac{\bar{x}(\theta)f(\theta)}{\sum_\Theta \bar{x}(\tau)f(\tau)} = f^{\bar{x}}(\theta), \end{aligned}$$

which is condition (A.6).

To show the opposite direction, start with a conditional distribution of posterior beliefs over the winner's type $\eta \in \Delta(\Delta(\Theta))$, satisfying conditions (A.5) and (A.6) for a non-decreasing allocation rule $\bar{x}$. I will define a feasible ex-ante (unconditional) distribution of beliefs over the cutoff $\varrho \in \Delta(\Delta(C))$ such that $\varrho$ induces $\eta$.

First, for each $\bar{f} \in \text{supp}(\eta)$, define

$$G^{\bar{f}}(\theta) := \left(\bar{x}(\bar{\theta})\frac{f(\bar{\theta})}{\bar{f}(\bar{\theta})}\right)\frac{\bar{f}(\theta)}{f(\theta)}, \forall \theta \in \Theta, \tag{A.9}$$

where $\bar{\theta} = \max\{\Theta\}$. Because $\bar{f}$ likelihood-ratio dominates $f$, the function $G^{\bar{f}}(\theta)$ is non-decreasing and bounded above by 1 on $\Theta$. Thus, it defines a non-decreasing allocation rule, and hence also a cdf of the corresponding distribution of the cutoff (after extending it to $C$). Define a distribution $\varrho \in \Delta(\Delta(C))$ supported on $\{G^{\bar{f}} : \bar{f} \in \text{supp}(\eta)\}$ and defined by

$$\varrho(\{G^{\bar{f}} : \bar{f} \in \mathcal{F}\}) = \frac{\int_{\mathcal{F}} \bar{f}(\bar{\theta})d\eta(\bar{f})}{\int_{\text{supp}(\eta)} \bar{f}(\bar{\theta})d\eta(\bar{f})}, \text{ for any measurable } \mathcal{F} \subseteq \Delta(\Delta(\Theta)). \tag{A.10}$$

With this, I have to verify that $\varrho$ is feasible, i.e., it satisfies (A.7), and induces $\eta$. We have

$$\begin{aligned}\int_{\text{supp}(\varrho)} G^{\bar{f}}(\theta)d\varrho(G^{\bar{f}}) &= \int_{\text{supp}(\eta)} \left(\bar{x}(\bar{\theta})\frac{f(\bar{\theta})}{\bar{f}(\bar{\theta})}\right)\frac{\bar{f}(\theta)}{f(\theta)}\frac{\bar{f}(\bar{\theta})}{\int_{\text{supp}(\eta)} \hat{f}(\bar{\theta})d\eta(\hat{f})}d\eta(\bar{f}) \\ &= \int_{\text{supp}(\eta)} \frac{\bar{f}(\theta)}{f(\theta)}\left(\sum_{\tau\in\Theta}\bar{x}(\tau)f(\tau)\right)d\eta(\bar{f}) = f^{\bar{x}}(\theta)\frac{1}{f(\theta)}\left(\sum_{\tau\in\Theta}\bar{x}(\tau)f(\tau)\right) = \bar{x}(\theta),\end{aligned} \tag{A.11}$$

where I have used condition (A.6) twice.

To show that $\varrho$ induces $\eta$, note that $f^{G^{\bar{f}}} = \bar{f}$. Moreover, using (A.8) and (A.10), the posterior distribution (conditional on the agent being the winner) over $G^{\bar{f}}$ is given by, for any measurable $\mathcal{F} \in \Delta(\Delta(\Theta))$,

$$\bar{\varrho}(\{G^{\bar{f}} : \bar{f} \in \mathcal{F}\}) = \frac{\int_{\mathcal{F}} \left(\sum_{\Theta} G^{\bar{f}}(\theta)f(\theta)\right)\bar{f}(\bar{\theta})d\eta(\bar{f})}{\int_{\text{supp}(\eta)} \left(\sum_{\Theta} G^{\bar{f}}(\theta)f(\theta)\right)\bar{f}(\bar{\theta})d\eta(\bar{f})} = \int_{\mathcal{F}} d\eta(\bar{f}) = \eta(\mathcal{F})$$

which shows that $\varrho$ induces the posterior distribution $\eta$ over the winner's type. □

### A.4.1 Proof of Proposition 2

The proof follows from Lemma 5 and Lemma 4 found in Appendix A.3. Fixing an agent $i$, I drop the subscripts $i$ to simplify notation. Starting from the objective function (A.3), interim allocation rule $\bar{x}$, and a feasible ex-ante distribution $\varrho$ of beliefs over $i$'s cutoff,

$$\begin{aligned}\mathbb{E}_{G\sim\varrho}\mathcal{V}(G) &\overset{(1)}{=} \int_{\text{supp}(\eta)} \mathcal{V}(G^{\bar{f}}) \frac{\bar{f}(\bar{\theta})}{\int_{\text{supp}(\eta)} \hat{f}(\bar{\theta}) d\eta(\hat{f})} d\eta(\bar{f}) \\ &\overset{(2)}{=} \int_{\text{supp}(\eta)} \left(\sum_{\theta\in\Theta} V(\theta;\, f^{G^{\bar{f}}}) G^{\bar{f}}(\theta) f(\theta)\right) \frac{\bar{f}(\bar{\theta})}{f^{\bar{x}}(\bar{\theta})} d\eta(\bar{f}) \\ &\overset{(3)}{=} \left(\sum_{\theta\in\Theta} \bar{x}(\theta) f(\theta)\right) \int_{\text{supp}(\eta)} \underbrace{\sum_{\theta\in\Theta} V(\theta;\, \bar{f}) \bar{f}(\theta)}_{\mathcal{W}(\bar{f})} d\eta(\bar{f}),\end{aligned}$$

where (1) follows from the proof of Lemma 5 and specifically (A.10), where $G^{\bar{f}}$ is defined in (A.9), (2) uses definitions (4.1) and (4.2), and (3) uses the definition of $\mathcal{W}$ and $f^{\bar{x}}$, in particular $f^{G^{\bar{f}}} = \bar{f}$. This proves that the objective function can be written as

$$\left(\sum_{\theta\in\Theta} \bar{x}(\theta) f(\theta)\right) \mathbb{E}_{\bar{f}\sim\eta} \mathcal{W}(\bar{f}),$$

where feasible $\eta$ satisfy conditions (A.5) and (A.6), by Lemma 5. Given this representation of the objective function and Lemma 5, the concave closure characterization follows immediately.

## A.5 Proof of Theorem 5

Consider a regular mechanism frame $(\boldsymbol{x}, \boldsymbol{\pi})$ that is SDS implementable. Fix agent $i \in \mathcal{N}$. We can assume without loss of generality that any two distinct signal realizations $s, \hat{s} \in \mathcal{S}_i$ sent conditional on agent $i$ being the winner induce posterior beliefs $f_i^s$ and $f_i^{\hat{s}}$ that lead to different payoffs for some type of agent $i$: There exists $\theta_i$ such that $u_i(\theta_i;\, f_i^s) \neq u_i(\theta_i;\, f_i^{\hat{s}})$. If this was not the case, we could merge signals $s$ and $\hat{s}$ without affecting the payoffs of any of the players (including the designer). Indeed, this follows from the assumption that whenever $u_i(\theta;\, \bar{f}) = u_i(\theta;\, \bar{g})$ for all $\theta \in \Theta_i$, then any convex combination of $\bar{f}$ and $\bar{g}$ yields the same aftermarket payoff to agent $i$ and the designer. By the regularity of the mechanism frame, we can assume that for any $s, \hat{s} \in \mathcal{S}_i$ (sent with positive probability), $s > \hat{s}$ implies that $f_i^s \succeq^{LR} f_i^{\hat{s}}$. That is, we can order signals in $\mathcal{S}_i$ in such a way that higher signals $s$ induce higher (in the sense of LR order) posterior beliefs over agent $i$'s type conditional on $i$ being the winner.

An SDS-IC mechanism can be represented as a randomization over mechanisms that are deterministic and DS-IC: For some measurable space $\Theta_0$ and cdf $F_0 \in \Delta(\Theta_0)$,

$$\pi_i(s|\, \theta_i,\, \boldsymbol{\theta}_{-i}) x_i(\theta_i,\, \boldsymbol{\theta}_{-i}) = \int_{\Theta_0} \hat{\pi}_i(s|\, \theta_i,\, \boldsymbol{\theta}_{-i};\, \theta_0) \hat{x}_i(\theta_i,\, \boldsymbol{\theta}_{-i};\, \theta_0) dF_0(\theta_0), \tag{A.12}$$

where $(\hat{\boldsymbol{x}}(\cdot\,;\,\theta_0),\,\hat{\boldsymbol{\pi}}(\cdot\,;\,\theta_0))$ is a deterministic DS-IC mechanism for any $\theta_0 \in \Theta_0$.

The following lemma establishes key properties of deterministic DS-IC mechanisms under a strictly monotone and submodular aftermarket.

**Lemma 6.** *If the aftermarket is strictly monotone, then* $\hat{x}_i(\theta_i,\,\boldsymbol{\theta}_{-i};\,\theta_0) = \mathbf{1}_{\{\theta_i \geq \theta_i^\star(\boldsymbol{\theta}_{-i},\theta_0)\}}$ *for some* $\theta_i^\star : \boldsymbol{\Theta}_{-i} \times \Theta_0 \to \Theta_i$*. If additionally the aftermarket is strictly submodular, then* $\hat{\pi}_i(s|\theta_i,\,\boldsymbol{\theta}_{-i};\,\theta_0) = \mathbf{1}_{\{s = s_i^\star(\theta_i,\boldsymbol{\theta}_{-i},\theta_0)\}}$ *for some function* $s_i^\star : \Theta_i \times \boldsymbol{\Theta}_{-i} \times \Theta_0 \to \mathcal{S}_i$ *that is non-increasing in* $\theta_i$ *(higher types receive beliefs that are ranked lower in the LR order).*

*Proof of Lemma 6.* Fix $\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$ and $\theta_0 \in \Theta_0$. I will suppress $i$, $\boldsymbol{\theta}_{-i}$, and $\theta_0$ from the notation and use $(\hat{x},\,\hat{\pi})$ to denote the corresponding deterministic single-agent IC mechanism frame for agent $i$. Because for any $\theta \in \Theta$, $\hat{\pi}(s|\,\theta) \in \{0,\,1\}$, I will write $s(\theta)$ for the (unique) signal sent when the agent reports type $\theta$. Incentive-compatibility implies that, for any $\theta,\,\hat{\theta} \in \Theta$,

$$u(\theta;\,f^{s(\theta)})\hat{x}(\theta) - u(\theta;\,f^{s(\hat{\theta})})\hat{x}(\hat{\theta}) \geq u(\hat{\theta};\,f^{s(\theta)})\hat{x}(\theta) - u(\hat{\theta};\,f^{s(\hat{\theta})})\hat{x}(\hat{\theta}). \tag{A.13}$$

First, consider two types $\theta,\,\hat{\theta}$ such that $\hat{x}(\theta) = 1$ but $\hat{x}(\hat{\theta}) = 0$. Equation (A.13) implies

$$u(\theta;\,f^{s(\theta)}) \geq u(\hat{\theta};\,f^{s(\theta)}).$$

By strict monotonicity of the aftermarket, $u(\theta;\,\bar{f})$ is strictly increasing in $\theta$ for any $\bar{f} \in \Delta(\Theta)$, so to avoid a contradiction we must have $\theta > \hat{\theta}$. Thus, the allocation rule $\hat{x}$ is non-decreasing. Because $\hat{x}(\theta) \in \{0,\,1\}$ for any $\theta \in \Theta$, there must exist some $\theta^\star$ such that $\hat{x}(\theta) = \mathbf{1}_{\{\theta \geq \theta^\star\}}$. Because $\boldsymbol{\theta}_{-i}$ and $\theta_0$ were fixed in an arbitrary way, this proves the first part of Lemma 6.

Next, consider two types $\theta$ and $\hat{\theta}$ that receive the object under $\hat{x}$, with $\theta > \hat{\theta}$. Equation (A.13) implies

$$u(\theta;\,f^{s(\theta)}) - u(\theta;\,f^{s(\hat{\theta})}) \geq u(\hat{\theta};\,f^{s(\theta)}) - u(\hat{\theta};\,f^{s(\hat{\theta})}). \tag{A.14}$$

We must have $s(\theta) \leq s(\hat{\theta})$ as otherwise we obtain a contradiction with strict submodularity of the aftermarket which states that if $s(\theta) > s(\hat{\theta})$, then $u(\tau;\,f^{s(\theta)}) - u(\tau;\,f^{s(\hat{\theta})})$ is strictly decreasing in $\tau$. Because $\boldsymbol{\theta}_{-i}$ and $\theta_0$ were fixed in an arbitrary way, this proves the second part of Lemma 6. □

By the representation (A.12) and Lemma 6,

$$\hat{\pi}_i(s|\,\theta_i,\,\boldsymbol{\theta}_{-i})x_i(\theta_i,\,\boldsymbol{\theta}_{-i}) = \int_{\Theta_0} \mathbf{1}_{\{s = s_i^\star(\theta_i,\boldsymbol{\theta}_{-i},\theta_0)\}}\mathbf{1}_{\{\theta_i \geq \theta_i^\star(\boldsymbol{\theta}_{-i},\theta_0)\}}dF_0(\theta_0),$$

with $s_i^\star : \Theta_i \times \boldsymbol{\Theta}_{-i} \times \Theta_0 \to \mathcal{S}_i$ non-increasing in the first argument $\theta_i$. Consider the reduced form of the mechanism frame $(\boldsymbol{x},\,\boldsymbol{\pi})$ for agent $i$,

$$\bar{\pi}_i(s|\,\theta_i)\bar{x}_i(\theta_i) = \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} \int_{\Theta_0} \mathbf{1}_{\{s = s_i^\star(\theta_i,\boldsymbol{\theta}_{-i},\theta_0)\}}\mathbf{1}_{\{\theta_i \geq \theta_i^\star(\boldsymbol{\theta}_{-i},\theta_0)\}}dF_0(\theta_0)\boldsymbol{f}_{-i}(\boldsymbol{\theta}_{-i}).$$

For any $r \in \mathbb{R}$, $\theta_i > \hat{\theta}_i$, we have

$$\sum_{s \in \mathcal{S}_i:\, s \leq r} \bar{\pi}_i(s|\,\theta_i)\bar{x}_i(\theta_i) \geq \sum_{s \in \mathcal{S}_i:\, s \leq r} \bar{\pi}_i(s|\,\hat{\theta}_i)\bar{x}_i(\hat{\theta}_i) \tag{A.15}$$

because

$$\sum_{s \in \mathcal{S}_i:\, s \leq r} \Big[\mathbf{1}_{\{s = s_i^\star(\theta_i,\, \boldsymbol{\theta}_{-i},\, \theta_0)\}} - \mathbf{1}_{\{s = s_i^\star(\hat{\theta}_i,\, \boldsymbol{\theta}_{-i},\, \theta_0)\}}\Big] \geq 0$$

by the fact that $s_i^\star$ is non-increasing in $\theta_i$. That is, the distribution of signals $s$ conditional on a lower type $\hat{\theta}_i$ first-order stochastically dominates the distribution of signals conditional on a higher type $\theta_i$.

Recall that for $s,\, \hat{s} \in \mathcal{S}_i$ such that $s > \hat{s}$, $f_i^s$ LR dominates $f_i^{\hat{s}}$. Because posterior beliefs over $i$'s type depend only on the reduced form of the mechanism frame for agent $i$, this property can be written as (see equation 2.1)

$$\frac{f_i^s(\theta)}{f_i^{\hat{s}}(\theta)} = \frac{\bar{\pi}_i(s|\,\theta)}{\bar{\pi}_i(\hat{s}|\,\theta)}\phi(s,\, \hat{s}) \;\text{ is non-decreasing in } \theta, \tag{A.16}$$

where $\phi(s,\, \hat{s})$ is a term that does not depend on $\theta$. Towards a contradiction, suppose that for some $r \in \mathcal{S}_i$ and $\theta > \hat{\theta}$ we have $\bar{\pi}_i(r|\hat{\theta})\bar{x}_i(\hat{\theta}) > \bar{\pi}_i(r|\theta)\bar{x}_i(\theta)$. For any $s < r$, by (A.16),

$$\frac{\bar{\pi}_i(s|\,\theta)\bar{x}_i(\theta)}{\bar{\pi}_i(s|\,\hat{\theta})\bar{x}_i(\hat{\theta})} \leq \frac{\bar{\pi}_i(r|\,\theta)\bar{x}_i(\theta)}{\bar{\pi}_i(r|\,\hat{\theta})\bar{x}_i(\hat{\theta})}.$$

Because

$$\frac{\bar{\pi}_i(r|\,\theta)\bar{x}_i(\theta)}{\bar{\pi}_i(r|\,\hat{\theta})\bar{x}_i(\hat{\theta})} < 1,$$

it follows that for all $s \leq r$, we have $\bar{\pi}_i(s|\,\theta)\bar{x}_i(\theta) < \bar{\pi}_i(s|\,\hat{\theta})\bar{x}_i(\hat{\theta})$, so that also

$$\sum_{s \in \mathcal{S}_i:\, s \leq r} \bar{\pi}_i(s|\,\theta)\bar{x}_i(\theta) < \sum_{s \in \mathcal{S}_i:\, s \leq r} \bar{\pi}_i(s|\,\hat{\theta})\bar{x}_i(\hat{\theta}).$$

This is a contradiction with equation (A.15) that holds for all $r$ and $\theta_i > \hat{\theta}_i$. Therefore, $\bar{\pi}_i(s|\hat{\theta})\bar{x}_i(\hat{\theta}) \leq \bar{\pi}_i(s|\theta)\bar{x}_i(\theta)$ for all $s$. Because $\theta > \hat{\theta}$ were arbitrary, we conclude that $\bar{\pi}_i(s|\theta)\bar{x}_i(\theta)$ is non-decreasing in $\theta$ for all $s \in \mathcal{S}_i$. By Proposition 1, $(\bar{x}_i,\, \bar{\pi}_i)$ is a reduced-form cutoff rule for agent $i$ (see Appendix A.3 for a definition):

$$\bar{\pi}_i(s|\,\theta_i)\bar{x}_i(\theta_i) = \sum_{c \leq \theta_i} \gamma_i(s|\,c)\Delta\bar{x}_i(c), \tag{A.17}$$

for some signal function $\gamma_i : C_i \to \Delta(\mathcal{S}_i)$. Because $i \in \mathcal{N}$ was arbitrary throughout the proof, it follows from Lemma 2 (found in the proof of Theorem 3 in Appendix A.3) that $(\bar{\boldsymbol{x}},\, \bar{\boldsymbol{\pi}})$ is a reduced form of a cutoff rule, and thus the original mechanism frame $(\boldsymbol{x},\, \boldsymbol{\pi})$

inducing that reduced form is payoff-equivalent to a cutoff rule.[35]

## A.6 Proof of Theorem 5'

Strict monotonicity and strict submodularity were only used in the proof of Lemma 6, so I only have to prove that the conclusion of Lemma 6 holds when the aftermarket is monotone and submodular (not necessarily strictly) and the deterministic mechanism frame $(\hat{x}_i,\, \hat{\pi}_i)$ for agent $i$ is strict-dominant-strategy implementable.

Using the same notation as in the proof of Lemma 6, note that equation (A.13) still holds. Consider two types $\theta$, $\hat{\theta}$ such that $\hat{x}(\theta) = 1$ but $\hat{x}(\hat{\theta}) = 0$. Equation (A.13) implies

$$u(\theta;\, f^{s(\theta)}) \geq u(\hat{\theta};\, f^{s(\theta)}).$$

By monotonicity of the aftermarket, we must have either $\theta \geq \hat{\theta}$, or $u(\theta;\, f^{s(\theta)}) = u(\hat{\theta};\, f^{s(\theta)})$. I will show that the latter case contradicts strict-dominant-strategy imlementability. Indeed, in this case, the transfer implementing the outcome $\hat{x}(\theta) = 1$ and $\hat{x}(\hat{\theta}) = 0$ must make both types $\theta$ and $\hat{\theta}$ indifferent between reporting $\theta$ and $\hat{\theta}$, despite the fact that they receive different allocations. This means that $\theta \geq \hat{\theta}$, and the rest of the proof of the first part of Lemma 6 is unchanged.

Next, consider two types $\theta$ and $\hat{\theta}$ that receive the object under $\hat{x}$, with $\theta > \hat{\theta}$. Equation (A.13) implies

$$u(\theta;\, f^{s(\theta)}) - u(\theta;\, f^{s(\hat{\theta})}) \geq u(\hat{\theta};\, f^{s(\theta)}) - u(\hat{\theta};\, f^{s(\hat{\theta})}). \tag{A.18}$$

By submodularity of the aftermarket, we must have either $s(\theta) \leq s(\hat{\theta})$ or

$$u(\theta;\, f^{s(\theta)}) - u(\theta;\, f^{s(\hat{\theta})}) = u(\hat{\theta};\, f^{s(\theta)}) - u(\hat{\theta};\, f^{s(\hat{\theta})}).$$

In the latter case, both types $\theta$ and $\hat{\theta}$ must be indifferent between reporting $\theta$ and $\hat{\theta}$; Indeed, since the implementing transfer $\hat{t}$ must satisfy

$$u(\theta;\, f^{s(\theta)}) - u(\hat{\theta};\, f^{s(\theta)}) = \hat{t}(\theta) - \hat{t}(\hat{\theta}) = u(\theta;\, f^{s(\hat{\theta})}) - u(\hat{\theta};\, f^{s(\hat{\theta})}),$$

we can conclude that

$$u(\theta;\, f^{s(\theta)}) - \hat{t}(\theta) = u(\hat{\theta};\, f^{s(\theta)}) - \hat{t}(\hat{\theta}) \text{ and } u(\hat{\theta};\, f^{s(\hat{\theta})}) - \hat{t}(\hat{\theta}) = u(\theta;\, f^{s(\hat{\theta})}) - \hat{t}(\theta).$$

By strict-dominant-strategy implementability, indifference implies that $\theta$ and $\hat{\theta}$ must receive the same outcome: $s(\theta) = s(\hat{\theta})$. The rest of the proof of the second part of Lemma 6 is unchanged.

[35] Point *(1)* in Lemma 2 is trivially satisfied given the above reasoning, and point *(3)* holds because $(\bar{\boldsymbol{x}},\, \bar{\boldsymbol{\pi}})$ is a reduced from of some feasible mechanism frame.