

Engel's Treadmill: A Theory of Balanced Growth and Perpetual Sectoral Turnover

Clement E. Bohr*

Martí Mestieri†

Emre Enes Yavuz‡

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Abstract

Modern economic growth is characterized by constant growth in income per capita and secular sectoral changes in the composition of the economy. We develop an endogenous growth model with directed technical change across sectors where these two facts emerge endogenously in equilibrium. Along the aggregate balanced growth path, there is perpetual unbalanced growth across sectors due to the two-way interaction between increasing income and directed technical change towards more income-elastic sectors. We refer to this perpetual process as Engel's Treadmill. To model agents' preferences, we introduce the heterothetic Cobb-Douglas (HCD) demand system, which allows us to isolate the market size effect driving this two-way interaction: HCD is the only demand system in which differences in expenditure shares are solely due to differences in income levels. We provide a sharp characterization of the sectoral dynamics of the model and evidence in favor of its predictions. For example, using disaggregated US price series since 1957, we show that sectoral prices have, on average, fallen more in more income-elastic sectors. We also fully characterize the sectoral dynamics when both price and income effects are present, and show that the same qualitative sectoral patterns emerge.

*Northwestern University. Email: cebohr@u.northwestern.edu

†UPF-CREI-BSE and FRB of Chicago. Email: mestieri.marti@gmail.com

‡Northwestern University. eey@u.northwestern.edu

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1 Introduction

Modern economic growth has brought unprecedented sustained increases in the standard of living to the Western world (DeLong, 2000; Crafts and O'Rourke, 2014). As pointed out in Kuznets' Nobel address (Kuznets, 1973) and later on formalized by endogenous growth theory, technological progress has played a critical role in generating this sustained growth in income per capita. Remarkably, this sustained aggregate growth has gone hand-in-hand with uneven growth across sectors of the economy.

In this paper, we present a theory that jointly rationalizes the steady growth in income per capita and the uneven growth patterns experienced by different sectors over time. We develop a model of endogenous growth with directed technical change across a continuum of sectors. Our theory emphasizes the role of demand non-homotheticities in determining the effective market size of different sectors along the development path (Pasinetti, 1981; Comin et al., 2016). Due to these differences in market size, innovation shifts towards more income-elastic sectors as income grows (Jaravel, 2018). We show that, in equilibrium, the two-way interaction between (i) income growth brought by technological progress and (ii) changes in the direction of innovation brought by income growth is consistent with an aggregate balanced growth path and the rise and fall in the relative importance of different sectors over time.

The core of our mechanism builds on embedding Engel's Law (i.e., the idea that the composition of households' consumption baskets moves towards more income-elastic goods as they become richer) into a directed technical change framework (Acemoglu, 2002; Gancia and Zilibotti, 2005). Taken together, these two elements enable demand to "pull" innovation across sectors (Schmookler, 1966).¹ Akin to the Hedonic Treadmill in psychology introduced by Brickman and Campbell (1971), which describes a perpetual pursuit of happiness due to humans' insatiable nature, households pursue the consumption of luxury goods only to eventually deem them a necessity when attained in sufficient amounts. They then shift their expenditures to the pursuit of other more luxurious goods. That is, consumers in our economy change their concept of what necessities are as they become richer, shifting their demand away from what was once considered a luxury, but now a necessity, towards other more luxurious goods. The income-growth induced conversion of luxury goods into necessities implies that innovation is continuously redirected toward other luxury goods spawning yet further income growth in a perpetual process that we coin

¹Pasinetti (1981) is, to the best of our knowledge, the first to argue that nonhomotheticities in demand should interact with the demand-pull effect of innovation. In this sense, our paper provides a neoclassical rendition of his insight.

"Engel's treadmill." As a result, the growth process, despite appearing stable in the aggregate, is intrinsically heterogeneous across sectors over time.

To capture Engel's treadmill mechanism in its simplest form, we introduce a non-homothetic demand system, which we call Heterothetic Cobb-Douglas (HCD). The HCD demand system allows us to isolate changes in sectoral market size arising from income effects. With a unitary own-price elasticity, the endogenous evolution of prices do not affect the sectoral shares of the economy. Instead, sectoral shares only depend on the level of household income. This allows for a sharp characterization of the sectoral evolution of innovation, prices, and market size along the balanced growth path. To our knowledge, we are the first to implement these preferences in a general equilibrium model while being initially examined in a broader context by [Hanoch \(1975\)](#). In a complementary paper, [Bohr et al. \(2023a\)](#) provide a detailed examination of HCD preferences.

Our model implies that prices should fall relatively faster for more income-elastic sectors. We combine price data for 150 product categories of the US Personal Consumption Expenditures (PCE) from 1958 through 2017 with sectoral income elasticities estimated from the Consumer Expenditure Survey (CE) using the method proposed by [Aguiar and Bils \(2015\)](#), and show that the predicted relationship holds in the US data. In this sense, our theory and empirical finding provide a rationale for the fact that inflation for rich households has grown not as much as for poorer households ([Jaravel, 2018](#); [Argente and Lee, 2020](#)). We also provide evidence consistent with other model predictions: innovation should grow relatively faster and firm employment turnover should be relatively higher in more income-elastic sectors.

On a more technical note, to our knowledge, the paper is the first to provide closed-form representations for the expenditure function and demand for both the heterothetic Cobb-Douglas and the nonhomothetic CES preferences. We further show that the price distribution necessary for this result to hold emerges endogenously in our framework. We view these results of independent interest since they can be used for a wide range of applications that go beyond the scope of the paper.²

Before proceeding, it is worth reiterating that the model presented here emphasizes the role of domestic demand in driving endogenous technological changes. We abstract from other potentially important forces driving sectoral growth and innovation, such as trade or technology shocks. We do not deem these other forces inconsequential and, we believe that extending our

²In a separate note ([Bohr et al., 2023b](#)), we provide further discussion about the closed form result and give some examples of potential applications.

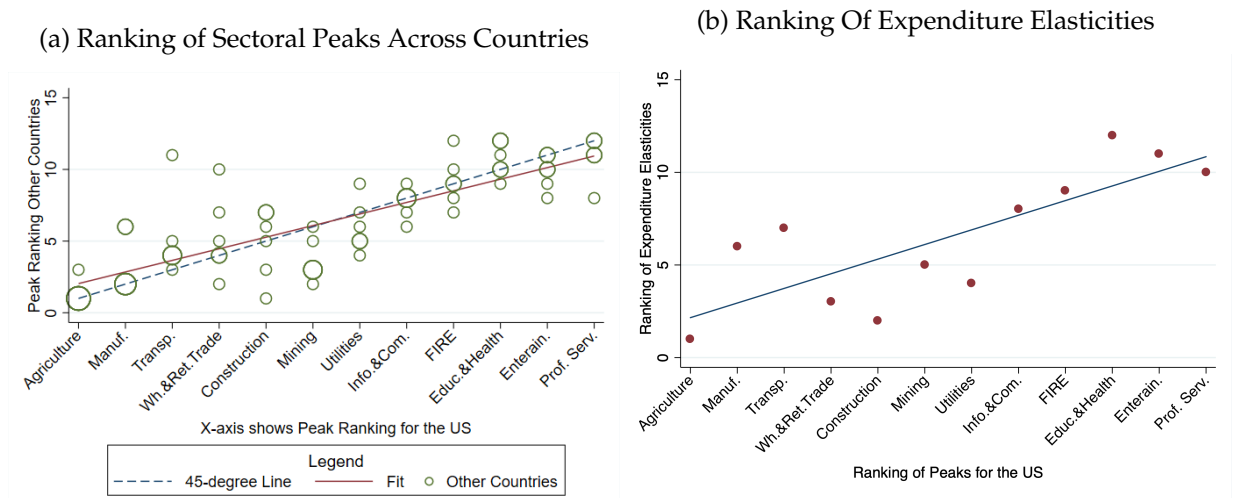
framework to account for these, e.g., an open-economy world would yield interesting results. However, we want to point out that an explanation based on technologies linked to different sectors exogenously appearing and being adopted is at odds with the following observation. Despite the fact that different sets of countries have gone through their development process at very different points in time, the ordering in the rise and fall of the various sectors across countries is highly correlated. For example consider on the one hand, early birds in modern growth, such as the United Kingdom, France and the US, and the late bloomers, Japan, South Korea and Taiwan, on the other hand who entered modern growth later on. We divide these economies into 12 sectors and compute the order in which these sectors have peaked in each country. Figure 1a shows the ranking for the US on the horizontal axis and for other countries on the vertical axis: the ordering of the peaks is highly correlated (0.82) across countries.³ In addition, we estimate the expenditure elasticities for the 12 sectors and rank them in order of their expenditure elasticity. Figure 1b shows the ranking for the US on the horizontal axis and the sectors' expenditure elasticity rank on the vertical axis: the ranking of a sector's peak and expenditure elasticity is highly correlated (0.79). This suggests a strong role for domestic demand in determining the evolution of sectoral shares.⁴

Detailed Outline of the Paper Section 2 presents our baseline model. At the aggregate level, the model appears identical to a standard expanding variety endogenous growth model à la [Grossman and Helpman \(1991a\)](#) (in particular, the aggregate behavior of our model mirrors that of chapter 13.4 in [Acemoglu, 2009a](#)). Its main departure is that it features a two-dimensional nested product space, representing goods across and within sectors, with HCD preferences used to model preferences across sectors. On top of isolating the market size effect driven by income effects, the HCD preferences allow us to be agnostic about the level of disaggregation and not take a stand on whether we should denote sectors as complements or substitutes. Within sectors, preferences are homothetic and goods are gross substitutes. Innovation is of the expanding variety type (horizontal innovation) à la [Romer \(1990\)](#). In our baseline model, labor is constant over time and it can be used to innovate or produce goods. Long-run growth is ensured thanks to knowledge spillovers.

³If we collapse the twelve sectors of this analysis to nine, we can extend our analysis to all OECD countries (joining prior to year 2000). In this case the correlation in rankings is 0.64. Figure 8 in appendix A depicts this correlation.

⁴[Matsuyama \(2002\)](#) makes a similar argument on the importance of the domestic market in driving sectoral take-offs along the development path. To be sure, domestic markets play a central role in models of technology adoption (see, among others, [Easterly et al., 1994](#) and [Comin and Mestieri, 2018](#)). In this sense, what we call innovation can be more broadly thought as encompassing also technology adoption.

Figure 1: Ranking of Sectoral Peaks and Expenditure Elasticities



Notes: Sectoral shares computed using World KLEMS data for all countries. Correlation of US peak ordering with that of UK, France, Japan, South Korea, and Japan is 0.82. Expenditure elasticity’s correlation with Sectoral Peak Ordering is 0.79.

The model features an aggregate balanced growth path (BGP) nearly identical to an off-the-shelf one-dimensional product space version without nonhomotheticities. Moreover, the BGP is stable in the sense that an economy with an arbitrary initial distribution of sectoral products converges to the distribution that generates a BGP. Despite the parsimony of the model in the aggregate, we show that sectors feature unbalanced growth at any point in time *along* the BGP. Sectors take-off sequentially as measured by their share in aggregate output according to their income elasticity ranking, featuring the “flying geese” pattern (Matsuyama, 2002). Sectors eventually decline in significance as they become necessities. We show that the expenditure share in each sector features a hump-shaped pattern with a rise and a subsequent fall.

Section 3 presents various extensions of the baseline model. In section 3.1 we show that our main results go through in Schumpeterian or semi-endogenous growth (Jones, 1995) renditions of the model. Section 3.2 shows that our theory above goes through if we allow for non-unitary price elasticities. In particular, we extend our analysis to Nonhomothetic CES (NhCES) preferences as described in Comin et al. (2021) and analyzed in Comin et al. (2016) in a context of directed technical change. In section 3.3, we extend our analysis to more general sectoral weighting functions than our baseline exercise. We fully characterize the model under the assumption that these weights are described by a gamma distribution. Finally, in section 3.4 we present a model variant of the model where we can attain similar analytically concise theory with aggregation across

heterogeneous households with expenditure levels that are distributed log-normally. The sectoral dynamics behave as if there is a representative household characterized by the average expenditure level.

Section 4 presents evidence from the US consistent with the key predictions of the model. First, we use disaggregated price data over 150 product categories from the PCE. We use the finest matching between PCE and the consumer expenditure survey categories and estimate the income elasticity of each industry following the [Aguiar and Bils \(2015\)](#) methodology. We show that there is a negative correlation between the price growth between 1959 and 2019 and income elasticity. The result is also robust to adding broad product category fixed effects (e.g., goods vs. services or durables vs. nondurables). Second, we show that innovation output across sectors as measured by patents has grown faster in more income-elastic sectors. Using patent classes matched to different 3-digit NAICS industries, we show that there is positive correlation between industry patent growth between 1974 and 2015 and our estimated sectoral income elasticity. This correlation persists when we add broad industry (1-digit) fixed effects.

Related Literature Our paper relates to several vast and rich literatures. First, the core result of our paper on endogenously-generated, demand-induced take-offs of different sectors along the growth path relates to a classical tradition in growth and development, notably [Nurkse \(1963\)](#). As in our model, this literature emphasized the role of market size and demand complementarities across sectors as key to understand the development process. Our model generates a sequential take-off of sectors, also known as the “flying-geese” pattern. This result is similar to [Matsuyama \(2002\)](#) and [Foellmi and Zweimüller \(2006\)](#). However, the economic mechanism in these papers necessarily relies on a trickle-down effect from rich to poor consumers (combined with learning-by-doing or an innovation-decision). By contrast, we assume a representative consumer and obtain the flying-geese result solely from an endogenous change in the direction of innovation over time.

The paper also relates to theories that have studied structural change with balanced growth. Following the seminal work of [Kongsamut et al. \(2001\)](#), most studies in this literature have taken sectoral productivity growth as exogenous.⁵ A notable exception is [Foellmi and Zweimüller \(2008\)](#). In the last section of their paper, they combine nonhomothetic, hierarchical preferences with an expanding variety model to generate a BGP. In their setup, however, there is no margin

⁵See [Ngai and Pissarides \(2007\)](#) and [Boppart \(2014\)](#) for other theories consistent with BGP and exogenous sectoral productivity growth.

for endogenous changes in the direction of innovation. Innovation only happens at the extensive margin, i.e., by adding more products to a one-dimensional product hierarchy. By contrast, we have two margins for innovation (within and across sectors). This allows us to study the endogenous evolution of the direction of innovation across sectors.^{6,7} Heterothetic Cobb-Douglas preferences also enable us to isolate the income effects from any price effects that are driving sectoral change.

Our paper builds on [Comin et al. \(2016\)](#), who also develop a model of directed technological change under nonhomothetic preferences.⁸ They develop a quantitative model incorporating richer elements than the present work to quantitatively assess the joint evolution of sectoral innovation and production. They show that the model can capture the key features of the development path of the US and other advanced economies. However, their model does not generate a balanced growth path. By contrast, our model shows how it is possible to have directed technical change across sectors along a balanced growth path. On the empirical side, [Comin et al. \(2016\)](#) document structural change in innovation along several measures (patenting, R&D expenditures, TFP). They show that TFP grows faster in more income-elastic sectors in the context of the broad sectors of the economy. They also provide evidence on patenting and R&D growing faster for the service sector. The more disaggregated evidence on patenting in this paper complements their findings. [Weiss and Boppart \(2013\)](#) also develop a model of directed technological change under nonhomothetic preferences with two sectors and a rich input-output network. They also provide evidence in favor of market sizes driving innovation. However, they restrict their attention to a model where nonhomotheticities apply to two sectors and, as a result, the perpetual two-way interaction between aggregate growth and demand-directed innovation generating Engel's treadmill emphasized in this paper is absent.

At a more technical level, this paper relates to the extant work that has used NhCES preferences ([Hanoch, 1975](#)) for the study of structural change in open and closed economies (e.g., [Matsuyama, 2019](#); [Comin et al., 2016, 2021](#); [Duernecker et al., 2017](#); [Sposi, 2019](#)). Relative to these papers, we show how they relate to HCD preferences and how to incorporate both HCD and NhCES preferences into an endogenous growth model that delivers aggregate implications identical to

⁶Another important technical difference is that we have a unique BGP. Moreover, all our results go through under endogenous growth, while they must rely on an exogenous growth version for some of their analytical results.

⁷[Herrendorf and Valentinyi \(2015\)](#) propose a theory to rationalize reallocation from goods to services along a BGP based on the assumption that the returns to variety are larger in the goods sector. Their theory does not make use of nonhomothetic preferences.

⁸See also [Acemoglu et al. \(2012\)](#) and [Aghion et al. \(2016\)](#) for a different application of demand affecting the direction of innovation in the context of clean and dirty technologies.

an off-the-shelf one-sector growth model. In so doing, we extend the home-market effect insight in Matsuyama (2019) to a dynamic setting, and we offer a tractable framework that complements the work in Comin et al. (2016). This enables us to study the two-way interaction between rising income and sectoral price and innovation dynamics. Moreover, to the best of our knowledge, this paper is also the first to provide a closed-form representation of the implicitly defined NhCES aggregator and show that the price distribution needed for the closed-form result arises as an equilibrium outcome.

As we have discussed, the idea of nonhomotheticities in demand affecting the direction of innovation goes back, at least, to Pasinetti (1981).⁹ There is recent empirical evidence supporting this mechanism. Beerli et al. (2020) document a sizable causal effect of changes in market size on the direction of innovation in the context of the Chinese durable good industry driven by heterogeneity in the slopes of Engel curves. In a more granular setting, Jaravel (2018) also finds substantial evidence of directed innovation towards higher income elastic sectors. He documents that more income elastic products have lower inflation because they have increasing demand, which leads to more entry and larger variety in these product categories.¹⁰

2 Baseline Model

This section presents our baseline model. The model is deliberately chosen to be parsimonious and to deliver equations that, on the aggregate, are analogous to a textbook one-sector endogenous growth model (in particular, section 13.4 in Acemoglu (2009b)). We first lay out the economic environment and then discuss how to solve the model. The ordering in solving the model is somewhat different from that in Acemoglu (2009b) to highlight the closed-form mapping between expenditures and utility that the model delivers.

2.1 Environment

Household Preferences, Endowments, and Demographics The economy is populated by a mass L of homogeneous households. Each household is endowed with one unit of labor that is inelastically supplied. Households have isoelastic preferences over an infinite stream of consumption

⁹See also Katona (1964) for a discussion of the age of mass consumption as redefining the concept of necessities and luxuries.

¹⁰There is additional evidence on the role of demand-pull effects driving innovation. For example, Acemoglu and Linn (2004) and Costinot et al. (2019) provide evidence on the effect of market size on innovation and product entry in the pharmaceutical industry.

bundles $\{C_t\}_{t=0}^{\infty}$ according to

$$\int_0^{\infty} e^{-\delta t} \frac{C(C_t)^{1-\gamma} - 1}{1-\gamma} dt, \quad (1)$$

where $\delta > 0$ is the discount factor, $\gamma > 0$ is the inverse elasticity of intertemporal substitution and $C(\cdot)$ is the intra-period utility aggregator over the consumption bundle C_t .

Households can smooth consumption over time through investments in an asset A_t with interest rate r_t which represents shares in the portfolio of all firms in the economy. The household budget constraint is thus given by

$$\dot{A}_t = r_t A_t + W_t - E_t, \quad (2)$$

where W_t and E_t denotes the wage rate and household expenditures in the economy.

At time t , the goods available to households to construct their consumption bundle belong to the product space $(\varepsilon, i) \in [0, \infty) \times [0, N_{\varepsilon,t}]$ with preferences over these goods given by a nested structure. The outer nest is indexed by ε and is defined by Heterothetic Cobb-Douglas preferences, while the inner nest is indexed by i and is defined by homothetic CES preferences with an elasticity of substitution $\sigma > 1$. Formally, within-period household preferences are defined by

$$0 = \int_0^{\infty} \alpha_{\varepsilon}(C_t) \ln \left(\frac{C_{\varepsilon,t}}{C_t} \right) d\varepsilon \quad \text{and} \quad C_{\varepsilon,t} = \left(\int_0^{N_{\varepsilon,t}} C_{\varepsilon i,t}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} \quad (3)$$

A natural interpretation of the outer nest is as different sectors in the economy, and the inner nest of goods within the sector.

While the homothetic CES aggregator used in the lower nest is standard, the use of HCD preferences in a general equilibrium setting is novel. To gain intuition on their formulation, note that if the weight α_{ε} was independent of C_t and satisfied that it integrates to one (i.e., $\int_0^{\infty} \alpha_{\varepsilon} d\varepsilon = 1$) the HCD preferences in Equation (3) become the standard homothetic Cobb-Douglas over a continuum of goods where α_{ε} simply denote the expenditure shares. The central difference of HCD is that expenditure shares are allowed to vary due to the level of utility C_t , and hence their nonhomothetic nature. In contrast to the nonhomothetic CES preferences used in [Comin et al. \(2021\)](#), HCD preferences are not defined for all levels of income. Instead, as with Stone-Geary or PIGL preferences, they are only well defined for a level of income that is above a certain threshold, see [Bohr et al. \(2023a\)](#) for further discussion. In what follows, we assume that preferences are well defined and thus we start above a minimum level of income.

In our baseline model, we characterize how expenditure shares across sectors evolve as a function of the household's utility level by specifying an exponential distribution for the sectoral weighting function.

$$\alpha_\varepsilon(C_t) = C_t^{-\beta} \exp(C_t^{-\beta} \varepsilon). \quad (4)$$

The sectoral preference weights are determined by the household's utility level C_t where the parameter $\beta \in [0, 1]$ controls the strength of the nonhomotheticities. Indeed, with $\beta = 0$, we are back to homothetic Cobb-Douglas preferences.

Innovation and Production Technologies Production of each intermediate in the production set is linear in labor

$$Y_{\varepsilon i,t} = L_{\varepsilon i,t}. \quad (5)$$

New products can be created in any ε sector through an innovation technology which is identical across sectors. The innovation flow of new products in sector ε is given by

$$\dot{N}_{\varepsilon,t} = \eta N_t L_{R\varepsilon,t} \quad (6)$$

where $L_{R\varepsilon,t}$ is total amount of labor hired for research in sector ε , $N_{\varepsilon,t}$ is the total number of product varieties in sector ε , and $N_t = \int_0^\infty N_{\varepsilon,t} d\varepsilon$ is the total number of product varieties in the economy at time t .

Markets and Patents Labor markets are competitive while firms selling to households engage in monopolistic competition. There is free entry in the innovation sector where firms are awarded a perpetual patent upon successful innovation of a new product.

2.2 Equilibrium Characterization

We begin our analysis with the competitive equilibrium of the economy, that is, households maximize utility given their budget constraint taking prices as given, firms maximize profits, innovation takes places until the cost of innovating a new product equals its return, and goods and labor markets clear.

Household Behavior We derive now household demand for each product and total expenditure over time. We begin with the within-period problem. Given total household expenditure, E_t , and

taking the price vector $\{P_{\varepsilon i,t}\}$ as given, household expenditure on good (ε, i) is

$$P_{\varepsilon i,t} C_{\varepsilon i,t} = \left(\frac{P_{\varepsilon,t}}{P_{\varepsilon i,t}} \right)^{\sigma-1} \alpha_{\varepsilon}(C_t) E_t \quad (7)$$

$$\text{with } P_{\varepsilon,t} = \left(\int_0^{N_{\varepsilon,t}} P_{\varepsilon i,t}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} \quad \text{and} \quad P_t = E_t / C_t = \int_0^{\infty} \alpha_{\varepsilon}(C_t) \ln \left(\frac{P_{\varepsilon,t}}{\alpha_{\varepsilon}(C_t)} \right) d\varepsilon. \quad (8)$$

In order to concisely characterize the household's optimal inter-temporal allocations, we first characterize a closed-form mapping between expenditures and aggregate utility which is implicitly defined in (75). Foreshadowing the properties of the equilibrium, let us assume (and later verify) that prices across sectors are characterized by an exponential function.

$$P_{\varepsilon,t} = \zeta_t \exp \left(\frac{\varepsilon}{\chi_t} \right). \quad (9)$$

Note that the parameters ζ_t and χ_t are only viewed as parameters from the perspective of the household and are taken as given. However, we show below that they are equilibrium objects determined by several economic forces. Note that as far as sectoral prices are nominal terms, ζ_t can be scaled up or down as suited and we thus refer to it as the overall price level, while χ_t fully characterizes the relative prices across sectors. We now show that the sectoral price distribution in Equation (76) allows for the aggregate price index to be solved for in closed form. Direct substitution of Equation (76) in (75) yields the following mapping between aggregate utility and expenditures for the household

$$\ln \frac{E_t}{\zeta_t} = (1 + \beta) \ln C_t + 1 + \frac{C_t^\beta}{\chi_t}. \quad (10)$$

Using this mapping, the household Euler equation can be solved by optimizing the intertemporal allocation of utility by choosing the flow of expenditures E_t and stock of assets A_t at any given moment. The Euler equation is

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma} \left(r_t - \frac{\dot{P}_t}{P_t} - \delta \right), \quad (11)$$

which includes the growth rate of the nonhomothetic price-index as the correct nominal adjustment of interest rate. From the household's perspective, the correct real interest rate is the one adjusted by the growth rate in the price-level of their changing consumption bundle. The house-

hold transversality condition is given by

$$\lim_{t \rightarrow \infty} \exp\left(-\int_0^t r_s ds\right) N_t V_t = 0, \quad (12)$$

where $N_t V_t = A_t$ is the combined present value of all firms.

Firm Optimality Equation (74) shows that the demand for good εi is isoelastic in its own price. Under monopolistic competition, the firm producing good εi finds it optimal to set its price $P_{\varepsilon i,t}$ at a constant markup over the marginal cost, W_t , determined by the within sector elasticity of substitution. Given the constant returns to scale technology, the associated profits $\Pi_{\varepsilon i,t}$ are fixed fraction of the total generated revenue.

$$P_{\varepsilon i,t} = \frac{\sigma}{\sigma - 1} W_t \quad \text{and} \quad \Pi_{\varepsilon i,t} = \frac{1}{\sigma} P_{\varepsilon i,t} Y_{\varepsilon i,t}. \quad (13)$$

All that is needed to characterize the distribution of prices across sectors is the distribution of products within sectors. Since all firms in a ε -sector are identical, we have that the sectoral index $P_{\varepsilon i}$ in Equation (74) is

$$P_{\varepsilon,t} = \frac{\sigma}{\sigma - 1} W_t N_{\varepsilon,t}^{-\frac{1}{\sigma-1}}. \quad (14)$$

We can derive a firm's equilibrium level of profits from its price and sector price index with market clearing for the good $Y_{\varepsilon i,t} = LC_{\varepsilon i,t}$ and the household's demand Equation (74), to obtain

$$\Pi_{\varepsilon i,t} = \frac{1}{\sigma} \frac{\alpha_\varepsilon(C_t) E_t L}{N_{\varepsilon,t}}. \quad (15)$$

Equation (84) shows how firm's profits depend on its sector aggregate expenditure $E_{\varepsilon,t} L = P_{\varepsilon,t} C_{\varepsilon,t} L = \alpha_\varepsilon(C_t) E_t L$ and its number of competitors $N_{\varepsilon,t}$, where the sectoral weights in the household preferences fully determine the sectoral expenditure share.

Free Entry and the Distribution of Products and Prices With a perpetual patent, the value of a product at any given time t is equal to the sum of its future discounted profits,

$$V_{\varepsilon i,t} = \int_t^\infty \exp\left(-\int_t^s r(\tau) d\tau\right) \Pi_{\varepsilon i}(s) ds. \quad (16)$$

The free entry condition endogenously determines the mass of firms $N_{\varepsilon,t}$ occupying each ε sector. Firms can select any ε sector in which to innovate. Firms innovate new product varieties in each ε sector until the flow value of doing research is equal to the labor cost of doing so, $\eta N_t V_{\varepsilon,t} = W_t$. Since the cost of research across sectors is the same (the wage W_t), firms enter each sector until the net present value of innovating in a product is identical across sectors, so $V_{\varepsilon,t} = V_t$. Since this is true at any given moment in time, it follows from the Hamilton-Jacobi-Bellman equation $r_t V_t = \Pi_{\varepsilon,t} + \dot{V}_t$ that $\Pi_{\varepsilon,t} = \Pi_t$ almost everywhere.

The symmetry in profits across sectors lets us obtain an expression for the number of products in each sector by rearranging (84) to obtain

$$N_{\varepsilon,t} = N_t \alpha_\varepsilon (C_t) \quad \text{where} \quad N_t = \frac{L E_t}{\sigma \Pi_t}. \quad (17)$$

Thus, the distribution of products across sectors exactly matches the expenditure share distribution. Given that sectoral prices are determined by the number of products in each sector, this product distribution implies an exponential product distribution verifying our earlier presumption,

$$P_{\varepsilon,t} = \frac{\sigma}{\sigma - 1} W_t (N_t \alpha_\varepsilon (C_t))^{-\frac{1}{\sigma-1}}. \quad (18)$$

Closed Form Mapping between Expenditures and Utility The symmetry in profits across products also implies that each product is produced in the same amount and employs the same amount of labor. For this reason the labor market clearing condition is given by

$$L = L_{Y,t} + L_{R,t} = N_t L_{\varepsilon,t} + L_{R,t} \quad (19)$$

where $L_{Y,t}$ is the total amount of labor used for production and $L_{R,t} = \int_0^\infty L_{R\varepsilon,t} d\varepsilon$ denote the total employment in research. The definition of aggregate expenditures as the sum of all individual purchases reduces to

$$L E_t = \frac{\sigma}{\sigma - 1} W_t L_{Y,t}. \quad (20)$$

With the endogenous price distribution in hand along with the definition of aggregate expenditures, we can solve the integral for the aggregate price index in Equation (75), which yields a log-linear mapping between utility and aggregate innovation,

$$C_t^{(1+\beta)\sigma-1} = e^{-\sigma} (L_{Y,t}/L)^{\sigma-1} N_t. \quad (21)$$

In turn, this implies a log-linear mapping between utility and expenditures by Equation (17).

2.3 Aggregate Balanced Growth Path

Next, we show that this economy features a balanced growth path (BGP). Along the BGP there is a constant interest rate, r , a constant share of workers in research and production, L_R and L_Y , and constant growth in expenditures g^* . Given Walras' law we can normalize one price. We choose to normalize the household price-index to one. The resulting BGP will exist when $\frac{\eta L}{\sigma-1} > \delta$, which ensures that there is positive growth. It is readily verified that the transversality condition is always satisfied within our specified parametric bounds. The BGP features constant growth in all aggregate variables:¹¹

$$g_E = g_W = g_C = g^*, \quad g_\Pi = g_V = -((1 + \beta)\sigma - 2)g^* \quad g_N = ((1 + \beta)\sigma - 1)g^*. \quad (22)$$

The interest rate and share of labor force in research are given by

$$r^* = \delta + \gamma g^* \quad , \quad L_R^* = \frac{(1 + \beta)\sigma - 1}{\eta} g^* \quad (23)$$

all of which above underlies the BPG growth rate of expenditures

$$g^* = \left(\frac{(1 + \beta)\sigma - 1}{\sigma - 1} + (1 + \beta)\sigma - 2 + \gamma \right)^{-1} \left(\frac{\eta}{\sigma - 1} - \delta \right). \quad (24)$$

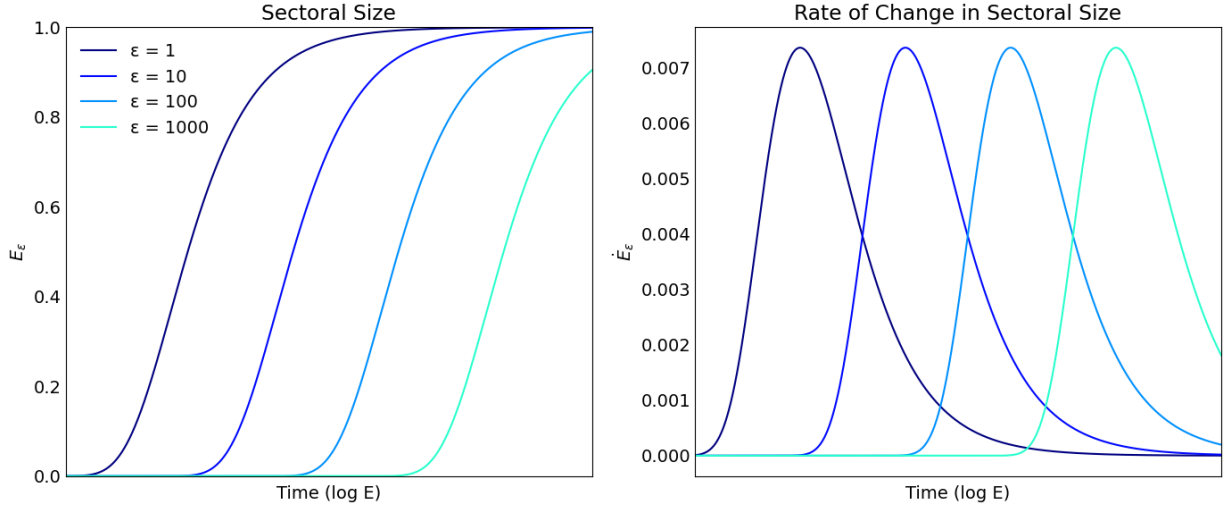
2.3.1 Optimal Growth

As is standard in expanding varieties models, the decentralized equilibrium (DE) characterized above suffers from a deficient level of growth relative to what a benevolent social planner (SP) maximizing welfare can attain. We characterize in the appendix this social planner problem. In particular, we show that the BGP of the SP problem and DE are:

$$\begin{aligned} \text{SP :} \quad & g^* = \left(0 + (1 + \beta)\sigma - 2 + \gamma \right)^{-1} \left(\frac{\eta}{\sigma - 1} - \delta \right) \\ \text{DE :} \quad & g^* = \left(\frac{(1 + \beta)\sigma - 1}{\sigma - 1} + (1 + \beta)\sigma - 2 + \gamma \right)^{-1} \left(\frac{\eta}{\sigma - 1} - \delta \right) \end{aligned}$$

¹¹Note that the growth rate of the profits and net present value of a product can be either positive or negative depending on the elasticity of substitution across and within sectors. This property of nested CES preferences and monopolistic competition is discussed in Matsuyama (1995).

Figure 2: Sectoral Dynamics along BGP



Notes: Parameterization $\beta = 1, g_N = 0.02$.

Importantly, however, we find that the direction and intensity of innovation across sectors along the BGP coincides for both the SP and DE. This means that a simple non-targeted subsidy to innovation suffices to attain the optimal allocation in the decentralized equilibrium.

2.4 Sectoral Dynamics along the BGP

After having shown that the aggregate behavior of the economy is analogous to an off-the-shelf one-sector expanding-variety endogenous growth model, we turn our attention to sectoral dynamics. We show that the dynamics of sectoral expenditures E_ϵ are concisely determined by the sector's expenditure elasticity, η_ϵ , and the aggregate growth rate of the economy, g^* . The expenditure elasticity of a sector, in turn, depends on the evolution of expenditure shares. Thanks to HCD's unitary own-price elasticity, these are fully determined by the sectoral weighting function α_ϵ in the HCD preferences. The sector's expenditure elasticity is given by

$$\eta_\epsilon(C_t) \equiv \frac{\partial \ln E_\epsilon}{\partial \ln E} = 1 - \beta + \beta C_t^{-\beta} \epsilon. \quad (25)$$

Equation (25) implies that sectors that are higher ranked in terms of their sectoral index ϵ feature higher expenditure elasticities, and the expenditure elasticity of any sector declines as income levels (and thus utility levels C_t) grow. Note how β tunes the strength of the income effect across sectors. Setting it to zero results in the homothetic case where all sectors feature an expenditure

elasticity of one. Setting it to one implies an expenditure elasticity of $\eta_\varepsilon(C_t) = \frac{\varepsilon}{C_t}$.

The sectoral growth rates are given by the product of aggregate growth rate and the sectoral expenditure elasticity,

$$\frac{\dot{E}_{\varepsilon,t}}{E_{\varepsilon,t}} = g^* \cdot \eta_\varepsilon(C_t). \quad (26)$$

Equation (26) implies that more expenditure-elastic sectors grow faster while all sectors' growth rates decline over time. This translates into sectors taking off subsequently according to their expenditure elasticity rank ε (with less income-elastic sectors taking off first). This is shown in the left panel of figure 2. The right panel shows the rate of change in the size of the sectors. After taking off, a sector initially features a high growth spurt. This rate of change peaks, along with the sector's total expenditure share, when the sector's expenditure elasticity equals one, which occurs when the households have attained a utility level $C^* = \varepsilon^{1/\beta}$.¹² Figure 2 shows the dynamics under the most extreme nonhomothetic parameterization of $\beta = 1$. In this case, sectors grow and converge toward a satiation level. For $\beta < 1$, all sectors converge towards a positive constant growth rate after having initially taken off in a rapid burst. Lastly, in order for monotonicity to hold, it can be seen from (25) that $\beta \leq 1$.¹³

Sectoral Price Dynamics Since the overall growth rate of prices depends on the chosen normalization, we focus on the relative growth rate across sectors. In particular, it is convenient to describe the evolution of sectoral prices relative to sector $\varepsilon = 0$ which features a constant expenditure elasticity below one (depending on the value of β). Let $\hat{P}_\varepsilon = \frac{P_\varepsilon}{P_0}$ be price of sector ε relative to sector 0. Then,

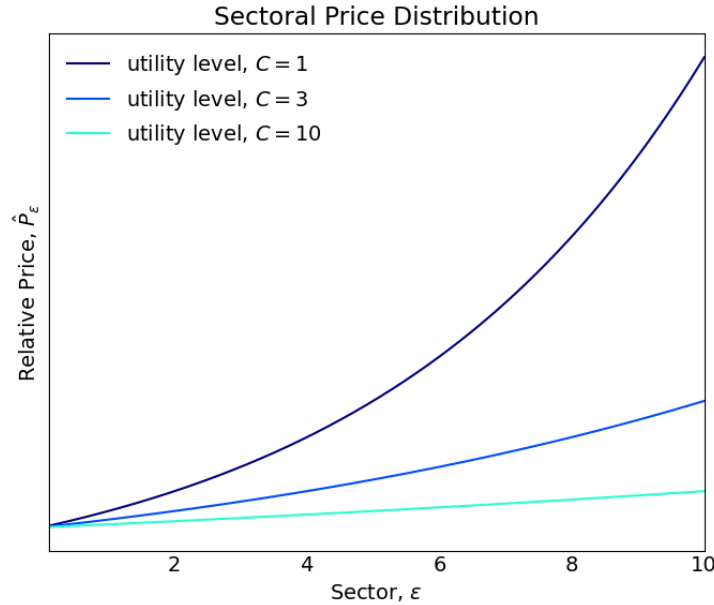
$$\frac{\dot{\hat{P}}_{\varepsilon,t}}{\hat{P}_{\varepsilon,t}} = \frac{g^*}{\sigma - 1} ((1 - \beta) - \eta_\varepsilon(C_t)). \quad (27)$$

Since more expenditure-elastic sectors, by definition, feature higher levels of growth in market size, Equation (27) implies that there will be more innovation and firm entry into them. This results in the price levels of more expenditure-elastic sectors declining faster relative to sector zero's price level. Figure 3 shows the sectoral price levels under various levels of utility along the BGP relative to the sector $\varepsilon = 0$. Sectors that are more expenditure-elastic feature higher price

¹²This is no longer true when we generalize the model to include price effects on demand. The price-effects result in sectors peaking when their expenditure elasticities differ from 1, as we show below.

¹³We do not only restrict ourselves to having positive expenditure elasticities across all sectors due to it being a sufficient condition for the preferences to be well defined. It is also necessary for the price distribution to preserve its inverted mirror image of the expenditure distribution. This is necessary for our closed-form results under the implicitly defined preferences.

Figure 3: Sectoral Price Distribution along BGP



Notes: Parameterization, $\beta = 1, g_N = 0.02$.

levels. They are more expensive because little innovation has been directed to these sectors yet. As utility levels rise, these sectors have relatively more innovation, making their price levels fall relative to the less expenditure-elastic sectors.

3 Model Extensions

3.1 Alternative Baseline Formulations: Schumpeterian and Semi-Endogenous Growth

Our baseline formulation is based on the expanding variety model à la Grossman and Helpman (1991a). However, it is possible to re-formulate Engel’s treadmill result in the context of a Schumpeterian growth model Aghion and Howitt (1992) with a continuum of products as in Grossman and Helpman (1991b) obtaining qualitatively identical results. Likewise, it is also possible to formulate the model as a semi-endogenous growth model and introduce population growth to sustain long-run growth. Given the similarity of the results in these two extensions, they are relegated to Appendix F.

3.2 Incorporating Baumol's Disease: Engel's Treadmill with Sectoral Price-Effects

Thanks to the unitary price elasticity of HCD preferences, we can derive a theory of balance growth with sectoral dynamics solely characterized by sectors' heterogeneous income elasticity. This allow us to isolate the central mechanism of the paper in the simplest way. Indeed, the literature on structural change also emphasizes the role of price effects in driving sectoral reallocation (Herrendorf et al., 2014). Here we show that our model generalizes neatly to the case with non-unitary own-price elasticities across sectors. The only deviation we make from the baseline model is to replace the outer HCD nest with its NhCES generalization,

$$1 = \int_0^\infty \alpha_\varepsilon(C_t)^{\frac{1}{\rho}} \left(\frac{C_{\varepsilon,t}}{C_t} \right)^{\frac{\rho-1}{\rho}} d\varepsilon, \quad (28)$$

with the condition that $0 < \rho < \sigma$. Sectors may be either substitutes or complements.¹⁴ The structural change literature emphasizing Baumol's disease focuses on the case of gross complements, $\rho < 1$.

Performing the same analysis as in the main section but with nhCES, we find that the non-unitary price elasticity modifies the equilibrium price-distribution. It no longer exactly mirrors the distribution of sectoral weights in the upper nest preference aggregator, instead we have that

$$\frac{N_{\varepsilon,t}}{N_t} = \frac{\sigma-1}{\sigma-\rho} C_t^{-\beta} \exp\left(-\frac{\sigma-1}{\sigma-\rho} C_t^{-\beta} \varepsilon\right). \quad (29)$$

Substituting this result in the price index equation (80), this property is also inherited by sectoral prices and they also thus have a slightly modified price distribution relative to the baseline. For both distributions, note that setting $\rho = 1$ yields the baseline equations, as expected.

The aggregate balanced growth path is qualitatively identical to that under HCD.¹⁵ The sectoral dynamics are given by

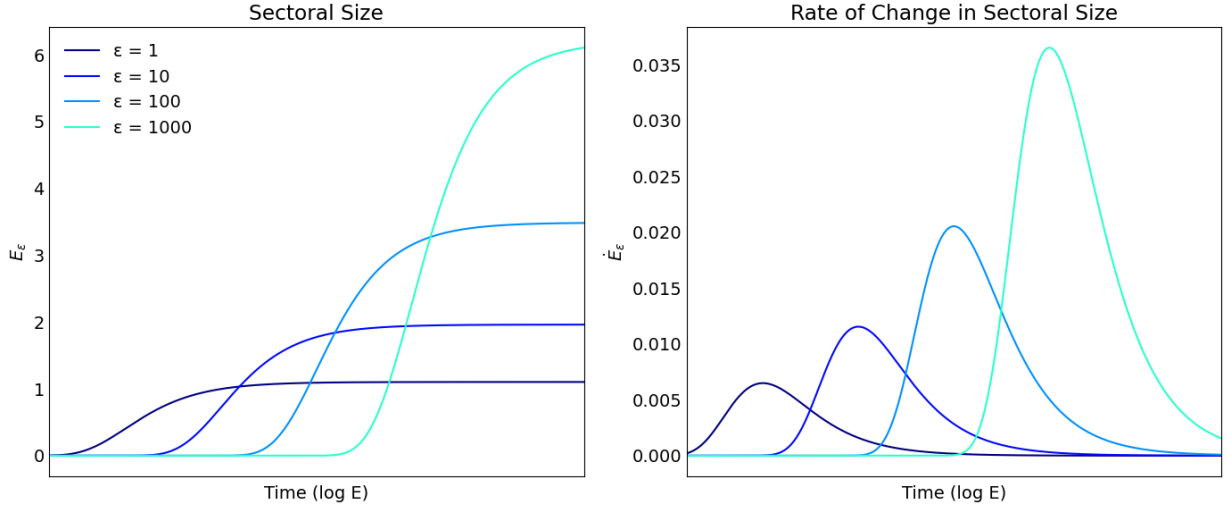
$$\frac{\dot{E}_{\varepsilon,t}}{E_{\varepsilon,t}} = g^* \cdot (\eta_\varepsilon(C_t) - \iota_\varepsilon(C_t)). \quad (30)$$

The income effect remains the same as described by the expenditure elasticity in (25). In addition,

¹⁴The limit of $1 = \left(\int_0^\infty \alpha_\varepsilon(C_t)^{\frac{1}{\rho}} \left(\frac{C_{\varepsilon,t}}{C_t} \right)^{\frac{\rho-1}{\rho}} d\varepsilon \right)^{\frac{\rho}{\rho-1}}$ as $\rho \rightarrow 1$ converges to the HCD preferences in Equation (3). Likewise, $\rho \rightarrow 0$ yields a nonhomothetic version of Leontief preferences.

¹⁵The sole difference in the growth rate is that the utility grows at a slightly different proportional rate to the stock of innovation. Appendix E provides a full derivation of the model under NhCES preferences and more a general weighting function α_ε given by a gamma distribution. To obtain the specific results with exponentially distributed sectoral weights as described here, we need to set the parameter of the gamma distribution $k = 0$.

Figure 4: Example of Sectoral Dynamics under Gamma-Distributed Sectoral Weights



Notes: Parameterization $\beta = 0.8, k = 0.25, g_N = 0.02$.

the sectoral dynamics now feature a price effect,

$$\iota_\varepsilon(C_t) = \frac{1-\rho}{\sigma-\rho} \beta C_t^{-\beta} \varepsilon. \quad (31)$$

Note that the price effect is zero for the HCD case, $\rho = 0$. The price-effect amplifies or diminishes the sectoral growth rates depending on whether sectors are substitutes or complements, respectively. A sector's share of the economy will peak at a utility level $C^* = \left(\frac{\sigma-1}{\sigma-\rho} \varepsilon\right)^{1/\beta}$, and at its peak, its expenditure elasticity will be $\eta_\varepsilon(C^*) = 1 + \frac{1-\rho}{\sigma-1} \beta$. Whether this peak occurs at an income elasticity above or below one depends on the substitutability across sectors, with complementary sectors peaking with an income elasticity greater than one. Qualitatively, the sectoral dynamics are the same as those under HCD preferences.

3.3 Generalization of Sectoral Weights to a Gamma Distribution

We explicated our baseline theory with an exponential distribution on sectoral preference weights α_ε because its ease of derivation and simple analytical results. One feature of this is weights is that the size of less income elastic sectors strictly dominates that of more income elastic sectors.¹⁶

¹⁶This is not necessarily a counterfactual assumption. The researcher must decide how the continuum of sectors is grouped and aggregated into the finite number of sectors featured in their data. For instance, [Foellmi and Zweimüller \(2008\)](#) partition an infinite continuum (which also features a decaying density) into three sectors to mimic the dynamics of the value-added shares of agriculture, manufacturing, and services. By defining services as the infinite measure of the top partitioning, its aggregated value-added share eventually overlaps that of the other two sectors despite the density at any equal-sized interval being less.

We can generalize the expenditure distribution to a gamma distribution without other modifications and still obtain closed-form results. Let the following gamma distribution give the sectoral preference weights

$$\alpha_\varepsilon(C_t) = \frac{\varepsilon^k \exp\left(-C_t^{-\beta} \varepsilon\right)}{\Gamma(k+1)C_t^{\beta(k+1)}}, \quad (32)$$

where $\Gamma(\cdot)$ denotes the gamma function. Note that when $k = 0$, we recover our baseline exponentially-weighted model. When $k > 0$, we allow sectoral preference weights to be hump-shaped across sectors when ordered by their income elasticity rank ε . This, in turn, modifies the equilibrium product distribution and prices relative to our baseline exercise. The model remains solvable in closed form, and the aggregate balanced growth path is identical to that under $k = 0$. Appendix E provides the full derivation of the model for both HCD and nhCES preferences.

However, the underlying sectoral dynamics are different and are shown in Figure 4. They still feature traveling waves that occur at the individual sectoral level. But now, higher income elastic sectors take off later, but they eventually overtake the less income elastic sectors in size. After overtaking them, the rate of change in sectoral size declines, to be surpassed by even more income-elastic sectors. The introduction of the new parameter $k \geq 0$ results in a slightly perturbed parameter space in which preference monotonicity is satisfied. Rather than $\beta \leq 1$ which still holds under $k = 0$, we need $k \leq \frac{\sigma-\rho}{\sigma-1} \frac{1-\beta}{\beta}$. Again in figure 4, we have parameterized the model to be on the edge of this parameter space, where each sector converges to a leveling off rather than continuing to grow or decline.

Formally, these dynamics are still characterized by Equation (30) where the income and price effects terms are modified to be

$$\eta_\varepsilon(C_t) = (1 - \beta)1 + \beta C_t^{-\beta} \varepsilon - \beta k \quad \text{and} \quad \iota_\varepsilon(C_t) = \frac{1 - \rho}{\sigma - \rho} \left(\beta C_t^{-\beta} \varepsilon - \beta k \right). \quad (33)$$

Sectoral peaks occur at utility level $C^* = \left(\left(\frac{\sigma-\rho}{\sigma-1} + k \right)^{-1} \varepsilon \right)^{1/\beta}$ with an expenditure elasticity of $\eta_\varepsilon(C_t^*) = 1 + \frac{1-\rho}{\sigma-1} \beta$. In sum, the qualitative behavior of the economy is that of our baseline model but enriched by the more complex shape of the underlying distribution of sectoral weights.

3.4 Aggregation under Gaussian-HCD Preferences

Our baseline model can be modified in several ways while obtaining a BGP and Engel's treadmill behavior. It is possible to add different sectoral weighting functions and production structures. For example, in [Bohr et al. \(2023a\)](#) we incorporate HCD into the framework of [Foellmi and Zweimüller \(2008\)](#) and obtain BGP and Engel's treadmill. Here, we present an example in which we modify sectoral weights to another sectoral weighting function and modify the production function. This particular example has the feature that makes it possible to model heterogeneous agents and still characterize the aggregate behavior of the economy.

We assume that households are of permanent types and differ in their labor endowments. These labor endowments are distributed according to a log-normal distribution. Along the balance growth path, this implies a log-normal expenditure distribution, which is well-documented to be a good empirical approximation. We denote \bar{E} the mean expenditure level and s^2 the variance of the log-normal distribution.

Households maintain HCD preferences across sectors with sectoral weights defined now by a Gaussian distribution on the space $\varepsilon \in (-\infty, \infty)$. We denote the mean of a household's h Gaussian sectoral weight in preferences α_ε as $\ln(C^h)$ and its variance, δ^2 .

Finally, we work with a variant of the production side of the economy that mutes the love-for-variety effect in the baseline model by redefining household preferences over products within a sector to be

$$C_{\varepsilon,t}^h = \left(\int_0^{N_{\varepsilon,t}} \left(\frac{1}{N_{\varepsilon,t}} \right)^{\frac{1}{\sigma}} C_{\varepsilon i,t}^h \frac{\sigma-1}{\sigma} \right)^{\frac{\sigma}{\sigma-1}} \quad (34)$$

Instead, we impose aggregate knowledge spillovers in production, as we already had in innovation, so the firm's production technology becomes

$$Y_{\varepsilon i,t} = N_t L_{\varepsilon i,t}. \quad (35)$$

This means that although innovation is directed across sectors according to market size, productivity gains are spread equally across sectors. Along with the fact that there are no sector-level gains from variety, the price of any given product in any given sector is equal to any other product and any sector's level price-index as well

$$P_{\varepsilon,t} = P_{\varepsilon i,t} = \frac{\sigma}{\sigma-1} \frac{W_t}{N_t} = p_t. \quad (36)$$

Given the identical price across sectors, the household-specific price index from (75) becomes

$$\ln P_t^h = \ln p_t + \ln(\sqrt{2\pi\delta^2}e) = 1. \quad (37)$$

Note that the price index is identical across households because the variance of the Gaussian sectoral preference weights α_ε is constant and identical for all households, so the entropy measure associated with the price index is constant across households. For convenience and in accordance with the rest of this paper, we normalize the household price index to one. Thus, we have a one-to-one mapping between the household's utility and expenditure levels, $E_t^h = C_t^h$.

The aggregate demand for any given sector is the integral over the households' demands for the sector,

$$C_\varepsilon = \int_0^\infty C_\varepsilon^h H(E) dE \quad (38)$$

where $H(E)$ denotes the log-normal distribution of household expenditures, and $C_\varepsilon^h = \alpha_\varepsilon (C^h) E^h$. Given the one-to-one mapping between household expenditures and utility, and the Gaussian expenditure share distribution, from the perspective of aggregation over expenditures in sector ε , it becomes the second moment of another log-normal density function in E with mean ε . Thus, we can multiply the two log-normal densities together and integrate over the combined log-normal. This results in aggregate demand across sectors to be distributed according to a Gaussian distribution with a mean equal to $\ln(\bar{E})$ and a variance $\bar{s}^2 = s^2 + \delta^2$, which is exponentially modified as follows

$$\ln C_{\varepsilon,t} = -\frac{1}{2} \ln(2\pi\bar{s}^2) + \frac{1}{2\bar{s}^2} (s^2\delta^2 + 2s^2\varepsilon + 2\delta^2 \ln \bar{E}_t) - \frac{1}{2\bar{s}^2} (\varepsilon - \ln \bar{E}_t)^2. \quad (39)$$

4 Empirics

This section presents three pieces of evidence consistent with the model dynamics implied by Engel's Treadmill. We show that the correlation between sectoral price growth and sectoral expenditure elasticities is negative. We also show that patenting growth and job creation due to entrants are higher in more income-elastic sectors. Before presenting these results we discuss two alternative measures of expenditure elasticities and how we compute them in the data. In so doing, we show an interesting property of HCD preferences: it is possible to retrieve differences in expenditure elasticities from differences in expenditure shares alone.

4.1 Estimation of Income Elasticities

To test the empirical predictions of the model, a central object to estimate is the income elasticity of different sectors ε . We derive two alternative measures of income elasticity based on household data from the CEX. First, we provide estimates based on the method from [Aguiar and Bils \(2015\)](#). Second, we use the structural equations of HCD to estimate the implied expenditure elasticities.¹⁷

[Aguiar and Bils \(2015\)](#) propose to estimate Engel curves for household n in sector s , quarter t according to

$$\ln \left(\frac{x_{st}^n}{\bar{x}_{st}} \right) = \alpha_{str} + \eta_s \ln E_t^n + \Gamma_s Z^n + u_{st}^n,$$

where x_{st}^n denotes expenditure in sector s , E_t^n is total household expenditure, \bar{x}_{st} denotes the average expenditure, α_{str} , sector-time-region FE and Z^n demographic controls¹⁸. The term η_s measures the expenditure elasticity of sector s . [Aguiar and Bils](#) propose to instrument E_t^n with household income quintiles and annual income. They argue that this specification makes it possible to deal with a number of measurement error issues that are likely to be pervasive in the CEX.

Model-consistent estimates To make a connection with our model, we assume sector s as a partition of the distribution of true sectors ε in our model. To keep things as simple as possible, assume that s corresponds to a range $\varepsilon \in (\underline{\varepsilon}_s, \bar{\varepsilon}_s) \equiv \mathcal{S}$ such that $\underline{\varepsilon}_s - \bar{\varepsilon}_s$ is constant.¹⁹ As a first step, we note that the expenditure share in sector ε is

$$\ln x_{\varepsilon,t}^n = -\ln C_t^n - \frac{\varepsilon}{C_t^n}. \quad (40)$$

And we note that for a given ε and ε' , the relative expenditure shares of household n is

$$\ln \left(\frac{x_{\varepsilon,t}^n}{x_{\varepsilon',t}^n} \right) = -\frac{(\varepsilon - \varepsilon')}{C_t^n} \quad (41)$$

where $C_t^n = E_t^n / P_t^n$. Using the definition of η it follows that

$$\ln \left(\frac{x_{\varepsilon,t}^n}{x_{\varepsilon',t}^n} \right) = -\frac{(\varepsilon - \varepsilon')}{C_t^n} = \eta_\varepsilon^n - \eta_{\varepsilon'}^n. \quad (42)$$

¹⁷We show that we can recover almost exactly [Aguiar and Bils](#) specification except for the fact that instead of household expenditure we need to use household expenditures deflated by a household-specific price index which we can readily construct from the data. In this case the parametrization of the weights is different from our baseline.

¹⁸These are dummies for household members and age of main earner.

¹⁹It is possible to have richer partitions, for example, s indexing a gamma distribution over existing varieties ε .

Suppose now that there are sector-time and region-sector effects, so that one would like to estimate a regression of the type

$$\ln x_{\epsilon,t}^n = \ln C_t^n - \frac{\epsilon}{C_t^n} + \delta_{st} + \delta_{sr} + u_{n,\epsilon,t}, \quad (43)$$

where δ_{st} and δ_{sr} denote the fixed effects and $u_{n,\epsilon,t}$ an error term. We observe that these fixed effects can be differenced out by considering the following double-differences in expenditure shares:

$$\ln \left(\frac{x_{\epsilon,t}^{top}}{x_{\epsilon',t}^{top}} \right) - \ln \left(\frac{x_{\epsilon,t}^{median}}{x_{\epsilon',t}^{median}} \right) = \eta_{\epsilon}^{top} - \eta_{\epsilon'}^{top} - (\eta_{\epsilon}^{median} - \eta_{\epsilon'}^{median}). \quad (44)$$

This exercise motivates using the expenditure share of the top quartile of households relative to the median as a proxy for the difference in expenditure shares across sectors. This measure has the advantage of being readily computed from the CEX.

Note that instead of observing sector ϵ we only observe partition \mathcal{S} . We have that

$$\alpha_s \equiv \frac{E_s}{E} = \int_{\epsilon \in \mathcal{S}} d\epsilon C^{-\beta} \exp(-\epsilon C_t^{-\beta}) = \int_{\underline{\epsilon}}^{\bar{\epsilon}} d\epsilon C^{-\beta} \exp(-\epsilon C_t^{-\beta}) = \exp(-\underline{\epsilon} C_t^{-\beta}) - \exp(-\bar{\epsilon} C_t^{-\beta}) \quad (45)$$

For a sufficiently small interval, we have that $e^{A(x+dx)} = e^{Ax}(1 + Adx)$ and thus, by choosing the mid-point $\tilde{\epsilon} = (\underline{\epsilon} + \bar{\epsilon})/2$ so that $\underline{\epsilon} = \tilde{\epsilon} - \Delta\epsilon$ with $\Delta\epsilon = \frac{\bar{\epsilon} - \underline{\epsilon}}{2}$, we have that:

$$\exp(-\underline{\epsilon} C_t^{-\beta}) - \exp(-\bar{\epsilon} C_t^{-\beta}) \simeq \exp(-\tilde{\epsilon} C_t^{-\beta}) (1 + C^{-\beta} \Delta\epsilon) - \exp(-\tilde{\epsilon} C_t^{-\beta}) (1 - C^{-\beta} \Delta\epsilon) \quad (46)$$

which implies that

$$\exp(-\underline{\epsilon} C_t^{-\beta}) - \exp(-\bar{\epsilon} C_t^{-\beta}) \simeq \exp(-\tilde{\epsilon} C_t^{-\beta}) C_t^{-\beta} 2\Delta\epsilon = \exp(-\tilde{\epsilon} C_t^{-\beta}) C_t^{-\beta} (\bar{\epsilon} - \underline{\epsilon}) \quad (47)$$

so if the partition becomes of equal sizes, we recover the previous result on relative shares. Armed with these two measures of expenditure elasticities, we proceed to analyze different model predictions.

4.2 Prices

A central prediction of the model is that prices should decline relatively faster for more income-elastic sectors. We showed in Equation (27) that the growth rate of the relative price of more-income elastic sectors, $\frac{\dot{P}_{\epsilon,t}}{P_{\epsilon,t}}$, declined relatively faster.

Table 1: Average Price Growth by PCE Category and Expenditure Elasticity

	Dep. Var.: Ave. Annual Price Growth in %					
	Ave. 1959-2020			Ave. 1980-2020		
	(1)	(2)	(3)	(4)	(5)	(6)
Exp. Elasticity η_ϵ	-1.20 (0.6)	-2.10 (0.56)	-1.67 (0.64)	-1.20 (0.71)	-1.82 (0.41)	-1.38 (0.46)
Goods vs. Servs FE	N	Y	Y	N	Y	Y
2-digit Prod. Cat. FE	N	N	Y	N	N	Y
(partial) R^2	0.05	0.2	0.11	0.05	0.18	0.09
Top Quintile Share relative to median share	-2.8 (1.3)	-3.8 (1.2)	-3.2 (1.5)	-4.3 (1.5)	-4.6 (1.2)	-3.4 (1.2)
Goods vs. Servs FE	N	Y	Y	N	Y	Y
2-digit Prod. Cat. FE	N	N	Y	N	N	Y
(partial) R^2	0.05	0.11	0.09	0.06	0.11	0.06

Notes: Numb. Obs. is 124. Robust std. errors. Obs. weighted by initial expenditure.

To test this prediction, we compute the expenditure elasticities for granular PCE categories. We do so by matching the Consumer Expenditure Survey UCC categories (over 600 categories) for the period 2000-2004 to the PCE disaggregated categories (which are over 150). We then compute the average price category growth over the period 1950-2020 period, and then regress the average price growth against sectoral expenditure elasticities, that is $\hat{P}_\epsilon / \hat{P}_\epsilon = \beta_0 + \beta_1 \eta_\epsilon + \text{error}_\epsilon$.

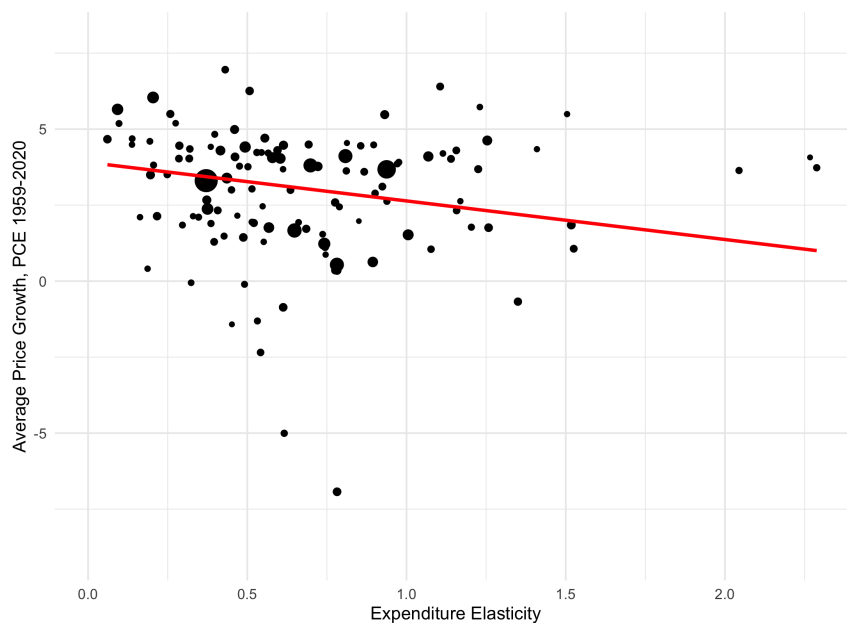
Table 1 reports the results. We find a consistently negatively estimated coefficient. This is true also if we restrict our attention to the second part of the sample. The estimated magnitude implies that a difference in the interquartile range of sectoral inflation implies a difference in yearly price growth of $-1.2 \cdot (0.85 - 0.36) = -0.59\%$. This number is similar in magnitude to that reported by Jaravel (2018). Figure 5 depicts the results graphically, we see that there is a clear downward trend in price growth but substantial dispersion around it. Indeed, there are numerous other factors that affect price growth.²⁰

4.3 Further evidence: Innovation and Churning

Another implication that mirrors the evolution of prices is that innovation should be growing in income elasticity. Under the assumption that the probability of patenting a new idea depends on a sector- and time-fixed effect, we have that the growth rate of patenting across sectors reflects the

²⁰Figure 9 shows the same figure separately for goods and services. We see that both have a downward negative slope.

Figure 5: Income-elastic sectors experience lower inflation



differences in expenditure elasticities across sectors.²¹ Thus, under these assumptions, we can test whether the implication of the model of higher patent growth in more income-elastic sector holds in the data.

We proceed by using patent data from the USPTO over the period 1974-2015. We match patent classes to 3-digit NAICS using the correspondence based on a probabilistic assignment of patent classes to industry codes that we construct from using firms in Compustat, for which we have both information on their NAICS code and patenting activity. Through this match, we end up with 88 industries.

Table 2 reports the results of these regressions. We find a positive and significant correlation regardless of whether we use the raw number of patents or if we account for the quality of patents by weighting by forward citations or sectoral value-added. Using the estimates of raw patents, we have that the median patent growth in a sector is 1.8% over the period, while the interquartile range (IQR) in patenting growth is 2.8%. Taken at face value, this correlation implies that a change in the IQR of expenditure elasticity (which is 0.12, implies an increase in patent growth from $2.4\% \times 0.12 = 0.29\%$ to $4.6\% \times 0.12 = 0.55\%$.

²¹Comin et al. (2016) provide a microfoundation for patenting in a model of directed technological change across sectors. This result follows directly from their analysis.

Table 2: Patenting growth and income elasticity

	Raw Citations		VA Weights		Citation Weights	
	(1)	(2)	(3)	(4)	(5)	(6)
Exp. Elasticity η_ε	2.4 (0.7)	1.6 (0.7)	2.5 (0.6)	1.6 (0.8)	7.9 (1.5)	4.6 (1.9)
Broad Ind. FE	No	Yes	No	Yes	No	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes

Notes: Std. Err. robust and clustered at NAICS 1, respectively. Numb. Obs. is 3002.

Job Churning Results The steady-state dynamics of the Schumpeterian growth model are identical to that of the expanding varieties model, with aggregate average quality level taking the role of the total number of product varieties. The dynamics are driven by alternative forces, however, which we again can test in the data. In the Schumpeterian growth paradigm the driver of growth is a mix of innovation by incumbent firms and their competitors who seek to replace said incumbents. To be sure, the Schumpeterian growth paradigm also implies declining price growth and increasing patenting in more income elastic sectors. Arguably, the evidence on price growth may be better rationalized by the Schumpeterian benchmark since it naturally implies firms undercutting each other with the price they set.

To test the additional prediction of the Schumpeterian growth model, we compute the share of job creation shares by incumbent firms. Our theory predicts more entry in more income-elastic sectors, which imply more job creation by entrants. Thus, we expect that the share of job creation by incumbents is declining in the expenditure elasticity of the sector. This is indeed what Table 3 reports. There is more job creation by incumbents in less-income elastic sectors. This correlation holds unconditionally and when we look within one-digit industries. Also, the correlation appears to be stable over-time. Figure 6 depicts the correlation in 1993 and 2013, and we see that the relationship appears to be very similar.

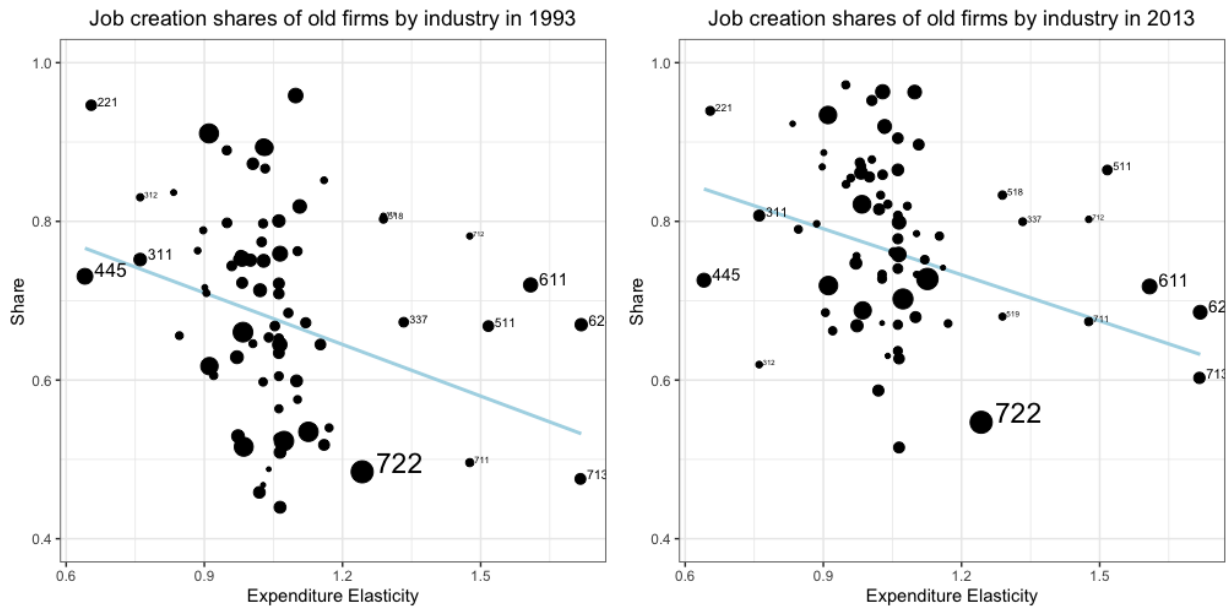
Taken together, the three correlations that we have shown in this section (price growth, patenting growth, and job creation shares), paint a picture consistent with the view that the income elasticity of a sector plays a role consistent with that predicted by our theory. Indeed, these are not causal relationships and they should be interpreted with caution.

Table 3: Correlation of Incumbent Job Creation Share and Expenditure Elasticity

	Job Creation Share		
	(1)	(2)	(3)
Expenditure Elasticity η_ε	-0.15	-0.13	-0.08
ε indexes 3-digit NAICS industries	(0.05)	(0.05)	(0.05)
Time FE	Y	Y	Y
Goods vs. Servs FE	N	Y	Y
1-digit NAICS FE	N	N	Y
(partial) R^2	0.06	0.04	0.01

Notes: 616 observations. Industry clustered robust standard errors.

Figure 6: Incumbent Job Creation Shares



5 Conclusion

This paper provides an endogenous growth model that features a unique and stable aggregate balanced growth path while also capturing the perpetual, non-linear dynamics across sectors resulting from directed innovation. The growth mechanism results from the two-way interaction between non-homotheticities in demand with directed technical change in what we have called “Engel’s Treadmill.” In search of profits, innovation tends toward where markets are expanding and drives overall growth. Due to nonhomotheticities in demand, the resulting income growth induces households to change their demand patterns creating yet newer markets into which innovation will flow.

We introduce the usage of the Heterothetic Cobb-Douglas preferences. They have the feature of isolating the changes in market size due to income effects. Indeed, since they have price elasticity equal to one, all changes in expenditure shares are due to income effects by construction. We also show that our results extend to a price elasticity that is non-unitary through the use of nonhomothetic CES, along the lines of [Comin et al. \(2016\)](#) and [Comin et al. \(2021\)](#). We also provide a closed-form representation for the expenditure and demand functions for both demand systems that may be of separate interest. Moreover, our model shows that the price distribution that emerges in equilibrium through free-entry yields a closed-form solution. In [Bohr et al. \(2023b\)](#) we provide further discussion on this result and extend it.

Our theory relies on and implies several testable predictions. First, at its core the theory deems that a sector’s defining characteristic is its rank in the ordering of expenditure elasticities, which determines when the sector will take-off, peak, and decline. Second, our theory predicts that prices should decline more in sectors with higher expenditure elasticities. Third, the intensity of innovation is greater in sectors with higher expenditure elasticities. We find all of these correlations to be true in the data.

Notably, our theory is mute regarding the effects of trade and the potentially heterogeneous technologies across sectors on structural change. We by no means dismiss these as being important contributors. However, the ability of our model to explain the consistent sectoral patterns across modern economies as well as the empirical predictions regarding price and innovation growth underlines the importance of the home-market sizes across sectors in structural change and growth. Nonetheless, the expansion of this theory into an open-economy setting is an obvious next step in this line of research.

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APPENDIX

A Additional Figures and Tables

Figure 7: US Sectoral VA, 20-Sector Split, 1899-2005

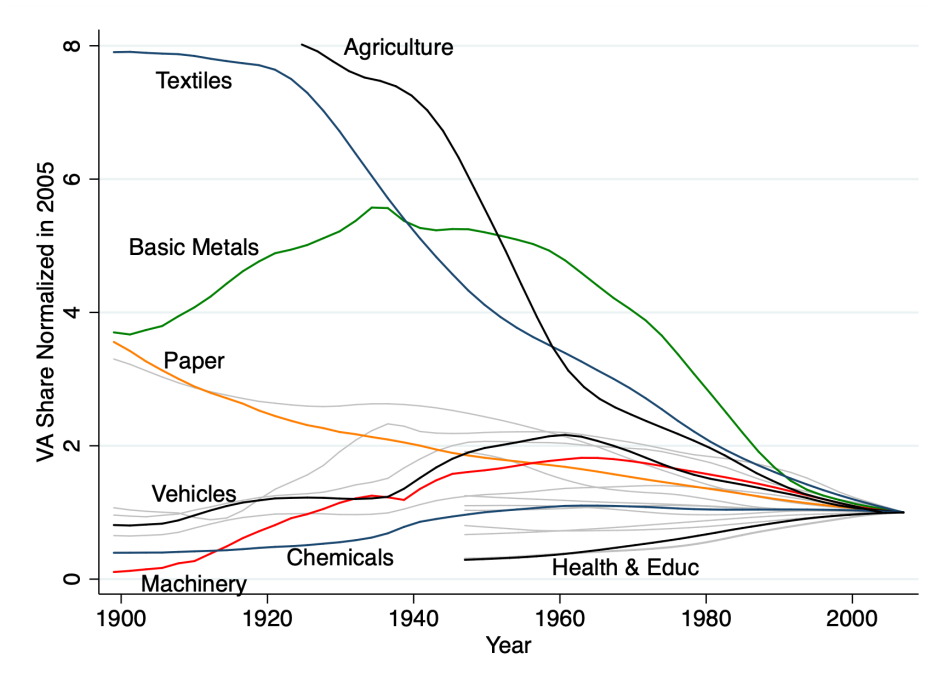


Figure 8: Sectoral Peak Ranking across OECD Countries

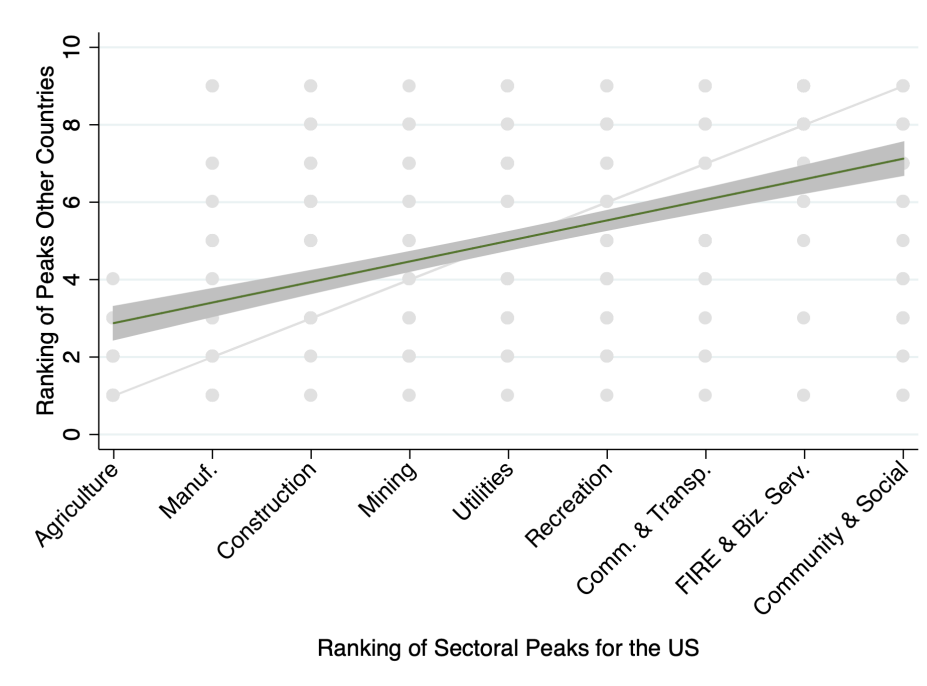
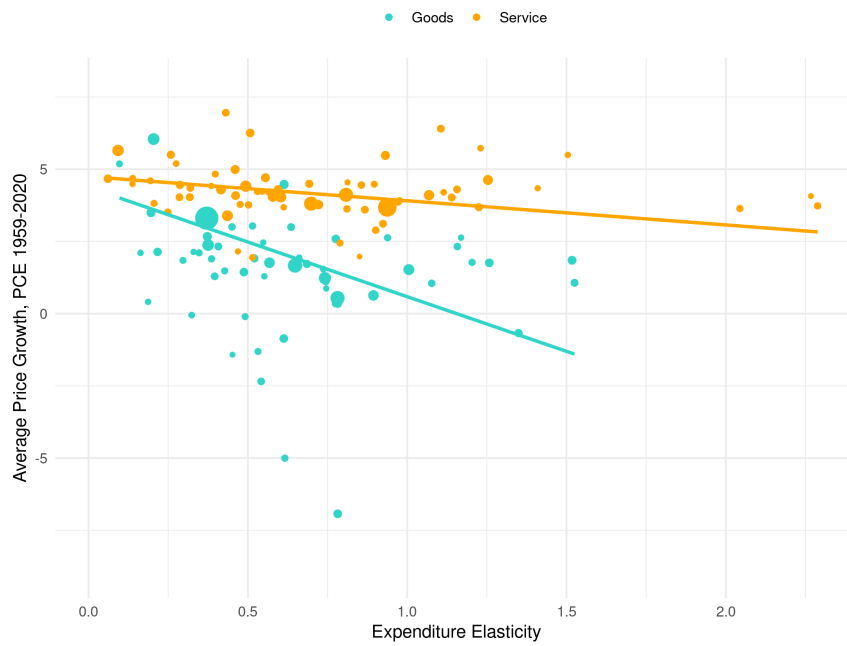


Figure 9: Income-elastic sectors experience lower inflation: Goods vs Services



B Uniqueness of the Equilibrium

We can show that the balanced growth path is indeed the unique equilibrium of the model. It is possible to show that the labor share in research has to be constant and this immediately implies that the growth rate of total number of varieties are constant. Here we refer to the equations in the next section which encompass both the cases of HCD and NhCES preferences. It is clear that from the equations 96 and 97, other growth rates and interest rate are constant because they can be written in terms of growth rate of total number varieties. Without imposing anything, using equations 78, 81, 85, 86, 87, 89 and price normalization, we can write the change of labor in the research in terms of its level.

$$\dot{L}_{R,t} = \frac{(L - L_{R,t})(L_{R,t} - L_R^*)}{\frac{1}{\sigma - \rho} [(2 - \rho)\sigma - 1]} \quad (48)$$

where L_R^* is the balanced growth path level, an expression in terms of model parameters provided in the equation 98. Notice that the denominator is positive given $\sigma > 1 > \rho$. We plot the differential equation in Figure 10 to illustrate its dynamics.

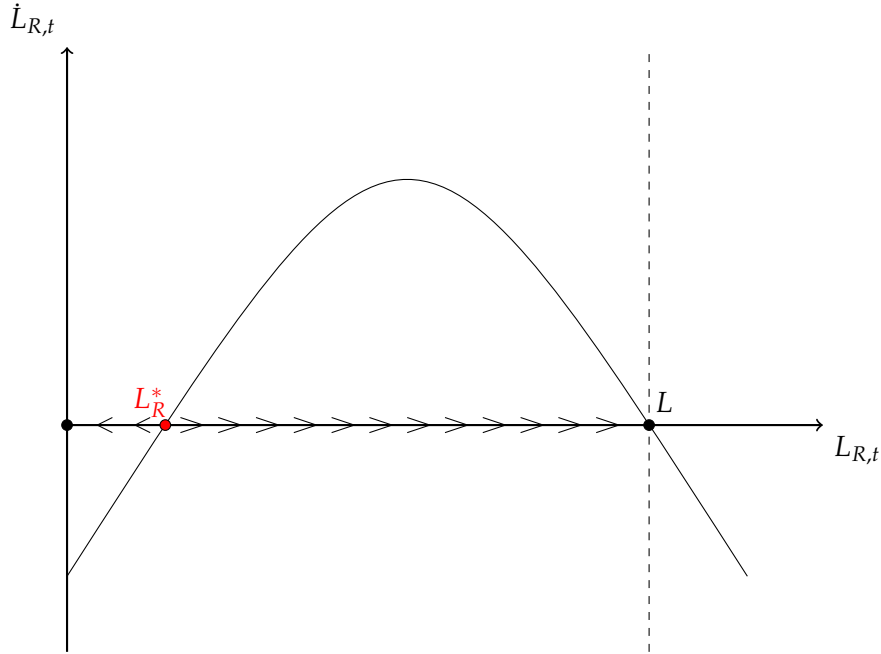


Figure 10: Change of labor employed in research sector in terms of its level.

First note that research labor cannot be higher than total labor, i.e. $L_{R,t} \in [0, L]$. If we have $L_{R,t} < L_R^*$ then $\dot{L}_{R,t} < 0$ hence the labor in research will always decrease and eventually we will

have negative labor which is not possible. Similarly if $L_{R,t} > L_R^*$, it will increase and eventually will be higher than the total labor, also not possible. Also we cannot have $L_{R,t} = L$, because that will leave no labor for production. Therefore the unique solution is $L_{R,t} = L_R^*$ and $\dot{L}_{R,t} = 0$, i.e. labor share in research sector has to be constant and the unique solution is the balanced growth path.

C Stability of the BGP Sectoral Distribution

The model implies an exponential distribution of varieties in the balanced growth path. What about if the economy has a different distribution in the initial state? It is possible to show that an arbitrary distribution converges to the exponential distribution implied by the BGP as the economy grows. Using equation (90), the ratio of number of varieties in BGP for sector an arbitrary sector $\varepsilon = \tilde{\varepsilon}$ to the sector with $\varepsilon = 0$ can be written by,

$$\frac{N_{\tilde{\varepsilon},t}}{N_{0,t}} = C_t^{\tilde{\varepsilon}(\sigma-1)\alpha}. \quad (49)$$

Assume $\frac{N_{\tilde{\varepsilon},t_0}}{N_{0,t_0}} > C_{t_0}^{\tilde{\varepsilon}(\sigma-1)\alpha}$ and suppose towards contradiction that $\frac{N_{\tilde{\varepsilon},t}}{N_{0,t}} \frac{1}{C_t^{\tilde{\varepsilon}(\sigma-1)\alpha}}$ is increasing for $t \in [t_0, t_0 + \epsilon)$. Equation (84) gives the ratio of profits,

$$\frac{\Pi_{\tilde{\varepsilon},t}}{\Pi_{0,t}} = \left[\frac{N_{\tilde{\varepsilon},t}}{N_{0,t}} \frac{1}{C_t^{\tilde{\varepsilon}(\sigma-1)\alpha}} \right]^{-\frac{\sigma-\rho}{\sigma-1}}. \quad (50)$$

Therefore we have $\frac{\Pi_{\tilde{\varepsilon},t}}{\Pi_{0,t}} < \frac{\Pi_{\tilde{\varepsilon},t_0}}{\Pi_{0,t_0}}$ for $t \in [t_0, t_0 + \epsilon)$ and it can be generalized to for all $t \geq t_0$.²² The assumption implies that there is innovation in sector $\varepsilon = \tilde{\varepsilon}$, but there might or might not be innovation in sector $\varepsilon = 0$, hence firm entry conditions are given by,

$$\eta N_t^\phi V_{\tilde{\varepsilon},t} = W_t \quad \text{and} \quad \eta N_t^\phi V_{0,t} \leq W_t. \quad (51)$$

This implies $V_{\tilde{\varepsilon},t} \geq V_{0,t}$, combining with the definition of value of a firm and $\frac{\Pi_{\tilde{\varepsilon},t}}{\Pi_{0,t}} < \frac{\Pi_{\tilde{\varepsilon},t_0}}{\Pi_{0,t_0}}$ for $t \geq t_0$, we can obtain the result, $\frac{\Pi_{\tilde{\varepsilon},t_0}}{\Pi_{0,t_0}} > 1$. This contradicts with the initial assumption of $\frac{N_{\tilde{\varepsilon},t_0}}{N_{0,t_0}} > C_{t_0}^{\tilde{\varepsilon}(\sigma-1)\alpha}$

²²Suppose it is violated for $\tilde{t} > t_0$, $\frac{\Pi_{\tilde{\varepsilon},\tilde{t}}}{\Pi_{0,\tilde{t}}} \geq \frac{\Pi_{\tilde{\varepsilon},t_0}}{\Pi_{0,t_0}}$. The profit is continuous in time, hence there exist \tilde{t} such that $\frac{\Pi_{\tilde{\varepsilon},\tilde{t}}}{\Pi_{0,\tilde{t}}} = \frac{\Pi_{\tilde{\varepsilon},t_0}}{\Pi_{0,t_0}}$ and profit ratio is increasing in the neighborhood of \tilde{t} . Following from the equation (50), we have $\frac{N_{\tilde{\varepsilon},\tilde{t}}}{N_{0,\tilde{t}}} \frac{1}{C_{\tilde{t}}^{\tilde{\varepsilon}(\sigma-1)\alpha}} = \frac{N_{\tilde{\varepsilon},t_0}}{N_{0,t_0}} \frac{1}{C_{t_0}^{\tilde{\varepsilon}(\sigma-1)\alpha}}$. By the initial assumption $\frac{N_{\tilde{\varepsilon},\tilde{t}}}{N_{0,\tilde{t}}} \frac{1}{C_{\tilde{t}}^{\tilde{\varepsilon}(\sigma-1)\alpha}}$ is increasing in the neighborhood of \tilde{t} and, hence, profit ratio is decreasing. This contradicts with the choice of \tilde{t} .

because it implies $\frac{\Pi_{\varepsilon,t_0}}{\Pi_{0,t_0}} < 1$.

D The Social Planner Problem

The social planner problem attempts to optimize the discounted flow of utility for households subject to their nonhomothetic CES preferences across goods, the production technology, and the innovation technology. It does so by determining how much labor to allocate each type of good for production and how much labor to allocate to each sector for innovation subject to the aggregate supply of labor.

$$\max_{\{L_{\varepsilon i,t}\}, \{L_{R\varepsilon,t}\}} L \int_0^\infty \exp(-\delta t) \ln(C_t) dt \quad (52)$$

$$\text{s.t.} \quad 1 = \left(\int_0^\infty (C_t^{-\varepsilon} C_{\varepsilon,t})^{\frac{\rho-1}{\rho}} d\varepsilon \right)^{\frac{\rho}{\rho-1}} \quad (53)$$

$$C_{\varepsilon,t} = \left(\int_0^{N_{\varepsilon,t}} C_{\varepsilon i,t}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} \quad (54)$$

$$Y_{\varepsilon i,t} = L_{\varepsilon i,t} \quad (55)$$

$$\dot{N}_{\varepsilon,t} = \eta N_t L_{R\varepsilon,t} \quad (56)$$

$$Y_{\varepsilon i,t} = L C_{\varepsilon i,t} \quad (57)$$

$$L = \int_0^\infty \int_0^{N_{\varepsilon,t}} L_{\varepsilon i,t} di d\varepsilon + \int_0^\infty L_{R\varepsilon,t} d\varepsilon \quad (58)$$

Similar to the household problem in the competitive equilibrium, the social planner problem can also be broken down into an intra- and an inter-temporal problem. The intra-temporal problem sets out to determine what the optimal distribution of goods across sectors is while the inter-temporal one determines how many new goods to create over time.²³ The intra-temporal problem can be written as follows. Given the labor allocation, $L_{Y,t}$ and total number of products,

²³The keen-eyed observer will note that the intra- versus inter-temporal breakdown of the problem is not entirely accurate since the intra-temporal problem of determining the optimal distribution of goods is indeed one that is enacted through the inter-temporal mechanism of research. This is innocuous, however, since the cost of innovation in any sector is the same. The problem is identical to one where the social planner determines the total number of new goods to innovate which are to be allocated to each sector in the next period. The problem of allocating new products across sectors is equivalent to allocation labor input to research across sectors.

N_t , the social planner seeks to

$$\max_{\{N_{\varepsilon,t}\}} C_t \quad \text{s.t.} \quad 1 = \int_0^\infty \left(C_t^{-\varepsilon} \frac{L_{Y,t}}{L} \frac{N_{\varepsilon,t}^{\frac{\sigma}{\sigma-1}}}{N_t} \right)^{\frac{\rho-1}{\rho}} d\varepsilon \quad (59)$$

$$N_t = \int_0^\infty N_{\varepsilon,t} d\varepsilon \quad (60)$$

Here, we have already imposed that the amount of labor for production of each product is identical and given by

$$L_{Y\varepsilon,t} = \frac{L_{Y,t}}{N_t} \quad (61)$$

This is optimal due to the households strict preference for variety. Any variation in labor input for production across sectors is absorbed by the variation in mass of products across sectors. Solving this problem yields the optimal distribution of products across sectors

$$\frac{N_{\varepsilon,t}}{N_t} = \left(C_t^{-\varepsilon(\sigma-1)} \left(\frac{L_{Y,t}}{L} \right)^{\sigma-1} N_t \right)^{-\frac{1-\rho}{\sigma-\rho}} \quad (62)$$

Plugging this back into the implicit preference constraint in (59), the constraint reduces to the optimal distribution of products across sectors

$$1 = \int_0^\infty \left(C_t^{-\varepsilon(\sigma-1)} \left(\frac{L_{Y,t}}{L} \right)^{\sigma-1} N_t \right)^{-\frac{1-\rho}{\sigma-\rho}} d\varepsilon = \int_0^\infty \frac{N_{\varepsilon,t}}{N_t} d\varepsilon \quad (63)$$

In hindsight, it is an obvious result that the optimal supplied basket of goods should perfectly match the shape of the preferences indicating the preferred basket of goods that is demanded. It also lets us define a mapping between utility C_t and the total mass of products N_t . Conditional on always having the optimal distribution of products, the total number of products is the indicator for the amount of development and wealth of the economy. Solving the integral above in (63) yields

$$\ln C_t = -\frac{\sigma-\rho}{(1-\rho)(\sigma-1)} \left(\left(\frac{L_{Y,t}}{L} \right)^{\sigma-1} N_t \right)^{-\frac{1-\rho}{\sigma-\rho}} \quad (64)$$

This appears identical to the equilibrium relationship we found in the competitive equilibrium in (91) which determined the distribution of product, but it will differ slightly from it due to the optimal labor share to L_Y/L is lower the optimal allocation. Moreover, it is not equivalent to the

mapping in partial equilibrium in (77) which households use when making their decisions. This is the source of the nonhomothetic CES externality. Households do not fully incorporate their spending's effect on the realized composition of the basket of products. Nor do they realize that when they allocate more labor to research that they are modifying their future basket of goods.

We next turn to analyzing the inter-temporal allocations. The social planner seeks to maximize discounted utility over time by deciding how much labor to allocate to production, which we know is evenly dispersed among products, and how much to allocate to aggregate product innovation when the new products are allocated optimally among sectors.

$$\max_{L_{Y,t}, L_{R,t}, \dot{N}_t} L \int_0^\infty \exp(-\delta t) \ln(C_t) dt \quad (65)$$

$$\text{s.t.} \quad \dot{N}_t = \eta N_t L_{R,t} \quad (66)$$

$$\ln(C_t) = -\frac{\sigma - \rho}{(1 - \rho)(\sigma - 1)} \left(\left(\frac{L_{Y,t}}{L} \right)^{\sigma-1} N_t \right)^{-\frac{1-\rho}{\sigma-\rho}} \quad (67)$$

$$L = L_{Y,t} + L_{R,t} \quad (68)$$

The amount of labor to allocate for research in each sector is accounted for in the optimal distribution of firms, and the amount of labor to allocate for production for each good is identical. The euler equation for social planner is provided in terms of the number of products which as an alternative to expenditure growth in the competitive equilibrium.

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\alpha + \frac{1}{\sigma-1}} \left(\frac{\eta L}{\sigma - 1} - \delta \right) \quad (69)$$

where $\alpha = \frac{1-\rho}{\sigma-\rho}$ as before.

E Baseline Model with Nonhomothetic CES Preferences and Gamma Distributed Sectoral Weights

E.1 Environment

Household Preferences, Endowments, and Demographics The economy is populated by a mass L of homogeneous households. Each household is endowed with one unit of labor that is inelastically supplied. Households have preferences over an infinite stream of consumption bundles

$\{C_t\}_{t=0}^\infty$ according to

$$\int_0^\infty e^{-\delta t} \ln C(C_t) dt \quad (70)$$

where $\delta > 0$ is the discount factor and $C(\cdot)$ is the intra-period utility aggregator over the consumption bundle C_t .

Households can smooth consumption over time through investments in an asset A_t which represents shares in the portfolio of all firms in the economy. The household budget constraint is thus given by

$$\dot{A}_t = r_t A_t + W_t - E_t,$$

where W_t , Π_t , and E_t denotes the wage rate, aggregate profits, and household expenditures in the economy.

At time t , the goods available to households to construct their consumption bundle belong to the product space $(\varepsilon, i) \in [0, \infty) \times [0, N_{\varepsilon,t}]$. The households' preferences over these goods are given by a nested CES structure. The outer nest is indexed by ε and defined through nonhomothetic CES preferences, while the inner nest is indexed by i and is a homothetic CES. Formally, within-period household preferences are defined by

$$1 = \left(\int_0^\infty \left(\varepsilon^{-\beta} g(U_t)^{-\varepsilon} C_{\varepsilon,t} \right)^{\frac{\rho-1}{\rho}} d\varepsilon \right)^{\frac{\rho}{\rho-1}} \quad \text{and} \quad C_{\varepsilon,t} = \left(\int_0^{N_{\varepsilon,t}} C_{\varepsilon i,t}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} \quad (71)$$

where $\rho, \sigma > 0$ and $g(\cdot) : \mathbb{R} \rightarrow [0, 1)$ is a monotonically increasing, continuously differentiable concave function that corresponds to the within-period aggregator in Equation (70), $C_t \equiv g(U_t)$.²⁴ For our baseline model we assume that goods are complements across the outer nest and substitutes within, $0 < \rho < 1 < \sigma < \infty$. A natural interpretation of the outer nest is as different sectors in the economy, and the inner nest of goods within the sector. The parameter $\beta \geq 0$ enables the existence of sector-dependent taste-parameters which vary monotonically in terms of a sector's expenditure elasticity rank ordering. The inclusion of this is not important for the main mechanism of our model. Therefore, for clarity, we will derive our model for $\beta = 0$, while providing the relevant results for $\beta > 0$ in tandem when necessary.

²⁴For example, the functional form $g(U_t) = 1 - \frac{1}{1+U}$ satisfies these conditions.

Innovation and Production Technologies Production of each intermediate in the production set is linear in labor

$$Y_{\varepsilon i,t} = L_{\varepsilon i,t}. \quad (72)$$

New products can be created in any ε sector through an innovation technology which is identical across ε sectors. The innovation flow of new products in sector ε is given by

$$\dot{N}_{\varepsilon,t} = \eta N_t L_{R\varepsilon,t} \quad (73)$$

where $L_{R\varepsilon,t}$ is total amount of labor hired for research in sector ε , $N_{\varepsilon,t}$ is the total number of product varieties in sector ε , and $N_t = \int_0^\infty N_{\varepsilon,t} d\varepsilon$ is the total number of product varieties in the economy at time t .

Markets and Patents Labor markets are competitive while firms selling to households engage in monopolistic competition. There is free entry in the innovation sector where Firms are awarded a perpetual patent upon successful innovation of a new product.

E.2 Equilibrium Characterization

We begin our analysis with the competitive equilibrium of the economy, that is, households maximize utility given their budget constraint taking prices as given, firms maximize profits, and goods and labor markets clear.

Household Optimality First, we derive household demand and expenditure along the lines of [Comin et al. \(2021\)](#). Given total household expenditure, E_t , and the price vector $\{P_{\varepsilon i,t}\}$, cost minimization implies

$$C_{\varepsilon i,t} = P_{\varepsilon i,t}^{-\sigma} P_{\varepsilon,t}^{\sigma-\rho} E_t^\rho C_t^{\varepsilon(1-\rho)} \quad (74)$$

$$\text{with } P_{\varepsilon,t} = \left(\int_0^{N_{\varepsilon,t}} P_{\varepsilon i,t}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} \quad \text{and} \quad E_t = \left(\int_0^\infty (C_t^\varepsilon P_{\varepsilon,t})^{1-\rho} d\varepsilon \right)^{\frac{1}{1-\rho}}. \quad (75)$$

In order to concisely characterize the household's optimal inter-temporal allocations, we need first to characterize a closed-form mapping between expenditures and aggregate utility which is implicitly defined in (75). Foreshadowing the properties of the equilibrium, we know that prices

across sectors are characterized by to an exponential function.

$$P_{\varepsilon,t} = \zeta_t \exp(\chi_t \varepsilon) \quad (76)$$

Note that the parameters ζ_t and χ_t are only viewed as parameters from the perspective of the household, and in general will be determined by equilibrium forces. Moreover, as far as sectoral prices are nominal terms, ζ_t can be scaled up or down as suited and we thus refer to it as the overall price-level, while χ_t fully characterizes the relative prices across sectors. With $\rho < 1$, the integral in (75) will be well defined when $\ln C_t + \chi_t < 0$. This will be verified later when equilibrium prices are determined. The closed-form mapping between aggregate utility and expenditures for the household is given by

$$C_t = \exp \left(-\chi_t - \frac{\zeta_t^{1-\rho}}{1-\rho} E_t^{-(1-\rho)} \right). \quad (77)$$

Using this mapping the household euler equation is given by

$$\frac{\dot{E}_t}{E_t} = \frac{1}{2-\rho} \left((r_t - \delta) + (1-\rho) \frac{\dot{\zeta}_t}{\zeta_t} \right), \quad (78)$$

We refer to ζ_t as the overall price level. The household transversality condition is given by

$$\lim_{t \rightarrow \infty} \exp \left(- \int_0^t r_s ds \right) N_t V_t = 0, \quad (79)$$

where $N_t V_t = A_t$ is the combined present value of all firms.

Firm Optimality Equation (74) shows that the demand for good εi is isoelastic in its own price. Under monopolistic competition, the firm producing good εi finds optimal to set a constant markup over the marginal cost, W , determined by the within sector elasticity of substitution

$$P_{\varepsilon i,t} = \frac{\sigma}{\sigma-1} W_t. \quad (80)$$

Note that all which is needed to characterize the distribution of prices across sectors is the distribution of products within sectors. The corresponding firm profits are

$$\Pi_{\varepsilon i,t} = \frac{1}{\sigma-1} W_t Y_{\varepsilon i,t}. \quad (81)$$

Since all firms in a ε -sector are identical, we have that the sectoral index $P_{\varepsilon i}$ in Equation (74) is

$$P_{\varepsilon,t} = \frac{\sigma}{\sigma-1} W_t N_{\varepsilon,t}^{-\frac{1}{\sigma-1}}. \quad (82)$$

Combining this result, with market clearing for good εi

$$Y_{\varepsilon i,t} = LC_{\varepsilon i,t} \quad (83)$$

and the demand Equation (74), a firm's profits is

$$\Pi_{\varepsilon i,t} = \frac{L}{\sigma} \left(\frac{\sigma}{\sigma-1} \right)^{1-\rho} W_t^{1-\rho} N_{\varepsilon,t}^{-\frac{\sigma-\rho}{\sigma-1}} E_t^\rho C_t^{\varepsilon(1-\rho)}. \quad (84)$$

Free Entry and the Distribution of Products and Prices With a perpetual patent, the value of a product at any given time, t , is equal to the sum of all its future discounted profits,

$$V_{\varepsilon i,t} = \int_t^\infty \exp\left(-\int_t^s r(\tau) d\tau\right) \Pi_{\varepsilon i}(s) ds. \quad (85)$$

The mass of firms N_ε occupying each ε sector is endogenously determined by the free entry. Firms can select any ε sector in which to innovate. Aggregating the research technology (73) over all sectors yields the aggregate research technology

$$\dot{N}_t = \eta N_t L_{R,t} \quad (86)$$

where $\dot{N}_t = \int_0^\infty \dot{N}_{\varepsilon,t} d\varepsilon$ and $L_{R,t} = \int_0^\infty L_{R\varepsilon,t} d\varepsilon$ denote the aggregate flow of new products and employment in research. Firms innovate new product varieties in each ε sector until the flow value of doing research is equal to the labor cost of doing so

$$\eta N_t V_{\varepsilon i,t} = W_t. \quad (87)$$

Since the cost of research across ε is the same, firms will enter until the value of products are identical across them too. The only difference across ε sectors is the number of products within each sector, i.e. N_ε . Using the definition of the net present value of an innovation across different products εi , it follows that $\Pi_{\varepsilon i,t} = \Pi_t$ almost everywhere. The symmetry in profits across products implies that each product is produced in the same amount and employs the same amount of labor.

For this reason the labor market clearing condition is given by

$$L = L_{Y,t} + L_{R,t} = N_t L_{\varepsilon i,t} + L_{R,t} \quad (88)$$

where $L_{Y,t}$ is the total amount of labor used for production. Furthermore, the definition of aggregate expenditures as the sum of all individual purchases reduces to

$$LE_t = \frac{\sigma}{\sigma - 1} W_t L_{Y,t} \quad (89)$$

The symmetry in profits across sectors also lets us obtain an expression for the number of products in each sector, by rearranging (84) and using (89) to get

$$N_{\varepsilon,t} = \left(\frac{LE_t}{\sigma \Pi_t} \left(\frac{L_{Y,t}}{L} \right)^{\rho-1} C_t^{\varepsilon(1-\rho)} \right)^{\frac{\sigma-1}{\sigma-\rho}}. \quad (90)$$

This expression still explicitly relies on the aggregate utility level C , so it remains to fully characterize the closed-form mapping between household aggregate utility and expenditures. Rather than use an exogenous assumed price distribution as we did for the household problem, we can now write it using (82) and (90). With the price distribution in hand, we can solve the integral in expenditure function implied by the nonhomothetic CES outer nest in (75), this yields the following mapping between utility and expenditures

$$\ln C_t = -\frac{\sigma - \rho}{(\sigma - 1)(1 - \rho)} \left(\frac{LE_t}{\sigma \Pi_t} \left(\frac{L_{Y,t}}{L} \right)^{\sigma-1} \right)^{-\frac{1-\rho}{\sigma-\rho}}, \quad (91)$$

which features strong monotonicity between utility and expenditures. This also verifies that that the same mapping in the partial equilibrium case for the household is indeed well defined.²⁵

We can now concisely characterize the distribution of products across sectors, ε , without needing to allude to utility levels. Substituting (91) into (90) and reorganizing results in the following

²⁵When $\beta > 0$, this closed-form mapping between utility and expenditures becomes

$$\ln C_t = -\frac{1}{\alpha(\sigma - 1)} \left(\Gamma(1 + \beta\alpha(\sigma - 1)) \left(\frac{LE_t}{\sigma \Pi_t} \left(\frac{L_Y}{L} \right)^{\sigma-1} \right)^{-\alpha} \right)^{\frac{1}{1 + \beta\alpha(\sigma-1)}}, \quad (92)$$

where $\Gamma(\cdot)$ is the standard gamma function and only enters as a normalization constant.

expression

$$N_{\varepsilon,t} = N_t \Psi_t(N_t) \exp(-\Psi_t(N_t) \cdot \varepsilon) \quad (93)$$

$$\text{where } \Psi_t(N_t) = \left(\left(\frac{L_{Y,t}}{L} \right)^{\sigma-1} N_t \right)^{-\frac{1-\rho}{\sigma-\rho}} \quad \text{and} \quad N_t = \frac{L E_t}{\sigma \Pi_t}. \quad (94)$$

Note how the shape of the distribution changes as expenditures increase. The mass point at zero becomes higher and the decay becomes slower indicating that there is always positive growth across all sectors as expenditures increase albeit faster growth in higher ε sectors.²⁶

With the product distribution in hand, the price distribution follows using (82),

$$P_{\varepsilon,t} = E_t \underbrace{\left(\frac{L_{Y,t}}{L} \right)^{-1} (N_t \Psi_t(N_t))^{-\frac{1}{\sigma-1}}}_{\zeta_t} \exp \left(\underbrace{\frac{1}{\sigma-1} \Psi_t(N_t) \cdot \varepsilon}_{\chi_t} \right), \quad (95)$$

which in turn provides us with equilibrium characterizations of the price-level, ζ_t , and the shape of the exponential price distribution, χ_t .

E.3 Balanced Aggregate Growth

Our balanced growth path (BGP) will be defined by having a constant interest rate, r , a constant share of workers in research production, L_R , and the number of products growing at a constant rate g_N . We solve for this BGP by normalizing the price-level $\zeta_t = 1$, which in our baseline model is equivalent to normalizing the sector zero good, $P_{0,t} = 1$. The resulting BGP will exist when $\frac{\eta L}{\sigma-1} > \delta$, which ensures that there is positive growth. The transversality condition is always satisfied within our specified parametric bounds.

From (87), (89), and (94), we have proportional growth rates in expenditures, wages, and the profits and present value of products,²⁷

$$g_E = g_W = \frac{1}{\sigma-\rho} g_N, \quad g_\Pi = g_V = \left(\frac{1}{\sigma-\rho} - 1 \right) g_N. \quad (96)$$

The growth rate of the overall price-level, g_ζ , is zero by normalization. The interest rate is thus

²⁶This relation between aggregate expenditures and total number of products also holds in a standard expanding varieties model as in Acemoglu (2009a). This can be seen from the fact that one can also derive it by combining (81), (88), and (89), rather than by integrating over the product distribution.

²⁷Note that the growth rate of profits and present value of a product can be either positive or negative depending on the relative elasticity of substitution across and within sectors. This property of nested CES preferences and monopolistic competition is discussed in Matsuyama, 1995.

determined by (78) to be

$$r = \delta + \left(\frac{1}{\sigma - \rho} + \alpha \right) g_N. \quad (97)$$

Lastly, the growth rate in the number of products and the research labor share is solved for using (81), (86), and (88) yielding

$$g_N = \eta L_R = \frac{1}{1 + \alpha + \frac{1}{\sigma - 1}} \left(\frac{\eta L}{\sigma - 1} - \delta \right). \quad (98)$$

E.4 Nonbalanced Sectoral Growth

The sectoral growth dynamics can be concisely described in terms of the total number of products. Since the balanced growth path features constant growth in total number of products, we will characterize how the mass of products across sectors N_ε grows with the total mass of products N . As mentioned earlier, there is always positive growth in products across all sectors due to the fact that goods are complements. From equations (93) and (94), we can fully describe the sectoral dynamics along the BGP. Recall that $\alpha = \frac{1-\rho}{\sigma-\rho} \in (0, 1)$. The sectoral dynamics are fully captured by the following two equations:²⁸

$$N_{\varepsilon,t} = \left(\left(\frac{L_Y}{L} \right)^{\sigma-1} N_t \right)^{-\alpha} \exp \left(- \left(\left(\frac{L_Y}{L} \right)^{\sigma-1} N_t \right)^{-\alpha} \cdot \varepsilon \right) N_t \quad (101)$$

$$\frac{\dot{N}_{\varepsilon,t}}{N_{\varepsilon,t}} = \left((1 - \alpha) + \alpha \left(\left(\frac{L_Y}{L} \right)^{\sigma-1} N_t \right)^{-\alpha} \cdot \varepsilon \right) \frac{\dot{N}_t}{N_t} \quad (102)$$

Since the total mass of products features exponential growth, we can plot the sectoral dynamics against the logarithm of it to understand how they vary as a function of time. Figure 11 depicts these dynamics. There is a sequential relationship in growth where sectors featuring smaller ε increase in mass initially, after which higher ε sectors begin to take off. This is shown in the top

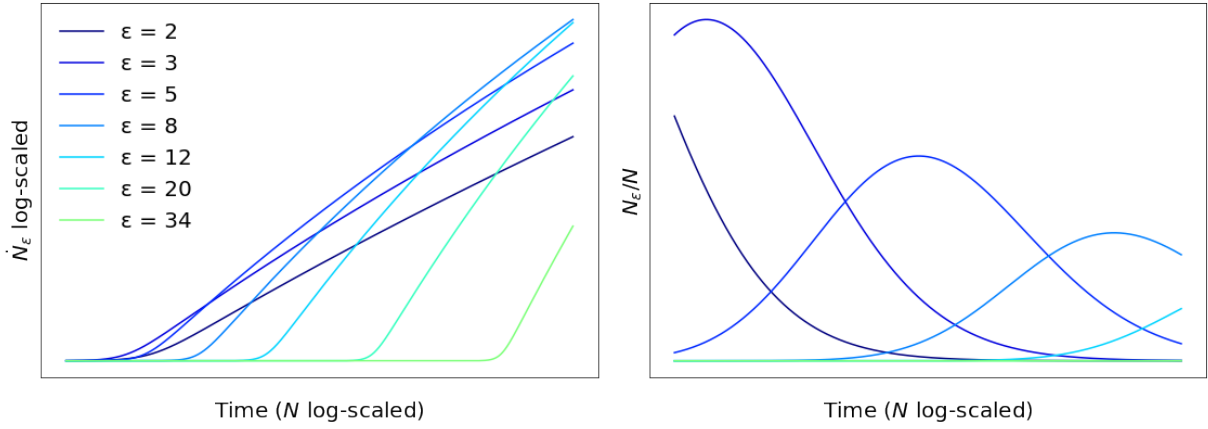
²⁸When $\beta > 0$ the sectoral product distribution and its dynamics are characterized instead by

$$N_{\varepsilon,t} = \Psi(N_t) \varepsilon^{\beta\alpha(\sigma-1)} \exp \left(- (\Gamma(1 + \beta\alpha(\sigma-1)) \Psi(N_t))^{\frac{1}{1+\beta\alpha(\sigma-1)}} \cdot \varepsilon \right) N_t, \quad (99)$$

$$\frac{\dot{N}_{\varepsilon,t}}{N_{\varepsilon,t}} = \left((1 - \alpha) + \frac{\alpha}{1 + \beta\alpha(\sigma-1)} (\Gamma(1 + \beta\alpha(\sigma-1)) \Psi(N_t))^{\frac{1}{1+\beta\alpha(\sigma-1)}} \cdot \varepsilon \right) \frac{\dot{N}_t}{N_t}. \quad (100)$$

The key difference is that when $\beta > 0$ the product distribution follows a gamma distribution rather an exponential distribution. This creates a non-monotonic relationship between the expenditure-elasticity rank ordered sectors and their absolute size, which in turn allows for the overlapping sectoral dynamics in value-added shares that we see in figure 11. As shown, this overlap is made more extreme by using a higher value of β . However, the peaks will always feature an exponential decay simply due to the total mass of products becoming ever larger, and therefore the peak value-added shares ε sectors will always be smaller for higher ε .

Figure 11:
Sectoral dynamics over time.



panels of each column. Here we see that sectors take off sequentially with sectors defined by larger ε growing slightly faster than their predecessors. Moreover, as expenditures increase in the limit, all ε sectors grow at the rate $(1 - \alpha)g_N$.²⁹

So far, we have characterized the sectoral dynamics along the BGP where the distribution of products always follows a gamma distribution. When the initial distribution is not the one induced by the BGP, one can show that the sectoral distribution converges to the gamma distribution featured by the BGP. Appendix C provides the proof.

E.5 Expenditure Elasticities and Expenditure Share Peaks

One can derive the expenditure elasticities by combining the household utility to expenditure mapping in (77) with the sectoral demand function,

$$C_{\varepsilon,t} = \left(\frac{P_{\varepsilon,t}}{E_t} \right)^{-\rho} C_t^{(1-\rho)\varepsilon}. \quad (103)$$

The expenditure elasticity of demand is defined by $\eta_{\varepsilon,t} = \frac{\partial \ln C_{\varepsilon,t}}{\partial \ln E_t}$. The chosen nominal normalization in this matters since an increase in expenditures may be associated with a large increase in well-being or not at all depending on how the price-level changes over time. For this analysis we will maintain the price-level normalization that we have used so far. The expenditure elasticity of

²⁹Note, however, that technically there are always new sectors with larger ε which take off for any level of expenditures and so the proportion that each sector covers goes to zero.

demand is then given by

$$\eta_{\varepsilon,t} = \rho + (1 - \rho)E_t^{\rho-2}\varepsilon. \quad (104)$$

Given the strongly monotonic and time-invariant mapping between ε and η_ε , it is clear that defining sectors by ε is equivalent to defining and ordering sectors by their associated expenditure elasticity. Moreover, the dependence on the aggregate expenditure level (hence utility level) implies that the expenditure elasticity of a given product or sector is diminishing as households get wealthier. At low income level's, nearly all products are luxuries goods, and as incomes grow, goods slowly turn from being luxuries into necessities in the spirit of Georgo Katona's *Mass Consumption Societies*.

Since sectors feature hump-shaped value-added shares over time, as depicted in the right panel of figure 11, it is of interest to explore how the sectoral peaks are associated with their expenditure elasticities. In other words, what is the expenditure elasticity of demand for a given sector at the moment it peaks in terms of it's value-added share? For a given ε sector, the household expenditure level at the time of it's peak can be derived to be

$$E_\varepsilon^{\text{peak}} = \frac{\sigma - 1}{\sigma} \varepsilon^{\frac{1}{1-\rho}}, \quad (105)$$

which captures what the right panel of figure 11 qualitatively does: that ε sectors peak sequentially at increasing expenditure levels. We can combine (105) and (104) to characterize at what expenditure elasticity different sectors peak. This is given by

$$\eta_\varepsilon^{\text{peak}} = \rho + (1 - \rho) \left(\frac{\sigma - 1}{\sigma} \right)^{2-\rho} \varepsilon^{-\frac{1}{1-\rho}}. \quad (106)$$

Thus we find that the expenditure elasticity of a given sector at the time of it's value-added share peak is smaller for larger ε sectors. One notable feature is that it implies some sectors will peak with an expenditure elasticity above 1 denoting them as luxury goods while "advanced" sectors will peak with expenditure an elasticity below one. Recall, again, however, that this result depends on which type of nominal normalization one uses to derive the BGP.

F Schumpeterian Growth Featuring Engel's Treadmill

Our growth mechanism may also be incorporated into a model of vertical innovation where incumbent and entering firms innovate in direct competition with each other. Rather than the development of new products, innovation leads to quality improvements in existing products. Maintaining a parallel to our baseline model, research and production requires a scarce resource as input, namely labor, and growth is ensured thanks to aggregate knowledge spillovers.

Preferences and Production Products are still defined over a two-dimensional space with the alteration that there now exists a normalized unit mass of products within each sector. The product space is given by $(\varepsilon, i) \in [0, \infty) \times [0, 1]$. Sectors are still complementary while product varieties within a sector are substitutes, with respective elasticities of substitution $0 < \rho < 1 < \sigma$. Each differentiated product within in a sector is associated with the quality level of the leading firm that is producing it. Only the highest quality version of a product is produced at any given time since different qualities are perfect substitutes. The quality level affects the household preferences for the product in the CES aggregator as follows

$$C_{\varepsilon,t} = \left(\int_0^1 q_{\varepsilon i,t} C_{\varepsilon i|q,t}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, \quad (107)$$

where $C_{\varepsilon i|q,t}$ is amount consumption of the variety i in sector ε with quality level q , and similarly $q_{\varepsilon i,t}$ is the quality level of variety i in sector ε . The preferences across sectors are defined according to the NhCES aggregator as in the baseline model and the sectoral demand in equation (103).

Quality is incorporated into the production and research technologies by requiring one unit of labor for quality unit of a product produced, scaled appropriately. That is, higher quality goods take more labor to produce. The production technology is thus given by

$$Y_{\varepsilon i,t} = \frac{L_{\varepsilon i,t}}{q_{\varepsilon i,t}}. \quad (108)$$

Firms are awarded a perpetual patent for their quality innovation and compete monopolistically. This implies that the price of a given product, which the firm optimally set as a constant markup over marginal cost, is

$$P_{\varepsilon i,t} = \frac{\sigma}{\sigma-1} W_t q_{\varepsilon i,t}. \quad (109)$$

Let the average quality within a sector be defined by $Q_{\varepsilon,t} = \int_0^1 q_{\varepsilon i,t} di$. From the within sector ex-

penditure minimization problem, the sector-specific price index is fully characterized by a sector's average quality level,

$$P_{\varepsilon,t} = \frac{\sigma}{\sigma-1} W_t Q_{\varepsilon,t}^{-\frac{1}{\sigma-1}} \quad (110)$$

which is analogous to the role the number of varieties played for the sectoral price-index in (82) in the baseline model. Note that in this setup, the equilibrium demand for any product within a sector is also identical, despite heterogeneous quality levels, and only depends on the sector's average quality level

$$C_{\varepsilon i|q,t} = C_{\varepsilon i,t} = Q_{\varepsilon,t}^{-\frac{\sigma}{\sigma-1}} C_{\varepsilon,t} \quad (111)$$

and thus also equal to average product demand in that sector. Profits for a specific product are dependent on its quality, however, and given by

$$\Pi_{\varepsilon i|q,t} = \frac{1}{\sigma-1} L W_t q_{\varepsilon i,t} C_{\varepsilon i,t}. \quad (112)$$

Incumbent and Entrant Innovation There are two types of innovation processes, which we denote the incumbent technology and the entrant technology.³⁰ The incumbent technology enables a firm to improve on its current quality level by a factor $\lambda^I > 0$. The flow rate of incumbent innovations depends on how much incumbent R&D the firm performs, which requires labor, and is given by

$$z_{\varepsilon i,t}^I = \eta^I Q_t \frac{L_{R\&D,t}^I}{q_{\varepsilon i,t}}. \quad (113)$$

As in the production technology, more labor is necessary to innovate for higher quality products. Q_t captures the same strong scale effects as in our baseline model. Rather than being a function of the total number of products, the scale effects are now determined by the aggregate quality level $Q_t = \int_0^\infty Q_{\varepsilon,t} d\varepsilon$.³¹ The leading incumbent firm will perform incumbent R&D until the labor cost equals the gains from the expected quality innovations of its product, that is, $W_t q_{\varepsilon i,t} = \eta^I Q_t (V_{\varepsilon i|\lambda^I q,t} - V_{\varepsilon i|q,t})$, where $V_{\varepsilon i|q,t}$ is the cumulative sum of the expected future discounted profits for a product of quality $q_{\varepsilon i,t}$. We will see below that $V_{\varepsilon i|q,t}$ is linear in $q_{\varepsilon i,t}$, and thus the condition above becomes

$$W_t q_{\varepsilon i,t} = \eta^I (\lambda^I - 1) Q_t V_{\varepsilon i|q,t}. \quad (114)$$

³⁰The names are given based on which type of firm ends up performing the type of innovation technology. It is not based upon who has access to a given technology.

³¹These scale effects may similarly be weakened while including population growth to get a semi-endogenous growth version of the model.

The entrant research technology enables a firm to innovate for a product quality level that it does not own. An entrant innovation improves on the existing quality level by a factor of $\lambda^E > \lambda^I$. The flow rate of entrant innovations depends on how much entrant R&D firms are performing, which also requires labor, and is given by

$$z_{ei,t}^E = \left(\eta^E Q_t \frac{L_{Rei,t}^E}{q_{ei,t}} \right)^\phi, \quad (115)$$

where $L_{Rei,t}^E$ is the total amount of entrant innovation for this product by all firms combined, and $\phi \in (0, 1)$ implies that there are decreasing returns in total entrant innovation.³² There is free entry in entrant research and, while taking other firm's research efforts as given, competing firms will perform research until marginal cost of research equals the expected marginal return from innovation, given by

$$W_t q_{ei,t} = \phi \lambda^E \eta^E Q_t V_{ei|q,t} \left(\eta^E Q_t \frac{L_{Rei,t}^E}{q_{ei,t}} \right)^{\phi-1}. \quad (116)$$

Combining this with the incumbent innovation condition in (114) implies that the flow rate of innovation by entrants is constant across time and all sectors³³

$$z_{ei,t}^E = z^E = \left(\frac{\phi \eta^E \lambda^E}{\eta^I (\lambda^I - 1)} \right)^{\frac{\phi}{1-\phi}}. \quad (117)$$

The growth rate of the aggregate quality level, Q_t , can then be backed out from the intensities of innovation by incumbent and entering firms. The expected amount of quality growth of a single product is $\dot{q}_{ei,t} = (\lambda^I - 1) z_{ei,t}^I q_{ei,t} + (\lambda^E - 1) z_{ei,t}^E q_{ei,t}$. Substituting in for (113) and noting that the entrant innovation intensity is constant, integrating across the entire product space yields the following law of motion of aggregate quality

$$\frac{\dot{Q}_t}{Q_t} = (\lambda^I - 1) \eta^I L_{R,t}^I + (\lambda^E - 1) z^E \quad (118)$$

where $L_{R,t}^I$ is economy-wide amount of labor that is allocated to incumbents' research efforts.

³²This may be interpreted as many different potential entrants partially performing the same research and crowding out each other's efforts.

³³While the intensity of innovation by entrants is constant across sectors and time, the amount of R&D labor dedicated is not since sectors with higher quality level require more labor for the same flow rate of innovation to take place.

Closed Form Utility Mapping As in the baseline model, we can similarly derive a closed form mapping between household utility and expenditures. Here, we can derive the distribution of average quality across sectors as a function of utility by inverting the firms Hamiltonian-Jacobi-Bellman (HJB) equation. Combining this with the expenditure level derived from a household's expenditure minimization problem, yields the closed-form mapping.

The HJB associated with a specific product is

$$r_t V_{ei|q,t} = \max_{L_{Rei,t}} \Pi_{ei|q,t} + \underbrace{(\lambda^I - 1) z_{ei,t}^I V_{ei|q,t} - w_t L_{Rei,t} - z^E V_{ei|q,t}}_{=0} + \dot{V}_{ei|q,t} \quad (119)$$

Note that in equilibrium the two middle terms cancel for any level of incumbent research due to the constant returns to scale in the incumbent research technology. Note also for (114) that the the growth rate of in the value of a product conditional on it's current quality level only depends on the aggregate growth rates of quality and wages. Any level of incumbent research is optimal from a cost-versus-benefit perspective, and the equilibrium amount will be pinned down by the demand-induced market-size of each sector. The HJB can thus be significantly simplified to³⁴

$$r_t + z^E - \frac{\dot{W}_t}{W_t} + \frac{\dot{Q}_t}{Q_t} = \frac{\Pi_{ei|q,t}}{V_{ei|q,t}}. \quad (120)$$

Substituting in for the profit condition (112), the product and sector demand conditions, (111) and (103), and the incumbent innovation condition (114), yields a formulation for the average quality levels across sectors as a function of household utility,

$$Q_{\varepsilon,t} = \left(\frac{\eta^I (\lambda^I - 1)}{\sigma - 1} \frac{L Q_t}{z^E + r_t - \frac{\dot{W}_t}{W_t} + \frac{\dot{Q}_t}{Q_t}} \left(\frac{L_{Y,t}}{L} \right)^\rho \right)^{\frac{\sigma-1}{\sigma-\rho}} C_t^{\frac{(1-\rho)(\sigma-1)}{\sigma-\rho} \varepsilon} \quad (121)$$

Analogous to derivation in the baseline model, we can take the household expenditure level in (75) and substitute in for the sectoral price index in (110) and the sectoral quality level just above. This yields a well-defined integral which has the following closed-form solution

$$\ln C_t = -\frac{\sigma - \rho}{(\sigma - 1)(1 - \rho)} \left(\frac{\eta^I (\lambda^I - 1)}{\sigma - 1} \frac{L Q_t}{z^E + r_t - \frac{\dot{W}_t}{W_t} + \frac{\dot{Q}_t}{Q_t}} \left(\frac{L_{Y,t}}{L} \right)^\sigma \right)^{-\frac{1-\rho}{\sigma-\rho}} \quad (122)$$

³⁴Given the linearity in $q_{ei,t}$ of profits in (112), the linearity of $V_{ei|q,t}$ in $q_{ei,t}$ follows immediately from this simplified HJB.

Plugging this back into the sectoral quality level above and integrating over sectors leads to simplified expressions in equilibrium for the HJB average quality distribution, and aggregate utility in (120), (121), and (122)

$$r_t + z^E - \frac{\dot{V}_t}{V_t} = \frac{\eta^I(\lambda^I - 1)}{\sigma - 1} L_{Y,t}, \quad (123)$$

$$Q_{\varepsilon,t} = Q_t \Psi_t(Q_t) \exp(-\Psi_t(Q_t)\varepsilon) \quad (124)$$

$$\ln C_t = -\frac{\sigma - \rho}{(\sigma - 1)(1 - \rho)} \Psi_t(Q_t) \quad (125)$$

where $\Psi_t(Q_t) = \left(Q_t \left(\frac{L_{Y,t}}{L} \right)^{\sigma-1} \right)^{-\alpha}$ and $\alpha = \frac{1-\rho}{\sigma-\rho}$ as before. Note the complete parallel between the equilibrium quality distribution and closed-form utility mapping with that in the baseline model in (93) and (91). The price distribution across sectors follows immediately from plugging (124) into (110),

$$P_{\varepsilon,t} = \underbrace{\left(\frac{L_{Y,t}}{L} \right)^{-\frac{\sigma-1}{\sigma-\rho}} E_t Q_t^{-\frac{1}{\sigma-\rho}}}_{\zeta_t} \exp \left(\underbrace{\frac{1}{\sigma-1} \Psi_t(Q_t) \cdot \varepsilon}_{\chi_t} \right). \quad (126)$$

The last piece of the model is the household euler equation which is unchanged from the baseline and given by (78).

Steady State Growth Path We again find a growth path characterized by a constant interest rate and labor shares, balanced growth in aggregate variables, and unbalanced growth across sectors. In parallel to the baseline model we normalize the price-level $\zeta_t = 1$. The relations between the aggregate growth rates are

$$g_E = g_W = \frac{1}{\sigma - \rho} g_Q, \quad g_\Pi = g_V = \left(\frac{1}{\sigma - \rho} - 1 \right) g_Q, \quad (127)$$

and the unbalanced sectoral growth rates are given by

$$\frac{\dot{Q}_\varepsilon}{Q_\varepsilon} = ((1 - \alpha) + \alpha \Psi(Q) \cdot \varepsilon) g_Q. \quad (128)$$

The growth rate of the aggregate average quality level, the interest rate, and labor shares to production, incumbent innovation, and entrant innovation are pinned down by

$$r = \frac{\eta^I(\lambda^I - 1)}{\sigma - 1} L_Y - z^E + \left(\frac{1}{\sigma - \rho} - 1 \right) g_Q, \quad (129)$$

$$g_Q = \left(\frac{1}{\sigma - \rho} + \alpha \right)^{-1} (r - \delta), \quad (130)$$

$$L_R^I = \frac{1}{(\lambda^I - 1)\eta^I} \left(g_Q - (\lambda^E - 1)z^E \right), \quad (131)$$

$$L_R^E = \frac{1}{\eta^E} (z^E)^{\frac{1}{\phi}}, \quad (132)$$

$$L = L_Y + L_R^I + L_R^E. \quad (133)$$