Frequent flyer programs and dynamic contracting
with limited commitment

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Abstract

I present a novel contract theoretic explanation of the profitability and management of loyalty programs. I study a model where an airline faces uncertain demand, consumers learn how much they are willing to pay for a flight over time, and the frequent flyer program provides the airline with a way to contract with consumers ex ante, before uncertainty is resolved. The firm lacks the ability to commit to future prices and allocations ex ante, and may only commit that it will allocate ex post unsold capacity to members of its loyalty program. I characterize the optimal mechanism in this setting: the airline sells miles to all consumers ex ante, it then posts a price for a guaranteed seat at the interim stage, and finally it freely gives away any unsold seats to members of its mileage program through a lottery. The optimal mechanism yields larger revenue than the canonical static screening mechanism, and is more efficient; the airline captures all of the resulting gains in total surplus through the sale of airline miles ex ante.

Job market paper

Keywords: Dynamic Contracting, Frequent Flyer Programs, Airlines

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1 Introduction

Frequent flyer programs are one of the most significant drivers of profitability in the airline industry today. At many US and foreign airlines the mileage program is arguably the only consistently profitable division of the firm. Economists generally view such programs as a tool for airlines to create switching costs for consumers and thus soften the effect of competition in the industry. I present a novel contracts-based theory of the management and value of frequent flyer programs. I consider a model where demand for seats is uncertain, consumers learn how much they are willing to pay for a flight over time, and the loyalty program provides the airline with a way to partially contract with consumers ex ante, before this uncertainty is resolved. I solve for the optimal frequent flyer program (FFP) mechanism, under the constraints that the airline cannot commit to future prices and capacity allocation, and can only commit to let members of its frequent flyer program redeem their airline miles for unsold seats. This mechanism induces a more efficient allocation of seats than the standard static monopolistic screening mechanism, and allows the airline to capture more of the total surplus that is created. Frequent flyer programs are thus a valuable yield management tool, and not only a loyalty-building tool.

American Airlines introduced the first frequent flyer program in 1981, and since then such programs have increasingly become a significant source of revenue for the airline industry. Today every major U.S. airline runs such a program, often at significant profit. In contrast, many airlines’ actual flying operations and revenue from ticket sales have struggled, especially since the late 2000s. In 2007, for example, financial analysts estimated the value of United’s MileagePlus program to be approximately $7.5 billion, and concluded that the loyalty program, if it operated as a stand-alone business, could be worth more than United itself, which at the time had a market value of $5.45 billion.1

Economists’ understanding of frequent flyer programs has traditionally been that airlines use them to create switching costs for the consumer and induce lock-in, softening the effect of competition. I show that mileage programs are also a valuable dynamic price discrimination tool. Miles are essentially a currency which allows the airline to sell shares of seats and to price them dynamically. They are generally bundled into the price of flights, or sold directly

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to consumers, or earned as a result of other activities with airline partners such as banks, hotels, car rental companies, etc. In practice, an airline issues miles, it distributes them to consumers directly or indirectly, and finally it controls the rate at which they are redeemed by setting the mileage price of award tickets, and by choosing how much capacity on any given flight to allow consumers to book with their miles. Hence the airline controls both the supply of miles and the redemption opportunities that it will allow.

Interestingly, the majority of airline miles in the U.S. today are earned not by customers who fly with the airline, but through banks and credit card issuers who award miles as incentives to their own customers. These incentives provide consumers with a way of indirectly buying miles from the airline. American Airlines for example reported in its annual 10-K filings to the SEC during 2010-2013 that it sold 62-66% of all miles it issued to partners. In 2009 Citibank pre-purchased $1 billion worth of miles from American while between 2011 and 2013 American Express purchased $675 million worth of miles from Delta Airlines. The fact that consumers can readily earn an airline’s miles independently of whether and how much they actually fly with that airline suggests that there is more to mileage programs than just their loyalty component.

In light of these facts, this paper proposes a novel explanation of the profitability of frequent flyer programs, beyond the traditional theory of switching costs. I show that frequent flyer programs provide a way for a capacity-constrained airline to partially contract with consumers ex ante, by selling airline miles, and later allowing these miles to be redeemed for flights, while also selling tickets at the same time. This mechanism operates in a setting where the airline cannot contract with consumers to sell tickets ex ante, and cannot commit to offer any particular future prices, or any specific value to members of its mileage program. Instead, in

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2Industry expert Ravindra Bhagwanani of Global Flight estimates that, for a typical major U.S. airline, approximately 30-45% of miles issued are awarded to the airline’s own flyers and 5-10% to partner airlines customers, 40-55% are sold to credit card issuers, 3-8% to hotel programs, 2-5% to car rental companies, and 2-10% to retail and other partners.  
3Airlines also sell miles directly to consumers. For instance, before its recent merger with American, US Airways periodically sold miles to members of its Dividend Miles program for a price of $0.011 – 0.018 each.  
my model the value of airline miles is determined endogenously in equilibrium.\footnote{Note that as a consequence of the Airline Deregulation Act of 1978, when an airline sells miles to consumers it is not subject to a “duty of fair dealing and reasonableness,” which is generally part of most contracts. In other words, the ability of the airline to contractually commit ex ante to deliver certain value to miles owners is significantly limited. See this report on a Supreme Court decision from 2013, for example: http://www.nytimes.com/2013/12/04/us/politics/supreme-court-considers-frequent-flier-contracts.html?_r=0.}

I consider an airline monopolist with fixed capacity, who serves a market where consumers learn over time how much they are willing to pay for a flight.\footnote{Notice that I focus on a monopolistic setting, which immediately rules out the switching costs theory of frequent flyer programs.} Aggregate demand and consumers’ private values, i.e. their types, are revealed in a two-period model: in the first period, neither side of the market has any information; in the second period the state of the economy, i.e. the size of the market, is publicly revealed, and each consumer privately learns her own type. Aggregate demand is uncertain ex ante: it can be either high or low, which determines the size of the market for airline tickets. I assume that consumer types are distributed according to a general CDF with standard regularity properties.

Period 1 represents the time when a consumer generally collects airline miles (e.g. though airline partners, or by buying previous tickets bundled with miles), with no particular plan to redeem them for a seat on a flight, i.e. before she has a specific valuation in mind.\footnote{In practice consumers collect miles over a significant period of time, before they have enough to redeem their miles for an “award seat.” On average this time is 30 months, according to an IATA special report available here http://airlines.iata.org/reports/special-report-the-price-of-loyalty.} Period 2 is when a consumer learns that she will want to take a flight and how much she would be willing to pay for a seat, and when the airline offers a mechanism to sell seats. The airline lacks the ability to commit to a mechanism ex ante, before uncertainty is resolved. Instead, in the first period it only interacts with consumers through its mileage program, by selling them frequent flyer miles. I assume that the airline can only commit ex ante to allow consumers to redeem their airline miles ex post for any seats that have not been sold. Specifically, in period 2 the airline offers a mechanism to sell seats, consumers decide whether to buy tickets or not, and then finally any unsold capacity is allocated among consumers who have miles and have not bought a ticket.\footnote{In practice airlines decide how many seats on a particular flight to allocate for mileage redemptions when they are confident that the flight will not sell out. Flights that are in high demand often have no availability at all, while flights in low demand often have seats available, especially closer to the date of the flight. At American Airlines, for instance, 8\% of seats are taken up by mileage redemptions, on average. I model the allocation of award seats as a simple ex post lottery, which is resolved after tickets are sold, because my focus is
with consumers ex ante, by selling miles, with a limited form of commitment.

The airline maximizes its revenue by choosing how many miles to sell in period 1 and at what price, and by offering a mechanism in period 2 where it sells tickets and allocates award seats to mileage program members. I solve for the optimal mechanism, which I refer to as a “FFP mechanism,” assuming that the distribution of consumer types satisfies a modified version of the Monotone Hazard Rate Property, which is the standard assumption in the literature. Notice that the airline cannot commit ex ante to a period 2 contract, so the mechanism must satisfy the constraint that the period 2 contract be sequentially optimal, given the state of the economy and the distribution of miles to consumers. It is easy to see that with full ex ante commitment the optimal mechanism for the airline would be one where it allocates seats efficiently in period 2 (e.g. through an efficient auction), and charges a price in period 1 for the right to participate in the period 2 stage. Such a mechanism would maximize total surplus and extract all of it as firm revenue, and hence could not be improved upon. However, when the airline lacks the ability to commit to a period 2 mechanism ex ante, allocating seats efficiently is clearly not sequentially optimal in period 2. While the limited commitment model does not restrict the tools available to the airline, the timing and lack of commitment imply that the fully efficient mechanism is not feasible.

The state of aggregate demand is uncertain, and I assume that the parameters of the model are such that the capacity constraint is binding when demand is high and not binding when demand is low. I first show that the airline sells all of its capacity in the form of tickets when realized demand is high; i.e., it does not allow consumers to book seats with their miles. In contrast, when demand is low some seats are given away to members of the mileage program, which makes buying miles worthwhile in period 1. I show that in this case the optimal period 2 mechanism features a “redemption fee” of 0, i.e. a consumer who obtains a seat through the award seat lottery does not pay anything to redeem her miles.

I show that the period 2 allocation of seats through ticket sales and points is monotonic in type, which will subsequently be helpful in characterizing the relationship between prices and each type’s equilibrium purchase decision. This monotonicity holds despite the fact that consumers have type-dependent endogenous outside options in period 2: any consumer who does not buy a ticket for a guaranteed seat has some equilibrium probability of obtaining a seat through the frequent flyer program as a limited commitment mechanism, and not on the particular way that award seats are released.

\footnote{This assumption is easily justifiable if we extend the model to one where the airline makes a costly capacity choice. For simplicity I impose this restriction as an assumption.}
seat with her miles, which she values according to her type. A consumer with a high type has a higher valuation for a ticket, but also a higher valuation for the probability of getting an award seat. I show that in equilibrium consumers decide whether to buy a ticket or not based on whether their type exceeds a common threshold: relatively high types buy tickets, whereas low types take the miles lottery instead. When aggregate demand is low some seats are indeed allocated through the lottery, which implies that consumers have an outside option that is better than 0. Therefore the price of a ticket must induce the marginal buyer type to buy a ticket, rather than taking the lottery, and so must leave that type some rents at the interim stage. Hence by selling frequent flyer miles ex ante the airline seemingly cannibalizes some of its demand in period 2: at any given ticket price, fewer types will buy a ticket than would if the frequent flyer program did not exist.

Importantly, this monotonicity allows me to show that the optimal price when demand is low is in fact lower than the optimal price in the standard static monopolistic screening mechanism, and moreover the marginal buyer type is lower than in the static mechanism. In other words, if the airline sells miles in period 1 it will subsequently offer a mechanism whereby it sells more seats, at a lower price (relative to the static screening mechanism), and in addition it will give away any remaining unsold capacity. The sequentially optimal mechanism in period 2 is thus more efficient than the static benchmark. Intuitively, in the optimal FFP mechanism the airline sets a lower price and sells more tickets, because it has an additional incentive to increase the quantity sold that is not present in the usual monopolistic screening trade-offs.

When deciding whether to sell an extra seat on the margin the airline takes into account the typical effects this will have: the additional unit sold, and the decrease in price that is required. But it now also faces an additional consideration: if it sells an extra seat, it lowers the ratio of unsold seats to consumer types who take the lottery, i.e. it decreases the equilibrium probability that a consumer who takes the miles lottery will obtain an award seat. This implies that the outside option of a type who buys a ticket decreases, and so the price that the airline can charge that type increases, providing it with an additional incentive to increase the quantity sold on the margin.\footnote{Notice that if the firm could commit to a period 2 mechanism ex ante, this consideration would be absent. In the optimal mechanism with commitment the firm fully internalizes the effect of the period 2 allocation of seats.}

While this suggests at first that the adoption of a miles program is costly, because it leads to some demand cannibalization and induces equilibrium prices that are below the optimal static screening prices, the airline does better with the FFP mechanism overall. In particular,
total surplus in the FFP mechanism is larger than in the optimal static mechanism, and the firm captures all of this incremental gain in surplus as revenue when it sells points in period 1. Intuitively, the price that the airline can charge a consumer for miles in period 1 depends on the consumer’s expected payoff from the continuation game where she buys miles, relative to her expected payoff in the continuation game where she does not. In the latter subgame, the consumer is excluded from the award seat lottery, and so the sequentially optimal contract that the airline will offer the consumer is precisely the optimal static screening contract. Hence the consumer’s expected payoff is her expected surplus from the optimal static contract. On the other hand, if she buys miles ex ante her payoff is her expected surplus from the period 2 FFP mechanism where the airline sells seats to more types, at a lower price. The difference between these two continuation payoffs is precisely the price that the airline can charge for miles in period 1. Hence the airline captures all of the gains in total surplus due to the adoption of the frequent flyer program and of the subsequently optimal ticket mechanism.

Moreover, I show that the optimal period 2 prices are independent of the number of consumers the airline sells miles to, provided the latter is feasible, i.e. it is above a particular threshold where the airline can carry out its commitment to give away all unsold seats. This observation implies that it is optimal for the airline to sell miles to all consumers in period 1. Consumers are homogeneous ex ante, so at the optimal price of miles in period 1 each of them is in fact indifferent between buying miles or not.

Overall, the optimal FFP mechanism is one where the airline sells miles to all consumers ex ante; it then sells all of its capacity if demand is high in period 2, or it runs an award ticket lottery ex post for any seats that are not sold when demand is low. The FFP mechanism is more efficient than the optimal static screening mechanism, and all of the incremental gains in surplus accrue to the airline, through the price that it sets for frequent flyer miles in period 1. Although the frequent flyer program provides a limited form of commitment, the airline benefits from the ability to partially contract with consumers ex ante. While I restrict my discussion of the model to the context of the airline industry, the results apply more broadly to other markets that share similar features: a fixed inventory of a perishable product, low marginal costs, volatile demand, repeat transactions with consumers, and heterogeneity in willingness to pay. Such markets include for example the hotels and car rentals industries, both of which prominently use loyalty programs as a revenue management tool.

The remainder of this paper is organized as follows. In Section 2 I review the literature related to frequent flyer programs and dynamic screening. In Section 3 I present the model and discuss my main assumptions. In Section 4 I characterize the optimal FFP mechanism and discuss its
revenue and efficiency properties, relative to the standard static screening mechanism; I also consider the special case of uniformly distributed types as an illustrative example. In Section 5 I study an extension of the main model where the airline chooses ex post how much of its unsold capacity to allocate through its mileage program, rather than committing ex ante to release all such unsold capacity; I show that the revenue-maximal equilibrium of this model is identical to the FFP mechanism from Section 4. Section 6 concludes. The Appendix contains the proofs of all lemmas and propositions.

2 Related literature

There is an extensive literature that studies frequent flyer programs as a type of contract used to create endogenous switching costs in a horizontally differentiated market. Such models typically study a two-period oligopolistic model where competing firms sell a product in both periods and the firm benefits either from the ability to write long term contracts that lock consumers in, or from the ability to learn about a buyer’s unchanging preferences. Fundenberg and Tirole (2000) discuss a setting where consumers’ preferences are fixed over time and firms cannot commit to future prices or write long term contracts. The firm learns about a consumer’s preferences based on her first period purchase, and uses that information to price discriminate in the second period. Caminal and Matutes (1990) study a model where consumer preferences change over time, so a buyer’s first period purchase reveals no information about her second period preferences. However the firm can commit to offer a discount to repeat buyers in period 2, which softens the effect of competition in the latter stage. Fundenberg and Tirole (2000) also consider a setting where the firm offers both long term and short term contracts, which induce consumers who strongly prefer one firm to purchase the former type of contract, while those with weaker preferences buy the latter. Stole (2007) provides a comprehensive overview of the literature on contractual switching costs in horizontal differentiation models.

While the above papers consider the loyalty-building aspect of frequent flyer programs, I focus instead on a monopolistic setting where there is no scope for switching costs to play any role, and I study the dynamic contracting aspect of such programs. Hence the approach of this paper fits within the context of dynamic contracts and mechanism design. I consider a two-period model of information where consumption happens in the second stage, while the first stage is only a contracting device. The model is thus most closely related to the literature on sequential screening. Courty and Li (2000), a seminal paper in this area, studies optimal
dynamic screening when consumers learn about their valuation for a product gradually. In their setting the firm has full commitment power when it contracts in the first stage, and the optimal contract is a refund contract whereby a consumer whose final realized valuation is low receives a refund for her purchase. Intuitively, the less informative a consumer’s initial private information is about her final valuation, the more surplus the firm can extract. Akan, Ata and Dana (2009) generalize this model and extend the intuition to a setting where consumers learn about their types at different times. In that case, the optimal mechanism is also an initial contract with a refund clause. In contrast to Courty and Li (2000) and Akan et al. (2009), I model the case where the firm cannot commit to a period 2 mechanism and cannot sell full contracts ex ante, and where the firm has fixed capacity.

Hua (2007) studies an auction where the seller has the ability to contract with one uninformed buyer prior to the auction. The optimal contract in this setting induces less rent-seeking by the auctioneer in the auction stage, it increases the probability of trade by favoring the contracted buyer, and it may increase total surplus, which is intuitively similar to the effect of ex ante contracting that I consider. Unlike Hua (2007), in my model the firm can interact with all consumers ex ante, and has a much more limited form of commitment regarding the actual trade stage.

Deb and Said (2014) weaken some of the commitment assumptions of the sequential screening literature. In particular, in their model consumers arrive in two cohorts, and when the firm sells contracts to the first cohort it cannot commit to the prices that it will offer in the second cohort. The anticipated second-period contract thus offers an endogenous outside option to period 1 buyers. I further weaken the firm’s commitment power and contracting ability: I consider a setting where the firm cannot offer a long-term contract in the first stage and cannot commit to future prices. Instead, the firm’s commitment to a loyalty program in period 1 creates the type-dependent outside option that consumers have when contracting in period 2.

More generally, the paper relates to the extensive literature on dynamic mechanism design. Among the earliest papers on this subject, Baron and Besanko (1984) study a multiperiod model where a firm has private information about its costs and repeatedly interacts with a regulator who designs a mechanism to set the price of the firm’s output. In their paper, the regulator can fully commit to a dynamic mechanism. Pavan, Segal and Toikka (2014) and Battaglini and Lamba (2014) study incentive compatibility in dynamic mechanism design settings with commitment. The former provide a dynamic envelope formula for local incentive compatibility and discuss revenue equivalence and implementability, while the latter study global incentive compatibility in settings where local constraints are insufficient. In contrast to
the above papers, which feature dynamic information, Said (2012) and Board and Skrzypacz (2014) study models with a dynamic population of agents with fixed private information, where the designer can commit to a mechanism. Skreta (2006) considers optimal dynamic mechanisms in a model where the seller has no commitment, and shows that posted prices maximize revenue in this setting.

The results in this paper also share some intuitive features with models of monopoly pricing with demand uncertainty, such as in Nocke and Peitz (2007). The latter study clearance sales, whereby a monopolist sells to high consumer types with certainty at a high initial price, and later sells to low types with some equilibrium probability. Hence their results are qualitatively similar to the type separation that occurs in period 2 of the model in this paper. In contrast to Nocke and Peitz (2007), I study a two-period model where the firm deals with all consumers in the first period, and it is the interaction between period 1 actions and the period 2 mechanism that drives firm revenue.

3 Model

Environment

Consider an airline monopolist who serves a unit mass of consumers each of whom demands 1 seat. The monopolist has a capacity of $k < 1$ and a marginal cost of 0. Suppose the airline and its potential customers exist in two periods, $t \in \{1, 2\}$. In period 1 both sides of the market have no information. In period 2, the state of the economy is publicly realized: with probability $l \in (0, 1)$ demand is low, and with probability $1 - l \in (0, 1)$ demand is high; and each consumer privately learns her own type, which is her valuation for a flight. The airline only learns the state of aggregate demand, i.e. the overall distribution of types, in period 2.

- If demand is high in period 2, all of the mass 1 of consumers have their types distributed over $[0, 1]$ according to a CDF $F$ with positive density.

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13For convenience I will refer to the airline industry throughout the model and results, but it should be clear that the model applies more broadly to any industry or situation where firms use loyalty programs to manage revenues and capacity, such as hotels and car rental companies for example.

14The assumption of 0 marginal costs is made for simplicity of exposition only, and is not necessary for the results of the paper. Moreover, it is not an unreasonable assumption, as marginal costs are typically very low in the airline market and in other relevant industries.
• If demand is low in period 2, a mass $m < 1$ of consumers have their types distributed according to $F$ over $[0, 1]$, while the remaining $1 - m$ consumers have a valuation of 0.

**Contracting**

The timing of contracting is as follows: at $t = 1$ the airline sells “points” to consumers through its frequent flyer program (“FFP”); at $t = 2$ the airline announces a mechanism to sell tickets to consumers and a mechanism to let consumers redeem points to obtain a seat. The airline commits in period 1 to allocate any unsold capacity in period 2 through a lottery among those consumers who bought points in period 1 and did not buy a ticket through the airline’s sales mechanism. Hence buying a point gives the consumer some probability of obtaining an award flight through the FFP lottery, determined endogenously by the airline’s unsold capacity at $t = 2$.

In period 1 consumers only interact with the airline through its FFP, by buying points. The airline chooses what fraction $\phi$ of consumers to sell points to and the price of a point. Without loss of generality I can treat each consumer as buying a single point, which in practice represents a number of airline miles that are required for a seat obtained through the FFP, such as 25,000 miles for a domestic US round trip flight, for example. Period 1 represents the period of time before a consumer knows that she will need to buy a ticket, so her only interaction with the airline is through its FFP, when she collects airline miles from various activities. With this interpretation of the timing of the model, I naturally assume that the airline does not have the ability to sell tickets ex ante. That is, it cannot sell full contracts in period 1, before the consumers have learned the state of the economy and their own types.\(^\text{15}\)

Ticket sales and the allocation of seats to consumers only happen in period 2. At $t = 2$ the airline offers a mechanism to sell tickets and a mechanism to redeem points for a seat. Consumers simultaneously decide whether to buy a seat through the sales mechanism, and then the airline runs a lottery to allocate any unsold seats through the points redemption mechanism. I refer to prices in the sales mechanism as the “price of a ticket.” On the other hand, if a consumer does not buy a ticket in the sales mechanism and is chosen in the FFP

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\(^{15}\)This assumption rules out, for example, complex contracts sold ex ante, whereby the airline allows consumers to refund their booking when their realized type is low. Such refund contracts are generally not observed in practice over a period of time like 30 months before a flight, and the dynamics of such contracts are not the main focus of this paper. Notice also that a capacity-constrained airline would not want to sell simpler contracts ex ante, since the expected value of a consumer’s type is lower than the expected revenue the firm can get from ex post screening.
lottery, she may redeem her point for a seat and the airline may in principle charge her a redemption fee, which I will denote by $f_l$ and $f_h$ in the low and high states, respectively.

The commitment to give away any empty seats to FFP members is the only commitment that the airline may make at $t = 1$ regarding what happens at $t = 2$. I assume that the airline cannot commit ex ante to offer any particular contract in period 2, or to sell any particular number of seats. Hence the contract that is offered at $t = 2$ will have to be sequentially optimal for the airline. I will later relax this assumption and study an extension of the model where the airline decides ex post how much of its unsold capacity to give away to FFP members, and thus has no commitment power in period 1.

**Assumptions**

I will assume throughout the paper that the parameters of the model are such that the airline’s capacity constraint binds in the high demand period, but does not bind in the low demand period, and I present the precise statement of this assumption in the exposition of the results. I focus on this case because it is the most interesting of three possible cases, the other two being that the capacity constraint is always binding, and that the capacity constraint is never binding. In the former case the FFP would play no role at all in the solution to the airline’s pricing problem, and the optimal contract would be the solution to the standard static monopoly screening problem. In the latter case the results in this paper regarding the optimal contract in the low demand state would translate to both the low and the high demand states. More generally, this assumption on capacity can naturally be motivated if we explicitly model costly capacity choice by the airline.

4 **Analysis**

4.1 **Example: a simple FFP mechanism with uniform types**

To illustrate the general intuition of the model I will first consider a much simpler example. I make two simplifications: first, suppose $F$ is the uniform distribution over $[0, 1]$; second, consider a restricted class of mechanisms where $\phi = 1$ (i.e. the airline sells points to all consumers at $t = 1$), and $f_l = f_h = 0$ (i.e. a consumer pays nothing if she redeems her points, so award tickets are free). The latter set of assumptions simplifies the kind of mechanisms that the airline can offer. The restriction that $\phi = 1$ means that the airline cannot sell points
to a fraction of consumers in period 1, while the restriction that $f_l = f_h = 0$ means that it cannot screen among consumers who take a seat through the FFP lottery. The firm’s choices are then $p_{FFP}$, the price of a point at $t = 1$, and $p_h$ and $p_l$, the price of a ticket at $t = 2$ in the high or low state, respectively. As I will subsequently show in Proposition 1 in the more general contracting problem the airline in fact wants to set $\phi = 1$ and $f_l = f_h = 0$. Hence restricting the class of mechanisms in this way does not affect the optimal contract in this simpler example. It does however significantly simplify the derivation of the optimal contract.

To begin, suppose $k \in \left[1 - \sqrt{1 - m}, \frac{1}{2}\right]$ and $m \leq \frac{3}{4}$. This assumption will guarantee that the capacity constraint is binding at the optimal contract when demand is high, and is not binding when demand is low. Moreover, this condition is implied by the assumptions that we will make later on in the general contracting problem.\(^{16}\)

The static screening benchmark

As a benchmark for comparisons, first consider the canonical static monopoly screening contract, as if the firm and consumers only exist at $t = 2$ and there are no points. Suppose aggregate demand is high, i.e. there is a unit mass of consumers with types distributed uniformly over $[0, 1]$. The airline’s revenue maximization problem is

$$\max_{p_h} p_h (1 - p_h) \quad \text{s.t.} \quad 1 - p_h \leq k$$

Clearly the unconstrained optimum is at $p_h = \frac{1}{2}$, which violates the capacity constraint for $k < \frac{1}{2}$, so the constraint is binding at the optimum. Hence the optimal price and revenue when demand is high, denoted by $\bar{p}_h$ and $\bar{\Pi}_h$, are:

$$\bar{p}_h = 1 - k \quad \text{and} \quad \bar{\Pi}_h = k(1 - k).$$

Now suppose aggregate demand is low, i.e. there is a mass $m$ of consumers with types distributed uniformly over $[0, 1]$, while the remaining $1 - m$ consumers have a type of 0. The airline’s revenue maximization problem is

$$\max_{p_l} p_l m (1 - p_l) \quad \text{s.t.} \quad m(1 - p_l) \leq k$$

The unconstrained optimum is $p_l = \frac{1}{2}$, and demand at that price is $\frac{m}{2}$. Notice that this satisfies the capacity constraint, as $\frac{m}{2} \leq 1 - \sqrt{1 - m}$ for $m \in (0, 1)$. Hence the optimal price

\(^{16}\)Specifically, by Assumption 3.
and revenue when demand is low, denoted by $\bar{p}_l$ and $\bar{\Pi}_l$, are:

$$\bar{p}_l = \frac{1}{2} \quad \text{and} \quad \bar{\Pi}_l = \frac{m}{4}.$$ 

The optimal FFP mechanism

I now return to the two-period model where the airline sells points at $t = 1$ and commits to give away any unsold capacity to points owners at $t = 2$, with the constraints that $\phi = 1$ and $f_l = f_h = 0$. I can then characterize the optimal FFP mechanism in this restricted class of contracts, which will in fact be the optimal FFP mechanism in the unrestricted contracting problem, which I will subsequently characterize in Proposition 1. In period 1 the airline sells points to consumers at some price $p_{FFP}$ and commits to give away any unsold capacity to points owners in period 2. Consumers have no private information at this stage and the state of aggregate demand is uncertain. In period 2 the state of the economy is realized and each consumer privately learns her type, which is her valuation for a seat. The airline posts a price for a seat (denoted by $p_h$ or $p_l$), and all consumers simultaneously decide whether to buy a ticket or not. The airline then allocates any unsold seats among consumers who did not buy a ticket and who own points.

We proceed by backward induction. Suppose demand is high at $t = 2$. Notice that the capacity constraint would be binding in the canonical static screening problem. Any unsold capacity would be given away to consumers ex post, which would lower the price that each of them would be willing to pay for a ticket if they anticipated obtaining an award seat in equilibrium. Hence the airline now has even more of an incentive to increase the quantity it sells than in the case where points did not exist. Since the capacity constraint was binding in the latter case, it is clearly binding in the former case. The airline will thus sell all of its capacity, i.e. the optimal price and corresponding revenue when demand is high are:

$$p_h = 1 - k \quad \text{and} \quad \Pi_h = k(1 - k).$$

Now suppose demand is low at $t = 2$. At any posted price $p_l$ a consumer of type $v$ has two options: if she buys a ticket she gets a payoff of $v - p_l$; if she does not buy a ticket there is some equilibrium probability, denoted by $q$, that she will receive a free seat through the FFP, so she receives an expected payoff of $qv$. The fact that she owns points thus creates a type-dependent outside option in period 2. Clearly then a consumer buys a ticket if and only if $v - p_l \geq qv$, i.e. $v \geq \frac{p_l}{1-q}$. Hence the allocation of seats is monotonic in type: there
exists some cutoff such that all types above it buy a ticket, while all types below it take the FFP lottery. Denote this cutoff type by \( \hat{v} \). Now notice that the equilibrium probability of obtaining a seat in the FFP lottery must be consistent with this type \( \hat{v} \), and in particular the cutoff type \( \hat{v} \) must be indifferent between buying a ticket and taking the lottery. Hence we have the following indifference condition:
\[
\hat{v} - p_l = \frac{k - m(1 - \hat{v})}{1 - m(1 - \hat{v})} \hat{v}
\]
We can now re-write this condition to solve for \( p_l \) as a function of the cutoff type that is induced in equilibrium:
\[
p_l = \hat{v} \frac{1 - k}{1 - m(1 - \hat{v})}.
\]
We can then re-write the airline’s problem in terms of the cutoff type that it wants to induce in order to maximize revenue:
\[
\max_{\hat{v}} \frac{1 - k}{1 - m(1 - \hat{v})} \cdot m(1 - \hat{v}),
\]
where we have used the fact that \( p_l = \hat{v} \frac{1 - k}{1 - m(1 - \hat{v})} \) and that demand at a price \( p_l \) is \( m(1 - \hat{v}) \).

The first-order condition for this problem yields
\[
\hat{v} = \sqrt{1 - m} - (1 - m)
\]
and we can verify that the second-order condition holds. The optimal price \( p_l \) that induces this cutoff is then
\[
p_l = \hat{v} \frac{1 - k}{1 - m(1 - \hat{v})} = \frac{(1 - k)(1 - \sqrt{1 - m})}{m},
\]
and demand at this price is \( m(1 - \hat{v}) = 1 - \sqrt{1 - m} \). Hence revenue is
\[
\Pi_l = \frac{(1 - k)(1 - \sqrt{1 - m})^2}{m}.
\]

We can now turn to period 1 and find the optimal price that the airline can charge for a point, in the equilibrium where it sells points to all consumers. When a consumer decides whether to buy a point in period 1 or not, she compares two possible payoffs: her expected payoff from the continuation game where she owns a point at \( t = 2 \), and her expected payoff from the continuation game where she does not own a point at \( t = 2 \). In the former case, we have established that the consumer will face prices \( p_h \) and \( p_l \), depending on the state of the economy, given by:
\[
p_h = 1 - k \quad \text{and} \quad p_l = \frac{(1 - k)(1 - \sqrt{1 - m})}{m}.
\]
In the latter case, where she does not buy a point, the airline will offer her a different set of prices, since the consumer is excluded from the FFP lottery and hence her outside option when deciding whether to buy a ticket or not is just 0. Clearly the sequentially optimal prices for a consumer who has not bought a point are just the standard prices from the static monopoly screening problem:

\[
\bar{p}_h = 1 - k \quad \text{and} \quad \bar{p}_l = \frac{1}{2}.
\]

Notice that \( p_h = \bar{p}_h \) and, more importantly, \( p_l < \bar{p}_l \). In fact, we can also see that the marginal type who buys a ticket in the points mechanism (i.e. the type \( \hat{v} \)) is lower than the marginal type who buys a ticket in the standard no-points mechanism (i.e. the type \( \frac{1}{2} \)):

\[
\hat{v} = \frac{\sqrt{1 - m} - (1 - m)}{m} < \frac{1}{2}.
\]

Hence the consumer’s expected payoff in the continuation game where she owns a point is larger than in the continuation game where she does not, since in the former case the airline charges her a lower price, and it also gives her a seat through the FFP lottery with some probability if she does not buy a ticket. The difference between these two expected payoffs is precisely what each consumer is willing to pay for a point at \( t = 1 \), and is the optimal price that the airline charges for points at \( t = 1 \), denoted by \( p_{FFP} \).

In summary, the optimal FFP mechanism in this simple example is one where the airline sells points at a price \( p_{FFP} \) at \( t = 1 \), it posts prices \( p_h \) and \( p_l \) in period 2, and then it allocates any unsold capacity through a lottery if demand is low. Finally, we can verify that the FFP mechanism yields larger revenue than the standard static mechanism. Figure 1 illustrates the difference between revenue in the FFP mechanism and in the benchmark.

Notice that the allocation of seats is more efficient in the FFP mechanism than in the benchmark mechanism, because \( \hat{v} < \frac{1}{2} \), i.e. the airline sells more seats, and because all of the remaining unsold seats are allocated through the FFP lottery. Hence total surplus is larger.

---

17One can show that this price is given by:

\[
p_{FFP} = ml \left\{ \frac{1}{2} \left( \frac{1}{2} - p_l \right) + \left( \frac{1}{2} - \hat{v} \right) \left( \frac{1}{2} + \hat{v} - p_l \right) + \frac{(\sqrt{1 - m} - (1 - m))^2 \left( \sqrt{1 - m} - (1 - k) \right)}{2m^2 \sqrt{1 - m}} \right\} + \frac{m \left( 8(1 - \sqrt{1 - m}) - 4m + (3\sqrt{1 - m} - 4)m^2 + 4k(2\sqrt{1 - m} - 2 + m + m^2) \right)}{8m \sqrt{1 - m}}.
\]

Notice that this price is positive since \( p_l \leq \hat{v} \leq \frac{1}{2} \) and \( k \geq 1 - \sqrt{1 - m} \).
in the FFP mechanism than in the benchmark mechanism. Moreover, consumer surplus is the same in both mechanisms, since each consumer is made indifferent between the two when the airline charges a price $p_{FFP}$ for a point in period 1. Hence the airline captures all of the gains in surplus from the adoption of the FFP mechanism, relative to the benchmark static mechanism.

The main feature of the FFP mechanism is thus that the sale of points in period 1 induces a more efficient interim allocation of seats, even in this limited commitment setting. In particular, points give consumers a type-dependent outside option in period 2, which reduces the firm’s incentive to screen and induces it to sell more seats, at a lower price. Hence the FFP mechanism generates larger total surplus. All of the gains in surplus accrue to the firm, because the optimal price of points in period 1 leaves consumers indifferent between the FFP mechanism and the benchmark static mechanism.

In what follows I show that this simple intuition extends much more broadly. First, it naturally extends to the case where $F$ is a more general distribution. As is standard in the literature, I impose some regularity conditions on $F$ to guarantee that a solution to the contracting problem exists. Specifically, I impose on $F$ a version of the Monotone Hazard Rate Property, which is a sufficient condition to characterize the optimal contract. Second, I consider a broader class of mechanisms: I allow the airline to sell points to a fraction of consumers, which relaxes the restriction that $\phi = 1$, and I also allow the airline to charge a fee for the redemption of points, which relaxes the restriction that $f_l = f_h = 0$. This setting significantly enlarges the set of possible contracts that the airline can offer. Nonetheless, I show that the optimal FFP contract in this larger class of mechanisms indeed features $\phi = 1$ and $f_l = f_h = 0$. Setting $\phi = 1$ at $t = 1$ (i.e. selling points to all consumers) turns out to be optimal, because I show that the sequentially optimal period 2 allocation is in fact independent of $\phi$, provided $\phi$
is larger than a particular threshold whereby the airline sells points to enough consumers to be able to give away all unsold capacity through the FFP. Furthermore, in the general problem the airline has an incentive at the interim stage to make the FFP lottery as unappealing to consumers as possible, since the price that each type would be willing to pay for a ticket decreases in the equilibrium probability that a consumer gets a seat through the lottery. Hence lowering the point redemption fee to 0 (i.e. setting \( f_h = f_l = 0 \)) allows the airline to increase the pool of types who participate in the lottery, and thus to lower the outside option of those types who buy a ticket. The intuition from the uniform types example thus extends to a more general setting and to a much broader class of possible contracts.

4.2 The static screening benchmark

Now consider the general model outlined in Section 3. Before studying the optimal mechanism in the limited commitment model, I will first discuss the optimal mechanism in a static version of the model, which will be a useful benchmark for comparison later. Consider the model with only one period, \( t = 2 \), as if period 1 does not exist. Consumers privately learn their types and the airline learns the aggregate distribution of types. Contracting happens only in period 2, as if the FFP does not exist. The airline’s maximization problem in this case is clearly just the standard static screening problem, with a continuum of consumers, subject to a capacity constraint. As outlined in the description of the model I assume that the parameters are such that the capacity constraint binds in a high demand state and does not bind in a low demand state. Furthermore, I assume that the CDF of consumer types \( F \) satisfies the Monotone Hazard Rate Property, an assumption that is standard throughout the mechanism design and contracting literatures.\(^{18}\)

**Assumption 1.** Assume the hazard rate \( \frac{F'(v)}{1-F(v)} \) of \( F \) is increasing in \( v \), and that \( k \in [m(1-F(\bar{v})), 1-F(\bar{v})] \), where \( \bar{v} \) solves \( \bar{v} = \frac{1-F(\bar{v})}{F'(\bar{v})} \).

The following Remark characterizes the optimal posted price that solves the firm’s maximization problem: the airline sells all \( k \) of its capacity in a high demand state and sells less than \( k \) of its capacity in the low demand state. I consider posted prices without loss of generality, because the model features a continuum of consumers with unit demands. I denote by \( \bar{p}_a \)

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18More generally, we need the distribution \( F \) to be regular, i.e. the virtual value \( v - \frac{1-F(v)}{F'(v)} \) to be increasing. The MHRP is a more intuitive sufficient condition for \( F \) to be regular, and is satisfied for a variety of distributions of interest, such as the uniform, power, normal, logistic, exponential distributions, etc.
and $\Pi_s$ the optimal price and corresponding revenue in state $s \in \{l, h\}$ in the static screening benchmark.

**Remark 1.** Under Assumption 1, the optimal prices in the static benchmark are $\bar{p}_l$ and $\bar{p}_h$ in the low and high state, respectively:

$$
\bar{p}_l = \frac{1 - F(\bar{p}_l)}{F'(\bar{p}_l)} \quad \text{and} \quad \bar{p}_h = F^{-1}(1 - k)
$$

and the corresponding revenues are

$$
\bar{\Pi}_l = \frac{m(1 - F(\bar{p}_l))^2}{F'(\bar{p}_l)} \quad \text{and} \quad \bar{\Pi}_h = kF^{-1}(1 - k).
$$

Notice that in a low demand state a consumer of type $v$ buys a ticket if and only if $v \geq \bar{v}$, where $\bar{v}$ is the unique solution to $\bar{v} = \frac{1 - F(\bar{v})}{F'(\bar{v})}$. I will refer to this threshold type $\bar{v}$ in subsequent comparisons.

### 4.3 Contracting with limited FFP commitment

Now consider the full two-period model, as described in section 3 at $t = 1$ the airline chooses $\phi$, the quantity of points it sells, and $p_{FFP}$, the price it charges for a point; at $t = 2$ it offers a contract, consisting of $p_h$, $p_l$, $f_h$ and $f_l$, consumers simultaneously decide whether to buy a ticket or not, and finally any unsold seats are randomly allocated among those consumers who bought points in period 1 but did not buy a ticket. The only commitment power that the firm has at $t = 1$ is that it will allocate its unsold capacity through the points lottery in period 2. In particular, notice that the airline cannot commit to any period 2 mechanism at $t = 1$, so the mechanism in period 2 has to be sequentially optimal and incentive compatible, and hence must be a posted price mechanism.

I will characterize the revenue-maximizing mechanism in a series of steps. First, I consider the optimal contract that the airline will offer at $t = 2$ if demand is high. I show that the optimal contract, given that the airline has sold $\phi$ points at $t = 1$, is the same as in the static benchmark, i.e. the FFP does not affect ticket pricing. Second, I consider the optimal contract at $t = 2$ if demand is low. I begin by showing that it is optimal to set $f = 0$, i.e. the airline will not charge a fee when a consumer redeems her point for a seat, so FFP lottery seats are given away for free. I then show that it is optimal for the airline to set $\phi = 1$ at $t = 1$, i.e. to sell points to all consumers ex ante. I then solve for the optimal ticket price that the airline will charge at $t = 2$ and for the set of consumer types who buy a ticket.
Third, I solve for the price that the airline charges for a point at \( t = 1 \), which completes the characterization of the mechanism. Finally, I compare the revenue and efficiency properties of the limited commitment FFP mechanism and of the static benchmark.

**The optimal contract at \( t = 2 \) when demand is high**

To begin, consider period 2 and suppose demand is high. At \( t = 2 \) the airline posts ticket prices: \( p_h \) and \( \bar{p}_h \) for consumers who own points and who do not own points, respectively.\(^{19}\) A consumer who owns a point but does not buy a ticket enters the FFP lottery, and may, if chosen, redeem her point for a seat for a fee \( f_h \). The following Lemma states that the optimal contract when demand is high in period 2 is such that all capacity is sold out, i.e. there is no FFP lottery. Hence the contract is the same as in the static benchmark and the FFP does not affect pricing.

**Lemma 1.** Suppose demand is high at \( t = 2 \). Under Assumption 1 the optimal prices at \( t = 2 \) are \( p_h \) and \( \bar{p}_h \) for points owners and non-owners, respectively:

\[
p_h = \bar{p}_h = F^{-1}(1 - k),
\]

and total revenue is

\[
\Pi_h = kF^{-1}(1 - k).
\]

Any \( f_h \) is optimal, since no points owners obtain seats through the FFP lottery.

Notice that in the absence of points the capacity constraint would be binding in a high demand period. Intuitively, we can see that the introduction of points when the constraint would otherwise be binding means that there will be no spare capacity to be allocated through a FFP lottery. Hence consumers make their purchase choices as if the FFP did not exist. The revenue maximizing prices are then simply those prices which correspond to selling all \( k \)

\(^{19}\)I allow the airline to price discriminate between points owners and non-owners, as this type of mechanism is more general than one where we restrict the airline to selling tickets to both kinds of consumers at the same price. Moreover, the main results regarding the optimal contract when demand is low at \( t = 2 \) remain qualitatively similar, though less tractable, if we add this restriction, so instead I focus on the more general unrestricted class of contracts. In reality, while the posted price of a particular seat on a plane might be the same for a member of the airline’s FFP as for a non-member, the ticket typically bundles some number of airline miles with the seat, which effectively provides the FFP member with a discount off the posted price. Therefore the same nominal price implies a lower actual price of the ticket for a consumer who is a member of the airline’s FFP, which is consistent with the optimal contract that we derive in this subsection.
capacity, at a price that is equal for consumers who own and do not own points. In particular, the airline has no incentive to set prices such that different sets of consumer types will buy a seat, depending on whether they own points or not, since both points owners and non-owners have an outside option of 0, and are identically distributed.

The optimal contract at \( t = 2 \) when demand is low

Consider period 2 and suppose demand is low. A mass \( \phi \) of consumers arrive into period 2 having bought points at \( t = 1 \), while the remaining \( 1 - \phi \) have not bought points. At \( t = 2 \) the airline posts prices: \( p_l \) and \( \tilde{p}_l \) for consumers who own points and who do not own points, respectively.\(^{20}\) A consumer who owns a point but does not buy a ticket enters the FFP lottery and may, if chosen, redeem her point for a seat for a fee \( f_l \).

Analogously to the analysis of the static benchmark and of the optimal contract when demand is high, we will first impose conditions on the CDF \( F \) and the capacity constraint \( k \), which will guarantee that the optimal contract has a solution, and that the capacity constraint will be such that the optimal contract at \( t = 2 \) includes a lottery of some seats among points owners.\(^{21}\) I formalize the assumptions as follows.

**Assumption 2.** Assume the “modified hazard rate” \( \frac{F'(v)}{1-F(v)} \frac{1}{1-m(1-F(v))} \) is increasing in \( v \), and that \( k \geq m(1-F(\hat{v})) \), where \( \hat{v} \) solves \( \hat{v} = \frac{[1-F(\hat{v})][1-m(1-F(\hat{v}))]}{F'(\hat{v})} \).

More generally, we only need \( F \) to be such that \( v = \frac{1-F(v)}{F'(v)} (1 - m(1 - F(v))) \) has a unique solution, which is a condition analogous to the standard regularity assumption that \( F \) has monotone virtual valuations. Notice that this condition holds, for instance, when \( F \) is uniform, or more generally a power distribution. Assumption\(^2\) is a sufficient condition for this to hold.

The following Lemma states that the optimal contract in period 2 when demand is low features a redemption fee of \( f_l = 0 \), i.e. consumers who win the FFP lottery redeem their points for a seat for free.

**Lemma 2.** Suppose demand is low at \( t = 2 \). The optimal point redemption fee is \( f_l = 0 \).

The intuition behind this result is as follows. A consumer who owns a point chooses between

\(^{20}\)I assume, as I do in the section on the optimal contract when demand is high, that the airline may price discriminate between consumers who own points and those who do not.

\(^{21}\)If the capacity \( k \) did not satisfy this condition, it would be the case that the constraint is binding in the low demand state, and hence points do not affect pricing.
buying a ticket and taking the FFP lottery at $t = 2$, so her outside option is generally better than 0. If the airline wants to induce a consumer of type $v$ to buy a ticket, the maximum price it can charge that type is strictly less than $v$, and in particular it has to be low enough that the consumer’s payoff from buying the ticket is at least as high as her surplus from just taking the lottery. The size of this gap between the consumer’s type and the price that will induce her to buy depends on her outside option, i.e. on the probability that she would win a seat in the FFP lottery. The larger this probability is, the larger the gap between price and type must be. But notice that this probability is endogenously determined by the contract that the airline offers, since that determines the number of seats that will be allocated through the FFP lottery and the measure of consumer types who will participate in the lottery.

Hence in period 2 the airline has an incentive to make the lottery less appealing to the consumer, to decrease her endogenous outside option and thus decrease the rents she receives if she buys a ticket. The airline can decrease the probability that a consumer wins a seat in the FFP lottery either by reducing the price of tickets, $p_t$ (hence selling more of its seats and leaving fewer seats for the FFP lottery), or by reducing the redemption fee, $f_t$ (hence increasing the measure of consumers types who participate in the FFP lottery). The Lemma shows that it is in fact optimal to set $f_t = 0$, as this induces all types below a certain cutoff to participate in the lottery, which allows the airline to charge high types who buy tickets (i.e. the types above the cutoff) a higher price, since their probability of winning the lottery is low and hence their next best option is less valuable.

Next, I consider the airline’s choice of how many points to sell at $t = 1$, i.e. the optimal $\phi$. I will show that the airline is indifferent among all $\phi$ such that when demand is low at $t = 2$ it will allocate all of its capacity $k$ through ticket sales and through the FFP lottery. In particular, $\phi = 1$ is optimal.

**Lemma 3.** Under Assumption 2 any FFP mechanism with $\phi \geq \frac{k-m(1-F(\hat{v}))}{1-m(1-F(\hat{v}))}$ is feasible, and $\phi = 1$ is optimal.

The proof of this Lemma shows that for any feasible $\phi$ (i.e. any $\phi$ such that the number of points owners who do not buy a ticket is at least as large as the number of unsold seats) the airline sets prices at $t = 2$ such that it sells seats to the same sets of consumer types who buy points and who do not buy points. That is, in the optimal mechanism a consumer buys a ticket if and only if her type $v$ is greater than some threshold type, which does not depend on whether she owns a point or not. This is obvious in the case where demand is high in period 2, since in that case the airline sells all of its capacity, so no seats are allocated through the
FFP lottery, and points owners and non-owners face the same prices. However, the Lemma further proves a similar property of the optimal contract when demand is low in period 2, when in equilibrium some seats are allocated to points owners through the FFP lottery.

Figure 2 illustrates this feature of the mechanism, with a uniform distribution $F$ as an example. Let $p_l$ be the price that the airline charges a consumer who owns a point, and let $\tilde{p}_l$ be the price that it charges a consumer who does not. The latter kind of consumer then buys a ticket if and only if her type is $v \geq \tilde{p}_l$, since her outside option is 0. The former kind of consumer buys a ticket if and only if her type is greater than some cutoff, denoted by $\hat{v}$: i.e. if and only if $v \geq \hat{v}$, which is determined by the equilibrium probability of obtaining a seat through the FFP lottery, her outside option. Lemma 3 proves that it is optimal to set $\hat{v} = \tilde{p}_l$, and that the total number of tickets sold is independent of $\phi$, provided $\phi$ is above a certain feasibility cutoff. Hence the total surplus generated by any optimal mechanism is constant in $\phi$, so any other feasible $\phi$ (e.g. $\phi'$ in Figure 2) would induce the same $\hat{v}$ and $\tilde{p}_l$. Moreover, consumer surplus in any optimal mechanism is also constant in $\phi$, since the price of points in period 1 leaves all consumers indifferent between any FFP mechanism and the static benchmark mechanism. Hence the firm must be indifferent among all feasible $\phi$, provided it sets prices optimally in period 2.

Figure 2: Period 2 contracting as a function of $\phi$, with a uniform distribution of types.

Intuitively, when demand is low some seats are allocated through the FFP lottery and consumers who own points are charged a lower price than those who do not, when the airline sets prices optimally. But while the price that points owners pay is lower, the allocation of tickets as a function of type is in fact independent of $\phi$, and points owners and non-owners both purchase a ticket according to the same type threshold rule. Hence the airline’s optimal prices induce a points owner to buy a ticket if and only if her type is above a particular cutoff, and they induce a non-owner to buy a ticket if and only if her type is above that very same
cutoff. This feature of the mechanism is optimal for the airline, because both points owners and non-owners count against the airline’s capacity constraint equally, and selling a seat to either kind of consumer has the same effect on the total number of unsold seats which will be allocated through the FFP lottery. Thus the total number of seats that is optimally allocated through the FFP lottery is independent of $\phi$. This further implies that the total expected value of all seats allocated through the lottery is constant in $\phi$: decreasing $\phi$, for example, merely divides this surplus across a smaller mass of consumers. Notice also that the airline captures all of this surplus as revenue, through the sale of points ex ante, and hence the fact that it is constant in $\phi$ means that the airline cannot do better by decreasing $\phi$. Thus the airline is indifferent among all $\phi$ that guarantee that there will be enough consumers to take up all of the capacity given away through the FFP lottery. In particular, $\phi = 1$ is optimal.

To summarize, as a consequence of Lemma 3 we can restrict attention to the case where the airline sells points to all consumers in period 1, and by Lemma 2 it is optimal to charge a redemption fee of 0 when demand is low in period 2. We can now characterize the optimal ticket price at $t = 2$ when demand is low, given that $\phi = 1$ and $f_l = 0$.

**Lemma 4.** Suppose demand is low at $t = 2$, and wlog $\phi = 1$. Under Assumption 2 the optimal ticket price $p_l$ and redemption fee $f_l$ are, respectively:

$$p_l = \frac{(1 - k)(1 - F(\hat{v}))}{F'(\hat{v})}$$ and $f_l = 0$

Total revenue is

$$\Pi_l = \frac{m(1 - k)[1 - F(\hat{v})]^2}{F'(\hat{v})}$$

Intuitively, the ticket price when demand is low in period 2 has to be low enough to induce relatively high types to buy a ticket. The lemma shows that at any price there exists a threshold type, $\hat{v}$, such that types above it buy a ticket, while types below it take the FFP lottery. Since a consumer can take the lottery instead of buying a ticket, her outside option is better than 0, so if the type $\hat{v}$ buys a ticket, the maximum that the airline can charge that type is less than $\hat{v}$, i.e. the airline must set some $p_l < \hat{v}$. Figure 3 provides a graphical representation of how the cutoff type is determined and the rents that type receives due to her outside option. The solid line represents the payoff to a type $v$ from buying a ticket at price $p_l$. The dashed line represents the payoff to type $v$ from taking the FFP lottery, assuming that all other consumers play the strategy where types above $\hat{v}$ buy a ticket and types below $\hat{v}$ take the lottery. Notice that if $\hat{v}$ is the equilibrium cutoff type, the slope of the dashed line, which is the value of the outside option to a type $v$, must be equal to the equilibrium
probability of redeeming a point for a seat in the FFP lottery, which must be consistent with the cutoff type \( \hat{v} \). I.e. the slope of the dashed line itself depends on the cutoff \( \hat{v} \), so the latter is a fixed point.

![Figure 3: The cutoff type \( \hat{v} \) as a fixed point.](image)

The precise price \( p_l \) that is optimal is determined by the standard monopoly screening trade-offs, plus the additional fact that when the airline lowers its price on the margin, it induces more consumers to buy tickets, which lowers the endogenous equilibrium probability of obtaining a seat in the FFP lottery. Hence the outside option of those types that buy a ticket decreases, and the airline can charge a marginally higher price. Thus the airline now has an additional incentive to lower its price and increase the number of seats sold, as that allows it to charge marginally more, given the cutoff strategy that consumers play in equilibrium.

### The price of a point at \( t = 1 \)

Next, I consider the airline’s points pricing decision at \( t = 1 \). By Lemma 3 it is optimal to sell points to all consumers, i.e. to set \( \phi = 1 \), so I consider this case for simplicity. The following lemma characterizes the price that the airline can charge for a point ex ante.

**Lemma 5.** Suppose \( \phi = 1 \) wlog. Under Assumptions 4 and 5 the optimal price of a point at \( t = 1 \) is

\[
p_{FFP} = l \cdot m \cdot \left\{ (1 - F(\bar{v})) \cdot (\bar{p}_l - p_l) + (F(\bar{v}) - F(\hat{v})) \cdot [E(v|v \in (\hat{v}, \bar{v})) - p_l] + F(\hat{v}) \cdot E(v|v \in (0, \hat{v})) \frac{k - m(1 - F(\hat{v}))}{1 - m(1 - F(\hat{v}))} \right\} \equiv CS - CS
\]

where \( \bar{v} \) solves \( \bar{v} = \frac{1 - F(\bar{v})}{F(\bar{v})} \), \( \hat{v} \) solves \( \hat{v} = \frac{[1-F(\hat{v)}][1-m(1-F(\hat{v)))]}{F'(\hat{v})} \), \( \bar{p}_l = \bar{v} \), and \( p_l = \frac{(1-k)(1-F(\hat{v}))}{F'(\hat{v})} \).
In period 1 a consumer has no private information about her type or about the state of the economy, and her willingness to pay for a point is equal to the difference in expected payoffs from two continuation games: one where she buys a point and one where she does not, given that all other consumers buy points. Moreover, the airline knows which consumers buy points and which do not, so it can price discriminate in period 2 between these two kinds of consumers. Because consumers are ex ante homogeneous, the best that the airline can do at $t=1$ is to set the price of a point equal to the maximum that each consumer is willing to pay, which is given precisely by the difference in expected payoffs from the two continuation games above.

The lemma proves that this difference in expected payoffs can be broken down into three components, depending on the consumer’s realized type in period 2, if demand is low. These three distinct subsets of types are as follows: first, the set of high types who would buy a ticket in equilibrium even if they did not buy a point in period 1; second, the set of intermediate types who would not buy a ticket if they did not buy a point, but do buy a ticket if they buy a point; and third, the set of low types who would not buy a ticket in either case, but participate in the FFP lottery if they buy a point. I can then show that the first set of high types gain some surplus due to the fact that they face a lower equilibrium price; the second set of intermediate types gain some surplus due to the fact that they now buy a ticket in equilibrium, whereas they would otherwise get 0; the third set of low types gain some surplus because they now receive a seat from the FFP lottery with some probability. These three components are precisely what makes up the optimal price of a point that the airline sets in period 1.

The FFP mechanism

As a consequence of Lemmas 1–5 we can now fully characterize the optimal FFP mechanism, whereby in period 1 the airline sells points, while in period 2 it first sells tickets, and then allocates any unsold capacity through a lottery among points owners. The only commitment that the airline makes in period 1 regarding period 2 is that it will allocate any unsold capacity among points owners. In particular, it cannot commit ex ante to any particular period 2 contract, and hence the price and allocation of seats must be sequentially optimal in period 2.

While the preceding lemmas use Assumptions 1 and 2 there is some significant redundancy
between the two. First, notice that the latter assumes that the “modified hazard rate”

\[
\frac{F'(v)}{[1 - F(v)][1 - m(1 - F(v))]}
\]

is increasing, which implies that the hazard rate

\[
\frac{F'(v)}{1 - F(v)}
\]

is itself increasing, and hence this part of Assumption 1 is redundant.\(^{22}\) Moreover, notice that under Assumption 2 there exists a unique solution to \(\hat{v} = \frac{[1-F(\hat{v})][1-m(1-F(\hat{v}))]}{F'(\hat{v})}\), and a unique solution to \(\bar{v} = \frac{1-F(\bar{v})}{F'(\bar{v})}\). We can then re-write the conditions on \(k\) as

\[
 k \geq m(1 - F(\hat{v})) \quad \text{and} \quad k \leq 1 - F(\hat{v})
\]

So to summarize, we can combine and restate Assumptions 1 and 2 as follows.

**Assumption 3.** Assume the ‘modified hazard rate’

\[
\frac{F'(v)}{[1 - F(v)][1 - m(1 - F(v))]}
\]

is increasing in \(v\), and \(k\) satisfies the conditions

\[
 k \geq m(1 - F(\hat{v})) \quad \text{and} \quad k \leq 1 - F(\hat{v}),
\]

where \(\hat{v} = \frac{[1-F(\hat{v})][1-m(1-F(\hat{v}))]}{F'(\hat{v})}\) and \(\bar{v} = \frac{1-F(\bar{v})}{F'(\bar{v})}\).

The following proposition characterizes the optimal FFP mechanism, as a consequence of Lemmas \(\text{1}_2\) which hold under Assumption \(3\) I maintain Assumption 3 throughout the remainder of this section.

**Proposition 1.** Consider the optimal FFP mechanism under Assumption 3.

- In period 1 the airline sets \(\phi = 1\) and \(p_{FFP} = CS - CS\).
- If demand is high in period 2, the airline sets \(p_h = F^{-1}(1 - k)\), it sells all \(k\) of its capacity, and there is no FFP lottery.
- If demand is low in period 2, the airline sets \(f_l = 0\) and \(p_l = \frac{(1-k)(1-F(\hat{v}))}{F'(\hat{v})}\), where \(\hat{v}\) is the solution to \(\hat{v} = \frac{[1-F(\hat{v})][1-m(1-F(\hat{v}))]}{F'(\hat{v})}\), and allocates \(k - m(1 - F(\hat{v}))\) seats through the FFP lottery.

\(^{22}\)To see this, suppose that \(\frac{d}{dv} \frac{F'(v)}{[1 - F(v)][1 - m(1 - F(v))]} > 0\), and notice that

\[
\frac{d}{dv} \frac{F'(v)}{1 - F(v)} = \left[ \frac{d}{dv} \frac{F'(v)}{[1 - F(v)][1 - m(1 - F(v))]} + \frac{mF'(v)^2}{(1 - m(1 - F(v))^2(1 - F(v)))} \right] \cdot [1 - m(1 - F(v))],
\]

therefore \(\frac{d}{dv} \frac{F'(v)}{1 - F(v)} > 0\).

\(^{23}\)Note that the interval \([m(1 - F(\hat{v})), 1 - F(\hat{v})]\) is nonempty for \(m\) low enough.

27
The expected revenue from the mechanism is

\[
TR = p_{FFP} + l\Pi_l + (1-l)\Pi_h = p_{FFP} + l \frac{m(1-k)(1-F(\hat{v}))^2}{F'(\hat{v})} + (1-l)kF^{-1}(1-k).
\]

In words, the optimal FFP mechanism is one where the airline sells points to all consumers in period 1, and then allocates any unsold capacity in period 2 through a FFP lottery among those consumers who have not bought a ticket. If demand is high in period 2 all capacity is sold, so there is no lottery. If demand is low in period 2 the airline sells some of its capacity in the form of tickets, and gives away the rest for free through the FFP lottery. The sequentially optimal price of a ticket in period 2 is determined by the realized state of the economy, as well as by the endogenous outside option that the airline creates for the consumers through its commitment to give away any unsold seats.

Next, I discuss some interesting revenue and efficiency properties of the optimal FFP mechanism and compare it to the static benchmark from Remark 1.

**Proposition 2.** The optimal FFP mechanism is more efficient and yields larger revenue than the static benchmark.

Intuitively, there are two reasons why the FFP mechanism is more efficient than the static mechanism: first, more seats are sold to the highest types who buy tickets; second, some additional seats are allocated to lower types who do not buy tickets. Notice that full efficiency in this model means selling all $k$ capacity to the $k$ highest types in period 2, so the FFP mechanism is not fully efficient, and still features some distortion as a result of the usual incentive of a monopolist to screen consumers.

Allocations and prices are identical across the two mechanisms when demand is high in period 2, so I focus on the case where demand is low. In both mechanisms consumers play a cutoff strategy when deciding whether to buy a ticket in period 2, and the monopolist’s choice of price induces a particular cutoff type. In the static benchmark types between $\bar{v}$ and 1 buy a ticket, while types below $\bar{v}$ do not. In the FFP mechanism types between $\hat{v}$ and 1 buy a ticket, while types below $\hat{v}$ take the FFP lottery instead. Moreover, as Proposition 2 shows, $\hat{v} < \bar{v}$, which further implies

\[
p_t < \hat{v} < \bar{v} = \bar{p}_t
\]

Therefore the FFP mechanism proves to be more efficient, since the optimal price, and respectively the optimal type cutoff, are lower than in the static benchmark.
The monopolist optimally sets a lower price and cutoff type because, as shown in Proposition 1, there is now an additional incentive to increase the quantity sold, beyond the usual monopoly screening trade-offs. In particular, selling an additional ticket lowers the probability that a consumer who takes the FFP lottery obtains a seat, since the ratio of unsold seats to consumers who enter the lottery decreases. The outside option of a consumer who buys a ticket decreases, because her next best alternative, to take the FFP lottery, is now relatively less valuable. Therefore the monopolist can charge that consumer a marginally higher price, as determined by the willingness to pay of the cutoff type.

Proposition 2 also shows that the monopolist does better with the FFP mechanism than with the static mechanism. The intuition for this is fairly simple: because the monopolist sells points ex ante, the price she can charge consumers extracts all of the difference between the consumer’s period 2 surplus from the FFP mechanism and her period 2 surplus from the static mechanism. Notice that the airline cannot credibly threaten to exclude the consumer from the market, because period 2 prices have to be sequentially optimal. So if a consumer did not buy a point in period 1, in period 2 she would face prices equivalent to those in the static mechanism. Thus the price of a point is precisely such that it leaves her indifferent between that set of prices and the actual prices she will face when she buys a point. Therefore all of the gains in total surplus accrue to the firm, while the consumers are indifferent between the two mechanisms.

5 Extensions

5.1 The role of FFP commitment

A natural question in this setting is how important is the precise type of commitment that I assume the airline can make in period 1 regarding its use of points. Throughout the paper I assume that the airline cannot commit to a period 2 mechanism ex ante, and it cannot sell seats through type-contingent contracts at $t = 1$. Instead, period 1 is simply meant to represent a period of time when the consumer does not know whether she will be in the market at all, or what her type will be. I assumed that the only commitment the airline can make ex ante, i.e. at $t = 1$, regarding the mechanism offered at $t = 2$, is that it can sell frequent flyer “points” and commit to allocate its unsold capacity ex post to some consumers who participate in the FFP, i.e. who buy a point at $t = 1$. 

29
If the airline could commit to a period 2 mechanism ex ante, it is easy to see that the optimal mechanism would be a kind of “membership club,” whereby the airline allocates all of its capacity fully efficiently in period 2, and then charges a fee for the right to participate in the period 2 mechanism at $t = 1$. Such a fee would extract all of the surplus generated by the period 2 mechanism, which is itself maximized, since the allocation would be efficient ex post. Hence with full ex ante commitment the optimal mechanism would be fully efficient and the airline would capture all of the surplus generated.

On the other hand, as Remark 1 points out, if period 2 was the only period in which the airline could interact with consumers, i.e. if it could not make any commitments ex ante, then the optimal mechanism is simply the posted prices that solve the standard monopoly screening problem. In this case, to maximize revenue the airline would optimally sell a number of seats less than $k$ when demand is low.

Under the assumption that the airline can only commit to allocate unsold capacity among points owners, I showed in the previous section that the optimal mechanism produced an allocation that lies, in some sense, inbetween the allocation in the static benchmark and the allocation that would be optimal if the airline had full contractual commitment ex ante. In other words, under my main assumption the allocation in period 2 is not fully efficient, but is more efficient than in the static mechanism. Similarly, the airline’s total revenue is larger than in the static mechanism, but does not capture all of the total surplus generated.

Do these results continue to hold if we allow the airline to choose how much of its unsold capacity to allocate to points owners ex post? To answer this question we first have to generalize somewhat the timing of the events in the model. In particular, I assume the following: in period 1 the airline sells points through its FFP; in period 2 the state of aggregate demand is publicly realized and consumers privately learn their types; the airline then posts a price for tickets; consumers then simultaneously decide whether to buy tickets or not; finally, the airline decides how much of its unsold capacity to give away to consumers who have not bought tickets. This new timing of the model essentially takes my main model and appends at the end of it a choice by the airline of how much unsold capacity to give away. I denote by $r$ the number of seats that the airline gives away ex post, where

$$r \in [0, k - d(p)]$$

and $d(p)$ is the number of tickets sold at a price of $p$. For tractability, I will restrict attention to the case where the airline sells points to all consumers in period 1, i.e. it sets $\phi = 1$ at $t = 1$, and where it charges a redemption fee of 0 when a consumer redeems her point for a
seat through the FFP lottery, i.e. \( f_h = f_l = 0 \) at \( t = 2 \).

The following proposition shows that this extension of the model has a continuum of equilibria, which span the full range of allocations between the static benchmark in Remark 1 and the FFP mechanism in Proposition 1. Specifically, when demand is high in period 2 the airline sells all \( k \) of its capacity, just as it did in the static and FFP mechanisms; when demand is low in period 2, we have a continuum of equilibria, with price between \( p_l \) from the FFP mechanism and \( \bar{p}_l \) from the static benchmark. Correspondingly, in all equilibria consumers play a cutoff strategy when deciding whether to buy a ticket, and the equilibrium cutoff types cover the range between \( \hat{v} \) and \( \bar{v} \). In this sense, any outcome that is “between” the outcomes of the static mechanism and of the FFP mechanism is an equilibrium outcome. Moreover, the revenue-maximal equilibrium for the airline is the one that corresponds to a price of \( p_l \) and a cutoff type \( \hat{v} \). I.e. the optimal FFP mechanism in Proposition 1 is identical to the revenue-maximal equilibrium in this extension. Therefore the features of the optimal FFP mechanism do not depend critically on the exact form of commitment that I assume.

\[ \text{Proposition 3. Under Assumptions 1 and 2, the extension of the model where the airline chooses } r \text{ ex post has a continuum of equilibria. For any } r \in [0, k - m(1 - F(\hat{v}))] \text{ there exists an equilibrium with prices } \bar{p}_h \text{ and } \bar{p}_l \text{ in high and low demand states, respectively:} \]

\[ \bar{p}_h = F^{-1}(1 - k) \quad \text{and} \quad \bar{p}_l \in [p_l, \bar{p}_l] \]

where \( p_l = \frac{(1 - k)(1 - F(\hat{v}))}{F'(\hat{v})} \), \( \bar{p}_l = \frac{1 - F(\bar{v})}{F'(\bar{v})} \), and \( \hat{v} \) and \( \bar{v} \) solve:

\[ \hat{v} = \frac{[1 - F(\hat{v})][1 - m(1 - F(\hat{v}))]}{F'(\hat{v})} \quad \text{and} \quad \bar{v} = \frac{1 - F(\bar{v})}{F'(\bar{v})}. \]

The revenue-maximal equilibrium, which is in fact weakly Pareto dominant, coincides with the optimal FFP mechanism and has:

\[ \bar{p}_l = \frac{(1 - k)(1 - F(\hat{v}))}{F'(\hat{v})} \quad \text{and} \quad r = k - m(1 - F(\hat{v})) \]

Intuitively, when the airline chooses how much of its unsold capacity to give away, it is indifferent among all feasible \( r \), given that \( f_h = f_l = 0 \). This generates a multiplicity of equilibria, depending on which \( r \) the airline chooses at the final stage of the game. In each equilibrium consumers must correctly anticipate the airline’s choice of \( r \). For any \( r \) consumers must then play a cutoff strategy when deciding whether to buy a ticket or not, with the cutoff

\[ ^{24}\text{In this model it is natural to focus on the revenue-maximal equilibrium for the airline, as the model features one large player, the firm, and a continuum of non-atomic consumers.} \]
\( v'(p) \) depending on their belief about \( r \) and on the posted price \( p \). Given this, the airline chooses \( p \) to maximize its revenue, anticipating that consumers will decide according to the cutoff \( v'(p) \). When demand is high at \( t = 2 \), the optimal price is where the airline sells all \( k \) of its capacity, just as in Lemma [1] When demand is low at \( t = 2 \), different equilibrium values of \( r \) determine different equilibrium prices \( p^*_r \), ranging between the price \( p_l \) from Proposition [1] and the price \( \bar{p}_l \) from Remark [1]. Hence any outcome that is inbetween the outcomes of the FFP mechanism and of the static mechanism can be an equilibrium outcome.

However, the revenue-maximal equilibrium is the one where \( p^*_r = p_l \) and the airline releases all unsold capacity, i.e. \( r \) is such that \( m(1 - F(\hat{v})) + r = k \). This is because in all equilibria of the game consumers have an expected surplus of \( \overline{CS} \), and the airline extracts all of the remaining total surplus through the sale of points at \( t = 1 \). Thus the revenue-maximal equilibrium is the one that generates the largest total surplus, subject to prices being sequentially optimal at \( t = 2 \). Hence the airline is better off in the equilibrium where \( p^*_r \) is as low as possible, given that it is in the set \([p_l, \bar{p}_l] \), and \( r \) is as large as possible, given \( p^*_r \).

6 Conclusion

Frequent flyer programs are arguably the greatest recent invention in the airline industry. Today they account for much of the profitability of airlines in the U.S. and abroad. This paper offers a novel theory of such programs, and of loyalty programs more generally. I consider a model where a monopolist serves a market of consumers who privately learn their own willingness to pay over time. The frequent flyer program provides the airline with an opportunity to contract with consumers ex ante, before their uncertainty is resolved, although with a very limited form of commitment. I assume that the airline may only commit ex ante to allocate any unsold capacity among members of its program ex post, but it cannot commit in advance to future prices and capacity allocation, so the contract that is offered to consumers is constrained by sequential optimality.

I characterize the optimal FFP mechanism and show that despite this lack of commitment the firm benefits from the ability to partially contract with consumers through its loyalty program. Interestingly, in the FFP mechanism the airline allocates seats more efficiently than in the canonical static screening mechanism: it sells more seats, at a lower price, and in addition it randomly allocates any unsold seats to members of its frequent flyer program. The mechanism is therefore more efficient than the static benchmark that I compare it against.
Moreover, the airline captures all of the resulting gains in total surplus, through the optimal price that it charges ex ante for membership in the mileage program. While the introduction of a mileage program provides consumers with a better outside option at the interim stage, and effectively cannibalizes some of the demand for tickets, the firm extracts more in total revenue from the overall mechanism. I further show that the optimal FFP mechanism in the limited commitment setting in fact coincides with the revenue-maximal equilibrium of a game where the airline chooses ex post how much of its unsold capacity to give away.

The results provide a contract theoretic explanation for the profitability of miles and points programs and their prevalence in the airline industry today. The model that I discuss goes beyond the standard theory of frequent flyer programs as generators of contractual switching costs. In fact, I show that a firm benefits from the adoption of a frequent flyer program even in the absence of competition, where the switching costs theory cannot apply. The results also have some surprising implications for firm strategy: it is tempting to think of loyalty programs as a mechanism that induces demand cannibalization, but this is in fact a primary reason why they are profitable overall, and firms which use such programs as a dynamic contracting mechanism in fact benefit from increasing participation and redemptions through their loyalty program. Furthermore, the model and intuition apply more broadly to other industries that share features such as uncertainty of demand, fixed capacity, low marginal costs, perishable product, heterogeneity in consumer valuations, and repeat interactions with consumers.
References


Appendix

Remark 1

Proof. In a low demand state the airline solves:

$$\max_p \, p m (1 - F(p)) \quad \text{subject to} \quad m (1 - F(p)) \leq k$$

The unconstrained optimum is given by the FOC

$$m (1 - F(p)) - mp F'(p) = 0 \quad \Leftrightarrow \quad p = \frac{1 - F(p)}{F'(p)}$$

Notice that Assumption 1 implies that there is a unique $p$ that solves this condition. Let $\bar{p}_l$ be the solution to the above condition. Under Assumption 1, the capacity constraint is not binding at $\bar{p}_l$, so it is indeed the optimal price in a low demand state. Demand at $\bar{p}_l$ is $m (1 - F(\bar{p}_l))$, so the corresponding revenue is $\bar{\Pi}_l = \frac{m (1 - F(\bar{p}_l))^2}{F'(\bar{p}_l)}$.

In a high demand state the airline problem is analogous to the above, with $m = 1$. Under Assumption 1 the capacity constraint binds at the optimum. Hence the optimal price is $\bar{p}_h = F^{-1}(1 - k)$, demand at $\bar{p}_h$ is $k$, and revenue is $\bar{\Pi}_h = k F^{-1}(1 - k)$.

Lemma 1

Proof. First, notice that for any given $\tilde{p}_h$, demand from consumers who do not own points is $(1 - \phi)(1 - F(\tilde{p}_h))$. Notice that the capacity constraint implies the airline will only choose $\tilde{p}_h$ such that $(1 - \phi)(1 - F(\tilde{p}_h)) \leq k$. Let $\tilde{k} = k - (1 - \phi)(1 - F(\tilde{p}_h))$ be the remaining capacity that the airline can sell to points owners and possibly allocate through the FFP lottery.

Consider any incentive compatible pair $f_h, p_h$, i.e. such that $p_h > f_h$, offered to consumers who own points. A consumer of type $v$ receives a payoff of $v - p_h$ from buying a ticket, or $(v - f_h)q$ from the FFP lottery, where $q$ must be the equilibrium probability of winning a seat through the FFP lottery. Incentive compatibility requires $p_h > f_h$, and so all types below $f_h$ will not buy a ticket or take the FFP lottery. Moreover, in equilibrium a consumer of type $v \geq f_h$ must play a cutoff strategy, where she buys a ticket if her type is $v \geq \bar{v}$, and takes the FFP lottery if her type is $v \in [f_h, \bar{v})$, where $\bar{v}$ is the threshold type that satisfies the indifference condition

$$\bar{v} - p_h = (\bar{v} - f_h)q$$
and \( q \) must in equilibrium satisfy

\[
q = \frac{k - \phi(1 - F(\bar{v}))}{\phi(F(\bar{v}) - F(f_h))}
\]

After solving for \( \bar{v} \) and rearranging, we obtain

\[
\bar{v} - p_h = \frac{(p_h - f_h)(k - \phi(1 - F(\bar{v})))}{k - \phi(1 - F(f_h))}
\]

Note that \( k \leq \phi(1 - F(f_h)) \), or otherwise the airline does not allocate all \( k \) capacity to points owners, and hence violates the commitment to allocate all \( k \) of its capacity. Moreover, for type \( \bar{v} \) to buy a ticket, we require \( \bar{v} \geq p_h \), and incentive compatibility of the prices requires \( p_h > f_h \). Hence the condition is satisfied if and only if

\[
\bar{k} = \phi(1 - F(\bar{v}))
\]

Therefore the airline sells all \( k \) capacity to points owners, and there is no unsold capacity to be allocated to points owners through the FFP lottery. I.e. we have \( \bar{v} = p_h \).

Next, consider the airline’s optimal choice of \( \bar{k} \), that is, the fraction of \( k \) that it sells to points owners. Because the FFP lottery does not allocate any seats to points owners, their outside option when deciding whether to buy at \( p_h \) is 0. Moreover, the distribution of types among points owners is identical to that among non-owners, since they buy points ex ante, before any private information is revealed. Hence the airline serves two markets of consumers: points owners, with mass \( \phi \), and non-owners, with mass \( 1 - \phi \), which both have the same distribution of types and the same outside option of 0. The airline solves

\[
\max_{p_h, \tilde{p}_h} Q_h \left[ 1 - F(p_h) \right] + \tilde{p}_h (1 - \phi) \left[ 1 - F(\tilde{p}_h) \right] \quad \text{s.t.} \quad \phi \left[ 1 - F(p_h) \right] + (1 - \phi) \left[ 1 - F(\tilde{p}_h) \right] \leq k
\]

which implicitly pins down \( \bar{k} \).

Notice that under Assumption 1 the capacity constraint binds at the optimum. Hence the optimal prices are

\[
p_h = \bar{p}_h = F^{-1}(1 - k).
\]

Demand among points owners is then \( \phi k \), while demand among non-owners is \( (1 - \phi)k \), so total revenue is

\[
\Pi_h = kF^{-1}(1 - k).
\]

Because all capacity is sold at prices \( p_h = \bar{p}_h = F^{-1}(1 - k) \), there is no unsold capacity to be allocated through the FFP lottery, and hence any \( f_h \) is optimal.
Lemma 2

Proof. For any given $\tilde{p}_l$ demand from consumers who do not own points is $m(1-\phi)(1-F(\tilde{p}_l))$. The capacity constraint implies the airline will only choose $\tilde{p}_l$ such that $m(1-\phi)(1-F(\tilde{p}_l)) \leq k$. Let $\tilde{k} = k - m(1-\phi)(1-F(\tilde{p}_l))$ be the remaining capacity that the airline can sell to points owners and possibly allocate through the FFP lottery.

Consider any incentive compatible pair $f_l, p_l$, i.e. such that $p_l > f_l$, offered to consumers who own points. Suppose $f_l > 0$, and suppose a positive measure of consumers buy tickets and a positive measure of consumers participate in the FFP lottery. First, notice that the FFP lottery is only individually rational for a consumer of type $v \geq f_l$. Second, notice that in equilibrium consumers must play a cutoff strategy, whereby a consumer of type $v$ buys a ticket if and only if $v \geq \hat{v}$, and a consumer of type $v \in [f_l, \hat{v})$ takes the FFP lottery, where $\hat{v}$ is the cutoff type, which must satisfy the indifference condition:

$$\hat{v} - p_l = (\hat{v} - f_l)q$$

where $q$ is the equilibrium probability that a consumer who enters the FFP lottery gets a seat, i.e.

$$q = \frac{\tilde{k} - m\phi(1-F(\hat{v}))}{m\phi(F(\hat{v}) - F(f_l))}$$

Solving for $\hat{v}$ and rearranging, we obtain

$$\hat{v} - p_l = \frac{(p_l - f_l)(\tilde{k} - m\phi(1-F(\hat{v})))}{\tilde{k} - m\phi(1-F(f_l))}$$

We can now show that this contradicts the assumption that a positive measure of consumers buy a ticket at $p_l$ and a positive measure take the FFP lottery, therefore implying that $f_l = 0$. In particular, notice that $\tilde{k} < m\phi(1-F(f_l))$, or otherwise all consumers could obtain a seat from the FFP lottery with probability 1, and hence none would buy a ticket. Next, notice that incentive compatibility requires $\hat{v} \geq p_l$ and $p_l > f_l$. Hence the condition above may only hold if $\hat{v} = p_l$ and $\tilde{k} = m\phi(1-F(\hat{v}))$. Hence all $\tilde{k}$ capacity is sold at a price of $p_l$, and thus no consumers take the FFP lottery, which contradicts our assumption.

Therefore at $f_l > 0$, either all points owners buy a ticket, or all points owners take the FFP lottery. In the former case, the period 2 contract is a posted price with no FFP lottery. In the latter case, the airline only sells seats to points owners through a FFP lottery. It can clearly do better by selling the same number of seats, $\tilde{k}$, at a posted price $p_l = F^{-1}(1 - \tilde{k})$. Hence a posted price mechanism with no lottery weakly dominates any contract with $f_l > 0$. Therefore $f_l = 0$ is optimal at $t = 2$ when demand is low. \qed
Lemma 3

Proof. If demand is high at \( t = 2 \), no seats are allocated through the FFP lottery and points are irrelevant. Hence any \( \phi \) chosen in period 1 is optimal. We can thus restrict attention to the case where demand is low in period 2, to characterize the optimal \( \phi \).

As the proof of Lemma 2 points out, a consumer who does not own a point at \( t = 2 \) buys a ticket if and only if her type is \( v \geq \tilde{p} \), whereas a consumer who owns a point at \( t = 2 \) buys a ticket if and only if her type is \( v \geq \hat{v} \), where \( \hat{v} \) satisfies the threshold type’s indifference condition:

\[
\hat{v} - p_t = \hat{v} \frac{k - m(1 - \phi)(1 - F(\tilde{p})) - m\phi(1 - F(\hat{v}))}{\phi - m\phi(1 - F(\hat{v}))}
\]

Notice that the above condition uses the fact that \( f_t = 0 \) in the optimal contract at \( t = 2 \), from Lemma 2. Solving the condition for \( p_t \), we obtain

\[
p_t = \hat{v} \frac{\phi - k + m(1 - \phi)(1 - F(\tilde{p}))}{\phi - m\phi(1 - F(\hat{v}))}
\]

The airline solves the following problem, expressed in terms of \( \tilde{p}_t \) and \( \hat{v} \):

\[
\max_{\tilde{p}_t, \hat{v}} m(1 - \phi)(1 - F(\tilde{p}_t))\tilde{p}_t + m\phi(1 - F(\hat{v}))\hat{v} \frac{\phi - k + m(1 - \phi)(1 - F(\tilde{p}_t))}{\phi - m\phi(1 - F(\hat{v}))}
\]

The first order conditions for this problem give us the following system of equations:

\[
\hat{v} = \frac{1 - F(\hat{v}) - m + 2mF(\hat{v}) - mF(\hat{v})^2}{F'(\hat{v})}
\]

\[
\tilde{p}_t = \frac{(1 - m + mF(\hat{v}))(1 - F(\tilde{p}_t)) - m\hat{v}(1 - F(\hat{v}))F'(\tilde{p}_t)}{F'(\tilde{p}_t)((1 - m + mF(\hat{v})))}
\]

We can then verify that the solution to the system is

\[
\tilde{p}_t = \hat{v} = \frac{[1 - F(\hat{v})][1 - m(1 - F(\hat{v}))]}{F'(\hat{v})}
\]

Notice that under Assumption 2 there exists a unique \( \hat{v} \) that solves this condition.

Because in the optimal low demand contract \( \hat{v} = \tilde{p}_t \) at \( t = 2 \), the set of consumer types who own points and buy a ticket is the same as the set of consumer types who do not own points and buy a ticket. The airline charges each point owner \( p_t = \hat{v} \frac{k - m(1 - \phi)(1 - F(\tilde{p}_t))}{\phi - m\phi(1 - F(\hat{v}))} \), while it charges each non-owner \( \tilde{p}_t = \hat{v} \). Thus the total number of seats that the airline will sell when demand is low at \( t = 2 \) is \( m(1 - F(\hat{v})) \), which is independent of \( \phi \). The remaining unsold capacity, \( k - m(1 - F(\hat{v})) \) is allocated to points owners through the FFP lottery.
It follows that each consumer who buys a point at \( t = 1 \) expects the following surplus from the FFP lottery:

\[
mlF(\hat{v})E[v|v \in (0, \hat{v})] \frac{k - m(1 - F(\hat{v}))}{m\phi(1 - F(\hat{v})).}
\]

Notice that \( mlF(\hat{v}) \) is the probability that the consumer’s realized type is \( v \in (0, \hat{v}) \) in period 2 and demand is low, \( E[v|v \in (0, \hat{v})] \) is her expected type in that event, and \( \frac{k - m(1 - F(\hat{v}))}{m\phi(1 - F(\hat{v}))} \) is her equilibrium probability of obtaining a seat through the FFP lottery, given that her type is \( v \in (0, \hat{v}) \) and she participates in the lottery. Hence this surplus is included in what the consumer is willing to pay for a point at \( t = 1 \).

For any feasible \( \phi \), the airline charges \( mlF(\hat{v})E[v|v \in (0, \hat{v})] \frac{k - m(1 - F(\hat{v}))}{m\phi(1 - F(\hat{v}))} \) for each point, and therefore it captures the following surplus through points sales:

\[
mlF(\hat{v})E[v|v \in (0, \hat{v})] \frac{k - m(1 - F(\hat{v}))}{m(1 - F(\hat{v}))}
\]

Notice that this expression does not depend on \( \phi \), so the airline is indifferent among all feasible \( \phi \).

Moreover, we can now characterize which \( \phi \) are feasible, i.e. \( \phi \) such that the airline can carry out its commitment to allocate all unsold capacity among points owners. Since \( \hat{v} = \tilde{p}_t \), \( \phi \) is feasible if and only if

\[
\phi + (1 - \phi)m(1 - F(\hat{v})) \geq k
\]

Equivalently, \( \phi \) is feasible if and only if

\[
\phi \geq \frac{k - m(1 - F(\hat{v}))}{1 - m(1 - F(\hat{v}))}
\]

Finally, notice that \( \phi = 1 \) is feasible and the airline is indifferent among all feasible \( \phi \). Hence \( \phi = 1 \) is optimal at \( t = 1 \).

\[\text{Lemma 4}\]

\[\text{Proof.}\] By Lemma \[3\] we can restrict attention without loss of generality to the case where \( \phi = 1 \). By Lemma \[2\] it is optimal to set \( f_t = 0 \) at \( t = 2 \). So we only need to consider the optimal ticket price \( p_t \).

At any posted price \( p_t \) consumers simultaneously decide whether to buy a ticket or not, in which case they enter the FFP lottery. Let \( q \) be the equilibrium probability that a consumer
who enters the FFP lottery obtains a seat. If in equilibrium a type \( v \) buys a ticket, it must be that \( v - p_l \geq vq \), and hence any type \( v' > v \) must also buy a ticket, since \( v' - p_l - v'q > v - p_l - vq \geq 0 \), i.e. \( v' - p_l > v'q \). Analogously, if in equilibrium a type \( v \) does not buy a ticket, it must be that \( vq \geq v - p_l \), and hence any type \( v'' < v \) must also not buy a ticket, since \( v'' - p_l - v''q < v - p_l - vq \leq 0 \), i.e. \( v'' - p_l < v''q \). Hence at any price \( p_l \) there exists a cutoff type such that all types above it buy a ticket, and all types below it take the FFP lottery instead. Denote this cutoff type by \( \hat{v} \). Clearly \( \hat{v} \) must satisfy the indifference condition

\[
\hat{v} - p_l = \hat{v} \cdot \frac{k - m(1 - F(\hat{v}))}{1 - m(1 - F(\hat{v}))}
\]

where the last term is the equilibrium probability that a consumer who enters the lottery gets a seat, which must be consistent with \( \hat{v} \).

Let

\[
H(\hat{v}) = \frac{k - m(1 - F(\hat{v}))}{1 - m(1 - F(\hat{v}))}
\]

be the probability that a consumer gets a seat through the FFP lottery, given that all types above \( \hat{v} \) buy a ticket and all types below \( \hat{v} \) take the FFP lottery. Notice that

\[
p_l = \hat{v} \cdot (1 - H(\hat{v}))
\]

and also

\[
H'(\hat{v}) = \frac{(1 - k)mF'(\hat{v})}{[1 - m(1 - F(\hat{v)))]^2} = [1 - H(\hat{v})]\frac{mF'(\hat{v})}{1 - m(1 - F(\hat{v}))}
\]

We can now re-write the airline’s revenue maximization problem at \( t = 2 \) in terms of the cutoff type \( \hat{v} \) that it wants to induce in equilibrium to buy a ticket:

\[
\max_{\hat{v}} \hat{v}(1 - H(\hat{v})) \cdot m(1 - F(\hat{v}))
\]

The FOC for this problem imply that

\[
m[1 - H(\hat{v})]\left[(1 - F(\hat{v})) - \hat{v}F'(\hat{v}) - \frac{m\hat{v}(1 - F(\hat{v}))F'(\hat{v})}{1 - m(1 - F(\hat{v}))}\right] = 0
\]

and hence the revenue-maximizing \( \hat{v} \) solves the following equation:

\[
\hat{v} = \frac{[1 - F(\hat{v})][1 - m(1 - F(\hat{v))]}{F'(\hat{v})}
\]

Notice that under Assumption 2 this \( \hat{v} \) exists and is unique.

Next, since \( p_l = \hat{v}[1 - H(\hat{v})] \), we have that \( p_l = \frac{[1 - F(\hat{v})][1 - m(1 - F(\hat{v))]}{F'(\hat{v})[1 - m(1 - F(\hat{v}))]} \cdot \frac{1 - k}{1 - m(1 - F(\hat{v}))} \), so the optimal ticket price is

\[
p_l = \frac{1 - k}(1 - F(\hat{v}))
\]

41
Demand for tickets at \( p_t \) is equal to \( m(1 - F(\hat{v})) \), so revenue is

\[
\Pi_t = \frac{m(1 - k)[1 - F(\hat{v})]^2}{F'(\hat{v})}
\]

Lemma 5

Proof. Consider wlog the equilibrium where \( \phi = 1 \), i.e. all consumers buy a point at \( t = 1 \).

A consumer’s ex ante expected payoff from the continuation game where she buys a point is that of a continuation game where, by Lemmas 1 and 4, she will face

\[
p_h = F^{-1}(1 - k) \quad \text{and} \quad p_t = \frac{(1 - k)(1 - F(\hat{v}))}{F'(\hat{v})}
\]

where \( \hat{v} \) solves

\[
\hat{v} = \frac{[1 - F(\hat{v})][1 - m(1 - F(\hat{v}))]}{F'(\hat{v})}.
\]

Denote this expected payoff by \( CS \).

On the other hand, the consumer’s ex ante expected payoff from the continuation game where she does not buy a point is that of a continuation game where, by Remark 1, she will face

\[
\bar{p}_h = F^{-1}(1 - k) \quad \text{and} \quad \bar{p}_t = \bar{v}
\]

where \( \bar{v} \) solves

\[
\bar{v} = \frac{1 - F(\bar{v})}{F'(\bar{v})}.
\]

Denote this expected payoff by \( \overline{CS} \).

Notice that \( CS > \overline{CS} \), as \( \hat{v} < \bar{v} \) and \( p_s \leq \bar{p}_s \) for \( s \in \{l, h\} \). Every consumer is willing to pay at most \( CS - \overline{CS} \) for a point at \( t = 1 \). Thus the revenue-maximizing price of a point at \( t = 1 \), denoted by \( p_{\text{FFP}} \), is:

\[
p_{\text{FFP}} = CS - \overline{CS}
\]

Finally, we can characterize \( p_{\text{FFP}} \). Notice that the consumer’s payoff in the high demand state when she buys a point is the same as when she does not, since there is no FFP lottery when demand is high. Thus we can focus solely on the case where demand is low at \( t = 2 \). We can decompose the expected additional payoff from buying a point into three components,

\[\footnote{I discuss these properties of the FFP mechanism in detail in Proposition 2.}\]
depending on the realization of the consumer’s type in period 2. First, types between \( \bar{v} \) and 1 gain \( \bar{p}_l - p_l \), since they buy a ticket at a price of \( p_l \) instead of \( \bar{p}_l \). Second, types between \( \hat{v} \) and \( \bar{v} \) gain \( E(v|v \in (\hat{v}, \bar{v})) - p_l \), since they now buy a ticket, instead of not buying a ticket at all. Third, types between 0 and \( \hat{v} \) gain \( E(v|v \in (0, \hat{v}))/F'(\hat{v}) \), since they participate in the FFP lottery. Therefore we have:

\[
p_{FFP} = CS - \overline{CS} = \lim \left\{ (1 - F(\bar{v})) \cdot (\bar{p}_l - p_l) + (F(\bar{v}) - F(\hat{v})) \cdot [E(v|v \in (\hat{v}, \bar{v})) - p_l] \right. + \left. F(\hat{v}) \cdot E(v|v \in (0, \hat{v})) \frac{k - m(1 - F(\hat{v}))}{1 - m(1 - F(\hat{v}))} \right\}
\]

Proposition 1

Proof. The parts of the proposition follow immediately as a consequence of Lemmas 1-5. In particular, by Lemma 3 it is optimal in period 1 to set \( \phi = 1 \). By Lemma 5 it is optimal to set \( p_{FFP} = \frac{m(1-k)(1-F(\hat{v}))^2}{F'(\hat{v})} \) when demand is high, and the associated revenue is \( \Pi_h = kF^{-1}(1 - k) \). By Lemma 2 it is optimal to set \( f_l = 0 \) if demand is low in period 2, and by Lemma 4 it is optimal to charge \( p_l = \frac{(1-k)(1-F(\hat{v}))}{F'(\hat{v})} \), where \( \hat{v} \) is the solution to \( \hat{v} = \frac{[1-F(\hat{v})][1-m(1-F(\hat{v}))]}{F'(\hat{v})} \). Notice that under our assumptions \( \hat{v} \) exists and is unique. The corresponding revenue when demand is low in period 2 is \( \Pi_l = \frac{m(1-k)(1-F(\hat{v}))^2}{F'(\hat{v})} \).

Finally, total expected revenue from the FFP mechanism is the sum of the revenue from points sales in period 1, and the expected tickets revenue from period 2, when demand is low with probability \( l \) or high with probability \( 1 - l \).

Proposition 2

Proof. First, if demand is high at \( t = 2 \), all capacity \( k \) is sold at a price of \( F^{-1}(1 - k) \) under both mechanisms, so they generate the same revenue and are fully efficient ex post. Second, if
demand is low at $t = 2$, the airline sells $m(1 - F(\bar{v}))$ seats in the static benchmark, whereas it sells $m(1 - F(\hat{v}))$ in the FFP mechanism. A consumer in the static benchmark buys a ticket if and only if her type is $v \geq \bar{v}$, while in the FFP mechanism she buys a ticket if and only if her type is $v \geq \hat{v}$.

Recall that $\bar{v}$ is the solution to
$$\bar{v} = \frac{1 - F(\hat{v})}{F'(\hat{v})},$$
while $\hat{v}$ is the solution to
$$\hat{v} = \frac{1 - F(\hat{v})[1 - m(1 - F(\hat{v}))]}{F'(\hat{v})}.$$ 

Since the function $\frac{[1 - F(v)][1 - m(1 - F(v))]}{F'(v)}$ is everywhere below $\frac{1 - F(v)}{F'(v)}$, we have $\hat{v} < \bar{v}$. Hence more seats are sold in the FFP mechanism than in the static mechanism. Moreover, the remaining $k - m(1 - F(\hat{v}))$ capacity is allocated to consumers who do not buy tickets. Therefore the FFP mechanism is more efficient than the static mechanism, as the set of consumer types who are allocated a seat strictly contains the set of consumer types who are allocated a seat in the static mechanism.

Hence total surplus is larger in the FFP mechanism. But notice that consumers are indifferent among the two mechanisms, because the airline extracts all of the incremental gains in total surplus when it sells points at $t = 1$. Specifically, points revenue at $t = 1$ is $p_{FFP} = CS - CS$, and hence ex ante each consumer expects precisely $CS$ surplus from the FFP mechanism, the same as what she expects from the static mechanism.

Because total surplus is larger in the FFP mechanism and consumers are indifferent between the FFP and static mechanisms, it follows immediately that the airline’s revenue is higher under the FFP mechanism.

\begin{proof}
We proceed by backward induction. Consider the airline’s choice of $r$ ex post. Suppose the airline has posted some price $p$ and a measure $d(p)$ of consumers have bought tickets. The airline thus chooses $r \in [0, k - d(p)]$ ex post. Note that the airline is indifferent among all feasible $r$ at this stage. Therefore consider an equilibrium with some feasible $r \in [0, k - d(p)]$. When consumers decide whether to buy a ticket or not, in equilibrium they must accurately anticipate $r$.

\end{proof}
For any given equilibrium \( r \), a consumer expects some surplus \( CS^r \) from period 2, taking as sunk the cost of buying a point at \( t = 1 \). When the FFP sells points at \( t = 1 \), the consumer chooses between paying some price \( p_{FFP}^r \) and obtaining an expected surplus of \( CS^r \) in period 2, or not buying a point and obtaining a surplus of \( CS \), which is the surplus of a consumer who is excluded from the FFP lottery at \( t = 2 \), as discussed in Lemma 5. The optimal price of a point, \( p_{FFP}^r \), at \( t = 1 \) is thus

\[
p_{FFP}^r = CS^r - CS,
\]

for a given equilibrium \( r \). Hence in any equilibrium the consumer’s ex ante surplus is \( CS \), while the airline captures the remainder of the total surplus. Thus in the revenue-maximal equilibrium, \( p \) and \( r \) must maximize total surplus in period 2, subject to the constraint that the price offered is sequentially rational.

If demand is high at \( t = 2 \), it follows from the proof of Lemma 1 that \( r = 0 \) and the optimal price is \( p_h^r = F^{-1}(1 - k) \). If demand is low at \( t = 2 \), following the proof of Lemma 4 the sequentially optimal price for any \( r \) at \( t = 2 \) must be in the set \([p_l, \tilde{p}]\), where

\[
p_l = \frac{(1 - k)(1 - F(\hat{v}))}{F'(\hat{v})} \quad \text{and} \quad \tilde{p}_l = \frac{1 - F(\hat{v})}{F'(\hat{v})},
\]

and \( \hat{v} \) and \( \tilde{v} \) solve:

\[
\hat{v} = \frac{[1 - F(\hat{v})][1 - m(1 - F(\hat{v}))]}{F'(\hat{v})} \quad \text{and} \quad \tilde{v} = \frac{1 - F(\tilde{v})}{F'(\tilde{v})}.
\]

Notice that total surplus is maximized, subject to the constraint that \( p_l^r \) be sequentially optimal, when \( p_l^r \) is as large as possible, up to \( p_l \), and when \( r = k - m(1 - F(\hat{v})) \).

Therefore the revenue-maximal equilibrium is the one where:

\[
p_l^r = \frac{(1 - k)(1 - F(\hat{v}))}{F'(\hat{v})} \quad \text{and} \quad r = k - m(1 - F(\hat{v})).
\]